Cosmological Nonlinear Density and Velocity Power Spectra

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Perturbation method:

- Perturbation expansion
- All perturbation variables are small
- Weakly nonlinear
- Strong gravity; fully relativistic
- Valid in all scales
- Fully nonlinear and Exact perturbations

Excluding TT perturbation

Post-Newtonian method:

- Abandon geometric spirit of GR: recover the good old absolute space and absolute time
- Newtonian equations of motion with GR corrections
- Expansion in strength of gravity $\frac{\delta\Phi}{c^2} \sim \frac{GM}{Rc^2} \sim \frac{v^2}{c^2} \ll 1$
- Fully nonlinear
- No strong gravity; weakly relativistic
- Valid far inside horizon
- Case of the Fully nonlinear and Exact perturbations

Fully NL & Exact Pert. Theory

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Covariant (1+3) approach

The linear perturbations are so surprisingly simple that a perturbation analysis accurate to second order may be feasible using the methods of Hawking (1966). … One could then judge the domain of validity of the linear treatment and, more important, gain some insight into the non-linear effects.

Sachs and Wolfe (1967)

Convention: (Bardeen 1988)

Decomposition, possible to NL order (York 1973)

$$ds^2 = -a^2(1+2\alpha)d\eta^2 - 2a^2\left(\beta_{,i} + B^{(v)}_i\right)d\eta dx^i + a^2\left[(1+2\varphi)g^{(3)}_{ij} + 2\gamma_{,i|j} + C^{(v)}_{i|j} + C^{(v)}_{j|i} + 2C^{(t)}_{ij}\right]dx^i dx^j,$$

No TT-pert!

$$\chi \equiv a\beta + a^2\dot{\gamma}, \quad \Psi^{(v)}_i \equiv B^{(v)}_i + a\dot{C}^{(v)}_i,$$

Spatial gauge condition

No anisotropic stress

$$\widetilde{T}_{ab} = \widetilde{\mu}\widetilde{u}_a\widetilde{u}_b + \widetilde{p}\left(\widetilde{u}_a\widetilde{u}_b + \widetilde{g}_{ab}\right) + \widetilde{\pi}_{ab},$$

$$\widetilde{\mu} \equiv \mu + \delta\mu \equiv \mu(1+\delta), \quad \widetilde{p} \equiv p + \delta p, \quad \widetilde{u}_i \equiv a v_i, \quad v_i \equiv -v_{,i} + v^{(v)}_i,$$

Spatial gauge:

$$\gamma \equiv 0 \equiv C^{(v)}_i,$$

$$\chi_i \equiv \chi_{,i} + a\Psi^{(v)}_i = a\left(\beta_{,i} + B^{(v)}_i\right)$$

Temporal gauge still not taken yet!

→ Complete spatial gauge fixing.
Remaining variables are spatially gauge-invariant to fully NL order! ∴ Lose no generality!

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Metric:

$$ds^2 = -a^2(1+2\alpha)\,d\eta^2 - 2a\chi_i d\eta dx^i + a^2(1+2\varphi)\,g^{(3)}_{ij}dx^i dx^j,$$

Energy-momentum tensor:

$$\widetilde{T}_{ab} = \widetilde{\varrho}c^2\widetilde{u}_a\widetilde{u}_b + \widetilde{p}\left(\widetilde{g}_{ab} + \widetilde{u}_a\widetilde{u}_b\right), \qquad \widetilde{\varrho} \to \widetilde{\varrho}\left(1 + \frac{1}{c^2}\widetilde{\Pi}\right),$$

Internal energy

Four-vector:

$$\widetilde{u}_i \equiv a\frac{v_i}{c}, \quad \frac{1}{\widehat{\gamma}}v_i = \widehat{v}_i = \frac{1}{\mathcal{N}}\left[(1+2\varphi)\,\overline{v}_i - \frac{c}{a}\chi_i\right],$$

Coordinate three-velocity of fluid

Fluid three-velocity measured by Eulerian observer

$$\widehat{\gamma} \equiv \sqrt{1 + \frac{v^k v_k}{c^2(1+2\varphi)}} = \frac{1}{\sqrt{1 - \frac{\widehat{v}^k\widehat{v}_k}{c^2(1+2\varphi)}}}$$

: Lorentz factor

Scalar- & vector-type decomposition:

$$\chi_i = c\chi_{,i} + \chi^{(v)}_i, \quad \widehat{v}_i \equiv -\widehat{v}_{,i} + \widehat{v}^{(v)}_i, \quad \chi^{(v)i}{}_{,i} \equiv 0 \equiv \widehat{v}^{(v)i}{}_{,i}.$$

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Metric convention:

$$\widetilde{g}_{00} = -a^2 (1 + 2\alpha), \quad \widetilde{g}_{0i} = -a\chi_i, \quad \widetilde{g}_{ij} = a^2 (1 + 2\varphi) \delta_{ij},$$

Inverse metric:

$$\widetilde{g}^{00} = -\frac{1}{a^2} \frac{1 + 2\varphi}{(1 + 2\varphi)(1 + 2\alpha) + \chi^k \chi_k / a^2}, \quad \widetilde{g}^{0i} = -\frac{1}{a^2} \frac{\chi^i / a}{(1 + 2\varphi)(1 + 2\alpha) + \chi^k \chi_k / a^2},$$

$$\widetilde{g}^{ij} = \frac{1}{a^2 (1 + 2\varphi)} \left(\delta^{ij} - \frac{\chi^i \chi^j / a^2}{(1 + 2\varphi)(1 + 2\alpha) + \chi^k \chi_k / a^2} \right).$$

Exact!

Using the ADM and the covariant formalisms the rest are simple algebra. We do not even need the connection!

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Fully Nonlinear Perturbation Equations without taking temporal gauge condition:

Definition of κ:

$$\kappa \equiv 3\frac{\dot{a}}{a}\left(1-\frac{1}{\mathcal{N}}\right)-\frac{1}{\mathcal{N}(1+2\varphi)}\left[3\dot{\varphi}+\frac{c}{a^2}\left(\chi^k{}_{|k}+\frac{\chi^k\varphi_{,k}}{1+2\varphi}\right)\right].$$

ADM energy constraint:

$$-\frac{3}{2}\left(\frac{\dot{a}^2}{a^2}-\frac{8\pi G}{3c^2}\widetilde{\mu}+\frac{\overline{K}c^2}{a^2(1+2\varphi)}-\frac{\Lambda c^2}{3}\right)+\frac{\dot{a}}{a}\kappa+\frac{c^2\Delta\varphi}{a^2(1+2\varphi)^2}$$
$$=\frac{1}{6}\kappa^2-\frac{4\pi G}{c^2}\left(\widetilde{\mu}+\widetilde{p}\right)\left(\widehat{\gamma}^2-1\right)+\frac{3}{2}\frac{c^2\varphi^{|i}\varphi_{,i}}{a^2(1+2\varphi)^3}-\frac{c^2}{4}\overline{K}^i_j\overline{K}^j_i.$$

ADM momentum constraint:

$$\frac{2}{3}\kappa_{,i}+\frac{c}{a^2\mathcal{N}(1+2\varphi)}\left[\frac{1}{2}\left(\Delta\chi_i+\chi^k{}_{|ik}\right)-\frac{1}{3}\chi^k{}_{|ki}\right)+\frac{8\pi G}{c^4}\left(\widetilde{\mu}+\widetilde{p}\right)a\widehat{\gamma}^2\widehat{v}_i$$
$$=\frac{c}{a^2\mathcal{N}(1+2\varphi)}\left\{\left(\frac{\mathcal{N}_{,j}}{\mathcal{N}}-\frac{\varphi_{,j}}{1+2\varphi}\right)\left[\frac{1}{2}\left(\chi^j{}_{|i}+\chi_i{}^{|j}\right)-\frac{1}{3}\delta^j_i\chi^k{}_{|k}\right]\right.$$
$$\left.-\frac{\varphi^{,j}}{(1+2\varphi)^2}\left(\chi_i\varphi_{,j}+\frac{1}{3}\chi_j\varphi_{,i}\right)+\frac{\mathcal{N}}{1+2\varphi}\nabla_j\left[\frac{1}{\mathcal{N}}\left(\chi^j\varphi_{,i}+\chi_i\varphi^{|j}-\frac{2}{3}\delta^j_i\chi^k\varphi_{,k}\right)\right]\right\}.$$

Trace of ADM propagation:

$$-3\left[\frac{1}{\mathcal{N}}\left(\frac{\dot{a}}{a}\right)^{\cdot}+\frac{\dot{a}^2}{a^2}+\frac{4\pi G}{3c^2}\left(\widetilde{\mu}+3\widetilde{p}\right)-\frac{\Lambda c^2}{3}\right]+\frac{1}{\mathcal{N}}\dot{\kappa}+2\frac{\dot{a}}{a}\kappa+\frac{c^2\Delta\mathcal{N}}{a^2\mathcal{N}(1+2\varphi)}$$
$$=\frac{1}{3}\kappa^2+\frac{8\pi G}{c^2}\left(\widetilde{\mu}+\widetilde{p}\right)\left(\widehat{\gamma}^2-1\right)-\frac{c}{a^2\mathcal{N}(1+2\varphi)}\left(\chi^i\kappa_{,i}+c\frac{\varphi^{|i}\mathcal{N}_{,i}}{1+2\varphi}\right)+c^2\overline{K}^i_j\overline{K}^j_i.$$

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Tracefree ADM propagation (with linear tensor perturbation):

$$\ddot{h}_{ij} + 3H\dot{h}_{ij} - c^2\frac{\Delta - 2\overline{K}}{a^2}h_{ij} + \left(\frac{1}{\mathcal{N}}\frac{\partial}{\partial t} + 3\frac{\dot{a}}{a} - \kappa + \frac{c\chi^k}{a^2\mathcal{N}(1+2\varphi)}\nabla_k\right)\left\{\frac{c}{a^2\mathcal{N}(1+2\varphi)}\right.$$

Tensor-type to linear order

$$\left.\times\left[\frac{1}{2}\left(\chi^i{}_{|j} + \chi_j{}^{|i}\right) - \frac{1}{3}\delta^i_j\chi^k{}_{|k} - \frac{1}{1+2\varphi}\left(\chi^i\varphi_{,j} + \chi_j\varphi^{|i} - \frac{2}{3}\delta^i_j\chi^k\varphi_{,k}\right)\right]\right\}$$

$$-\frac{c^2}{a^2(1+2\varphi)}\left[\frac{1}{1+2\varphi}\left(\nabla^i\nabla_j - \frac{1}{3}\delta^i_j\Delta\right)\varphi + \frac{1}{\mathcal{N}}\left(\nabla^i\nabla_j - \frac{1}{3}\delta^i_j\Delta\right)\mathcal{N}\right]$$

$$= \frac{8\pi G}{c^2}(\widetilde{\mu} + \widetilde{p})\left[\frac{\widehat{\gamma}^2\widehat{v}^i\widehat{v}_j}{c^2(1+2\varphi)} - \frac{1}{3}\delta^i_j\left(\widehat{\gamma}^2 - 1\right)\right] + \frac{c^2}{a^4\mathcal{N}^2(1+2\varphi)^2}\left[\frac{1}{2}\left(\chi^{i|k}\chi_{j|k} - \chi_{k|j}\chi^{k|i}\right)\right.$$

$$\left.+\frac{1}{1+2\varphi}\left(\chi^{k|i}\chi_k\varphi_{,j} - \chi^{i|k}\chi_j\varphi_{,k} + \chi_{k|j}\chi^k\varphi^{|i} - \chi_{j|k}\chi^i\varphi^{|k}\right) + \frac{2}{(1+2\varphi)^2}\left(\chi^i\chi_j\varphi^{|k}\varphi_{,k} - \chi^k\chi_k\varphi^{|i}\varphi_{,j}\right)\right]$$

$$-\frac{c^2}{a^2(1+2\varphi)^2}\left[\frac{3}{1+2\varphi}\left(\varphi^{|i}\varphi_{,j} - \frac{1}{3}\delta^i_j\varphi^{|k}\varphi_{,k}\right) + \frac{1}{\mathcal{N}}\left(\varphi^{|i}\mathcal{N}_{,j} + \varphi_{,j}\mathcal{N}^{|i} - \frac{2}{3}\delta^i_j\varphi^{|k}\mathcal{N}_{,k}\right)\right].$$

Covariant energy conservation:

$$\left[\frac{\partial}{\partial t} + \frac{1}{a(1+2\varphi)}\left(\mathcal{N}\widehat{v}^k + \frac{c}{a}\chi^k\right)\nabla_k\right]\widetilde{\mu} + (\widetilde{\mu} + \widetilde{p})\left\{\mathcal{N}\left(3\frac{\dot{a}}{a} - \kappa\right)\right.$$

$$\left.+\frac{(\mathcal{N}\widehat{v}^k)_{|k}}{a(1+2\varphi)} + \frac{\mathcal{N}\widehat{v}^k\varphi_{,k}}{a(1+2\varphi)^2} + \frac{1}{\widehat{\gamma}}\left[\frac{\partial}{\partial t} + \frac{1}{a(1+2\varphi)}\left(\mathcal{N}\widehat{v}^k + \frac{c}{a}\chi^k\right)\nabla_k\right]\widehat{\gamma}\right\} = 0.$$

Covariant momentum conservation:

$$\frac{1}{a\widehat{\gamma}}\left[\frac{\partial}{\partial t} + \frac{1}{a(1+2\varphi)}\left(\mathcal{N}\widehat{v}^k + \frac{c}{a}\chi^k\right)\nabla_k\right](a\widehat{\gamma}\widehat{v}_i) + \widehat{v}^k\nabla_i\left(\frac{c\chi_k}{a^2(1+2\varphi)}\right) + \frac{c^2}{a}\mathcal{N}_{,i} - \left(1 - \frac{1}{\widehat{\gamma}^2}\right)\frac{c^2\mathcal{N}\varphi_{,i}}{a(1+2\varphi)}$$

$$+\frac{1}{\widetilde{\mu} + \widetilde{p}}\left\{\frac{\mathcal{N}c^2}{a\widehat{\gamma}^2}\widetilde{p}_{,i} + \widehat{v}_i\left[\frac{\partial}{\partial t} + \frac{1}{a(1+2\varphi)}\left(\mathcal{N}\widehat{v}^k + \frac{c}{a}\chi^k\right)\nabla_k\right]\widetilde{p}\right\} = 0.$$

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with

$$\mathcal{N} \equiv \sqrt{1 + 2\alpha + \frac{\chi^k \chi_k}{a^2(1+2\varphi)}} \equiv 1 + \delta\mathcal{N}, \quad \overline{K}^i_j \overline{K}^j_i = \frac{1}{a^4 \mathcal{N}^2 (1+2\varphi)^2} \left\{ \frac{1}{2} \chi^{i|j} \left(\chi_{i|j} + \chi_{j|i} \right) - \frac{1}{3} \chi^i{}_{|i} \chi^j{}_{|j} \right.$$

$$\left. - \frac{4}{1+2\varphi} \left[\frac{1}{2} \chi^i \varphi^{|j} \left(\chi_{i|j} + \chi_{j|i} \right) - \frac{1}{3} \chi^i{}_{|i} \chi^j \varphi_{,j} \right] + \frac{2}{(1+2\varphi)^2} \left(\chi^i \chi_i \varphi^{|j} \varphi_{,j} + \frac{1}{3} \chi^i \chi^j \varphi_{,i} \varphi_{,j} \right) \right\}.$$

To Background order:

$$\frac{\dot{a}^2}{a^2} = \frac{8\pi G}{3}\varrho - \frac{Kc^2}{a^2} + \frac{\Lambda c^2}{3}, \quad \frac{\ddot{a}}{a} = -\frac{4\pi G}{3}\left(\varrho + 3\frac{p}{c^2}\right) + \frac{\Lambda c^2}{3}, \quad \dot{\varrho} + 3\frac{\dot{a}}{a}\left(\varrho + \frac{p}{c^2}\right) = 0,$$

ADM energy-constraint — **Trace of ADM propagation** — **Covariant E-conservation**

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Temporal gauge (slicing, hypersurface):

$$\begin{aligned}
&\text{comoving gauge}: && v \equiv 0, \longleftrightarrow \widehat{v} \equiv 0\\
&\text{zero-shear gauge}: && \chi \equiv 0,\\
&\text{uniform-curvature gauge}: && \varphi \equiv 0,\\
&\text{uniform-expansion gauge}: && \kappa \equiv 0,\\
&\text{uniform-density gauge}: && \delta \equiv 0,\\
&\text{synchronous gauge}: && \alpha \equiv 0.
\end{aligned}$$

differs from third order in the presence of vector –type perturbation

~~synchronous gauge : $\alpha \equiv 0$.~~

Fully NL formulation Not available

Applicable to NL orders!

Complete gauge fixing.

Remaining variables are gauge-invariant to fully NL order!

Zero-pressure Irrotational Fluid

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Covariant energy-conservation:

Comoving gauge + irrotational   Zero-pressure

$$\left[\frac{\partial}{\partial t} + \frac{1}{a(1+2\varphi)}\left(\mathcal{N}\widehat{v}^i + \frac{c}{a}\chi^i\right)\nabla_i\right]\widetilde{\mu} + (\widetilde{\mu} + \widetilde{p})\left\{(3H-\kappa)\mathcal{N} + \frac{(\mathcal{N}\widehat{v}^i)_{|i}}{a(1+2\varphi)}\right.$$

$$\left. + \frac{\mathcal{N}\widehat{v}^i\varphi_{,i}}{a(1+2\varphi)^2} + \frac{1}{\widehat{\gamma}}\left[\frac{\partial}{\partial t} + \frac{1}{a(1+2\varphi)}\left(\mathcal{N}\widehat{v}^i + \frac{c}{a}\chi^i\right)\nabla_i\right]\widehat{\gamma}\right\} = 0.$$

Comoving gauge + irrotational ($v^i = 0$) + zero-pressure:

$$\left(\frac{\dot{\mu}}{\mu} + 3H\right)(1+\delta) + \dot{\delta} - \kappa = \delta\kappa - \frac{\chi^{,i}\delta_{,i}}{a^2(1+2\varphi)},$$

Exact!

= 0, Background order

To 12th-order perturbation, say:

$$\dot{\delta} - \kappa = \delta\kappa - \frac{1}{a^2}\chi^{,i}\delta_{,i}\Big(1 - 2\varphi + 4\varphi^2 - 8\varphi^3 + 16\varphi^4 - 32\varphi^5$$

$$+64\varphi^6 - 128\varphi^7 + 2566\varphi^8 - 512\varphi^9 + 1024\varphi^{10}\Big),$$

Zero-pressure fluid in the comoving gauge

Exact equations(flat backround):

Covariant energy-conservation:

$$\dot{\delta} - \kappa - \delta\kappa + \frac{1}{a^2}\chi^{,i}\delta_{,i} = \frac{2\varphi\chi^{,i}\delta_{,i}}{a^2(1+2\varphi)},$$

Trace of ADM propagation:

$$\dot{\kappa} + 2H\kappa - 4\pi G\delta\mu - \frac{1}{3}\kappa^2 + \frac{1}{a^2}\chi^{,i}\kappa_{,i} - \frac{1}{a^4}\left[\chi^{,ij}\chi_{,ij} - \frac{1}{3}(\Delta\chi)^2\right] = \frac{2\varphi\chi^{,i}\kappa_{,i}}{a^2(1+2\varphi)} - \frac{4\varphi(1+\varphi)}{a^4(1+2\varphi)^2}\left[\chi^{,ij}\chi_{,ij} - \frac{1}{3}(\Delta\chi)^2\right]$$

$$+ \frac{2}{a^4(1+2\varphi)^3}\left\{\frac{2}{3}(\Delta\chi)\chi^{,i}\varphi_{,i} - 2\chi^{,ij}\chi_{,i}\varphi_{,j} + \frac{1}{1+2\varphi}\left[\frac{1}{3}\left(\chi^{,i}\varphi_{,i}\right)^2 + \chi^{,i}\chi_{,i}\varphi^{,j}\varphi_{,j}\right]\right\},$$

ADM momentum constraint:

$$\left(\kappa + \frac{\Delta}{a^2}\chi\right)_{,i} = \frac{2\varphi\Delta\chi_{,i}}{a^2(1+2\varphi)} + \frac{1}{a^2(1+2\varphi)^2}\left[2(\Delta\chi)\varphi_{,i} + \frac{1}{2}\chi^{,k}\varphi_{,ik} - \chi_{,ik}\varphi^{,k} + \frac{3}{2}\chi_{,i}\Delta\varphi - \frac{3}{2}\frac{1}{1+2\varphi}\left(\chi_{,i}\varphi_{,k} + \frac{1}{3}\chi_{,k}\varphi_{,i}\right)\varphi^{,k}\right].$$

RHS = pure Einstein's gravity corrections, starting from the third order, all involving φ

Definition of kappa + ADM momentum constraint:

$$[\ln(1+2\varphi)]^{\cdot}_{,i} = \frac{1}{a^2(1+2\varphi)^2}\left[\chi^{,k}\varphi_{,ik} + \chi_{,i}\Delta\varphi - \frac{1}{1+2\varphi}\left(\chi_{,i}\varphi_{,k} + 3\chi_{,k}\varphi_{,i}\right)\varphi^{,k}\right],$$

Identify: $\kappa \equiv -\frac{1}{a}\nabla\cdot\mathbf{u}$

Linear-order:

$$\ddot{\delta} + 2\frac{\dot{a}}{a}\dot{\delta} - 4\pi G\mu\delta = 0,$$

Relativistic/Newtonian correspondence to second order.
This equation is valid to fully nonlinear order in Newtonian theory.

Second-order:

$$\ddot{\delta} + 2\frac{\dot{a}}{a}\dot{\delta} - 4\pi G\mu\delta = -\frac{1}{a^2}\frac{\partial}{\partial t}\left[a\nabla \cdot (\delta \mathbf{u})\right] + \frac{1}{a^2}\nabla \cdot (\mathbf{u} \cdot \nabla \mathbf{u}),$$

Pure relativistic correction appearing from third order. All involving $\boldsymbol{\varphi}$.

Third-order:

$$\ddot{\delta} + 2\frac{\dot{a}}{a}\dot{\delta} - 4\pi G\mu\delta = -\frac{1}{a^2}\frac{\partial}{\partial t}\left[a\nabla \cdot (\delta \mathbf{u})\right] + \frac{1}{a^2}\nabla \cdot (\mathbf{u} \cdot \nabla \mathbf{u})$$

$$+ \frac{1}{a^2}\frac{\partial}{\partial t}\{a[2\varphi\mathbf{u} - \nabla(\Delta^{-1}X)] \cdot \nabla\delta\} - \frac{4}{a^2}\nabla \cdot \left[\varphi\left(\mathbf{u} \cdot \nabla\mathbf{u} - \frac{1}{3}\mathbf{u}\nabla \cdot \mathbf{u}\right)\right]$$

$$+ \frac{2}{3a^2}\varphi\mathbf{u} \cdot \nabla(\nabla \cdot \mathbf{u}) + \frac{\Delta}{a^2}[\mathbf{u} \cdot \nabla(\Delta^{-1}X)] - \frac{1}{a^2}\mathbf{u} \cdot \nabla X - \frac{2}{3a^2}X\nabla \cdot \mathbf{u},$$

$$X \equiv 2\varphi\nabla \cdot \mathbf{u} - \mathbf{u} \cdot \nabla\varphi + \frac{3}{2}\Delta^{-1}\nabla \cdot [\mathbf{u} \cdot \nabla(\nabla\varphi) + \mathbf{u}\Delta\varphi].$$

Power spectra:

$$f(\mathbf{k}) = \int d^3x f(\mathbf{x}) e^{i\mathbf{k}\cdot\mathbf{x}}$$

$$\langle \delta(\mathbf{k}_1, t)\delta(\mathbf{k}_2, t)\rangle \equiv (2\pi)^3 \delta^{(3)}(\mathbf{k}_1 + \mathbf{k}_2) P_\delta(k_1, t)$$

$$\langle \theta(\mathbf{k}_1, t)\theta(\mathbf{k}_2, t)\rangle \equiv (2\pi)^3 \delta^{(3)}(\mathbf{k}_1 + \mathbf{k}_2) P_\theta(k_1, t)$$

$$\theta \equiv \frac{1}{a}\nabla \cdot \mathbf{u}$$

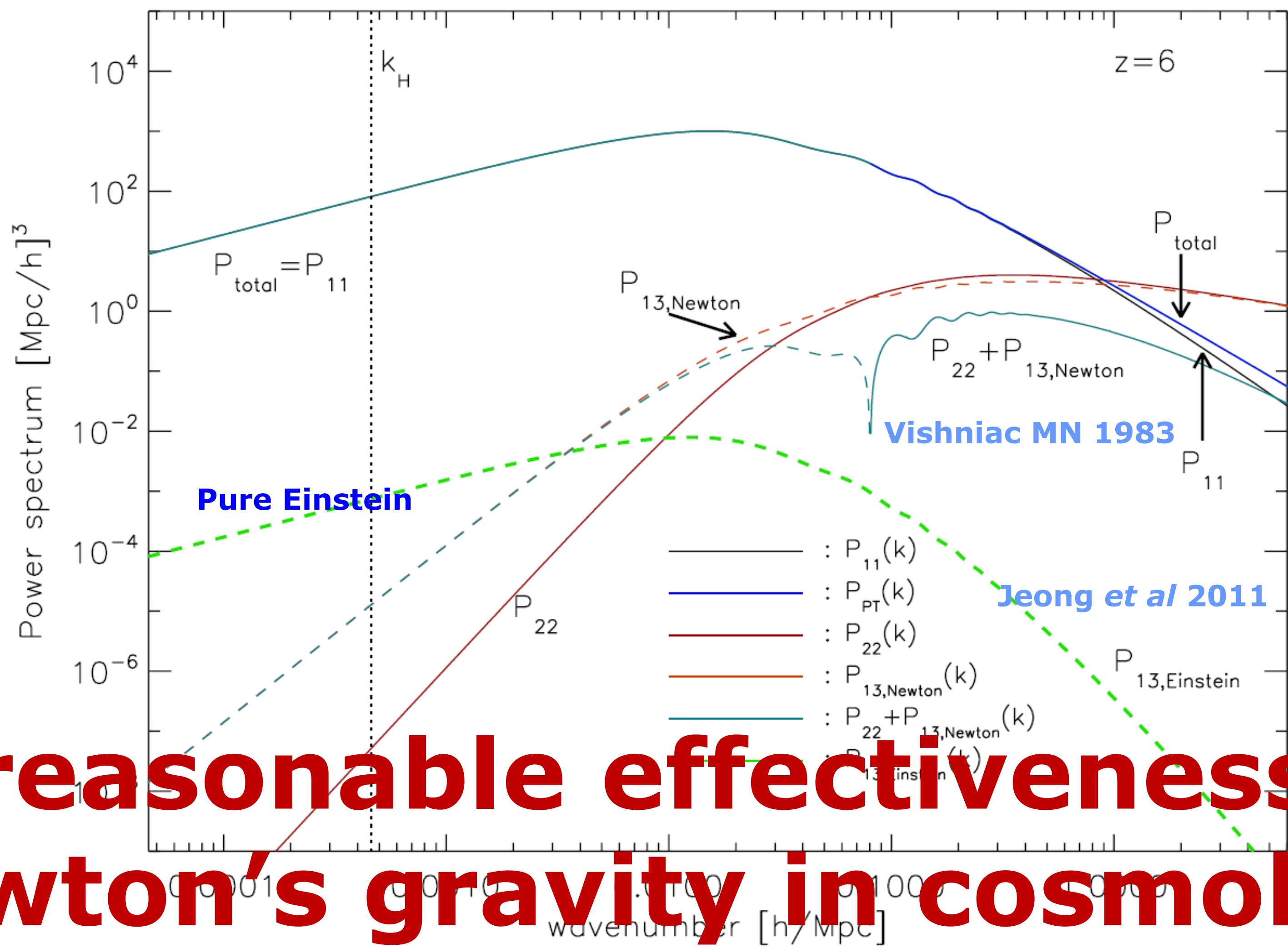
$$P \equiv P_{11} + P_{22} + P_{13}$$

$$P_{13} = P_{13}^{\mathrm{N}} + P_{13}^{\mathrm{E}}$$

$$P_{13}^{\mathrm{E}} = P_{13}^{\mathrm{ES}} + P_{13}^{\mathrm{EV}} + P_{13}^{\mathrm{ET}}$$

JH, Jeong & Noh , arXiv:1509.07534

Leading Nonlinear Density Power-spectrum in the Comoving gauge:



Unreasonable effectiveness of Newton's gravity in cosmology!

Jeong, et al., ApJ **722**, 1 (2011)

General Relativistic Continuity and Euler equations to Third order in the Comoving gauge:

Newtonian

$$\dot{\delta} + \frac{1}{a}\nabla\cdot\mathbf{u} + \frac{1}{a}\nabla\cdot(\delta\mathbf{u}) = \frac{1}{a}(\nabla\delta)\cdot\left[2\varphi\mathbf{u} - \Delta^{-1}(\nabla X + \mathbf{Y})\right],$$

$$\frac{1}{a}\nabla\cdot\left(\dot{\mathbf{u}} + \frac{\dot{a}}{a}\mathbf{u}\right) + 4\pi G\varrho\delta + \frac{1}{a^2}\nabla\cdot(\mathbf{u}\cdot\nabla\mathbf{u}) = \frac{1}{a^2}\left\{-\frac{2}{3}\varphi\mathbf{u}\cdot\nabla(\nabla\cdot\mathbf{u}) + 4\nabla\cdot\left[\varphi\left(\mathbf{u}\cdot\nabla\mathbf{u} - \frac{1}{3}\mathbf{u}\nabla\cdot\mathbf{u}\right)\right]\right.$$

$$\left. + \frac{2}{3}X\nabla\cdot\mathbf{u} + \mathbf{u}\cdot(\nabla X + \mathbf{Y}) - \Delta\left[\mathbf{u}\cdot\Delta^{-1}(\nabla X + \mathbf{Y})\right]\right\} + 2\frac{\dot{a}}{a}\frac{1}{a}u^{,ij}\Delta^{-1}\left(a^2 Z_{ij}\right).$$

Vector

Tensor

$$X \equiv 2\varphi\nabla\cdot\mathbf{u} - \mathbf{u}\cdot\nabla\varphi + \frac{3}{2}\Delta^{-1}\nabla\cdot(\mathbf{u}\cdot\nabla\nabla\varphi + \mathbf{u}\Delta\varphi),$$

$$\mathbf{Y} \equiv 2\left[\mathbf{u}\cdot\nabla\nabla\varphi + \mathbf{u}\Delta\varphi - \nabla\Delta^{-1}\nabla\cdot(\mathbf{u}\cdot\nabla\nabla\varphi + \mathbf{u}\Delta\varphi)\right].$$

Vector

$$Z_{ij} \equiv N_{ij} - 2\Delta^{-1}\nabla_{(i}N^k_{j),k} + \frac{1}{2}\Delta^{-2}\left(\nabla_i\nabla_j + \delta_{ij}\Delta\right)N^{k\ell}{}_{,k\ell},$$

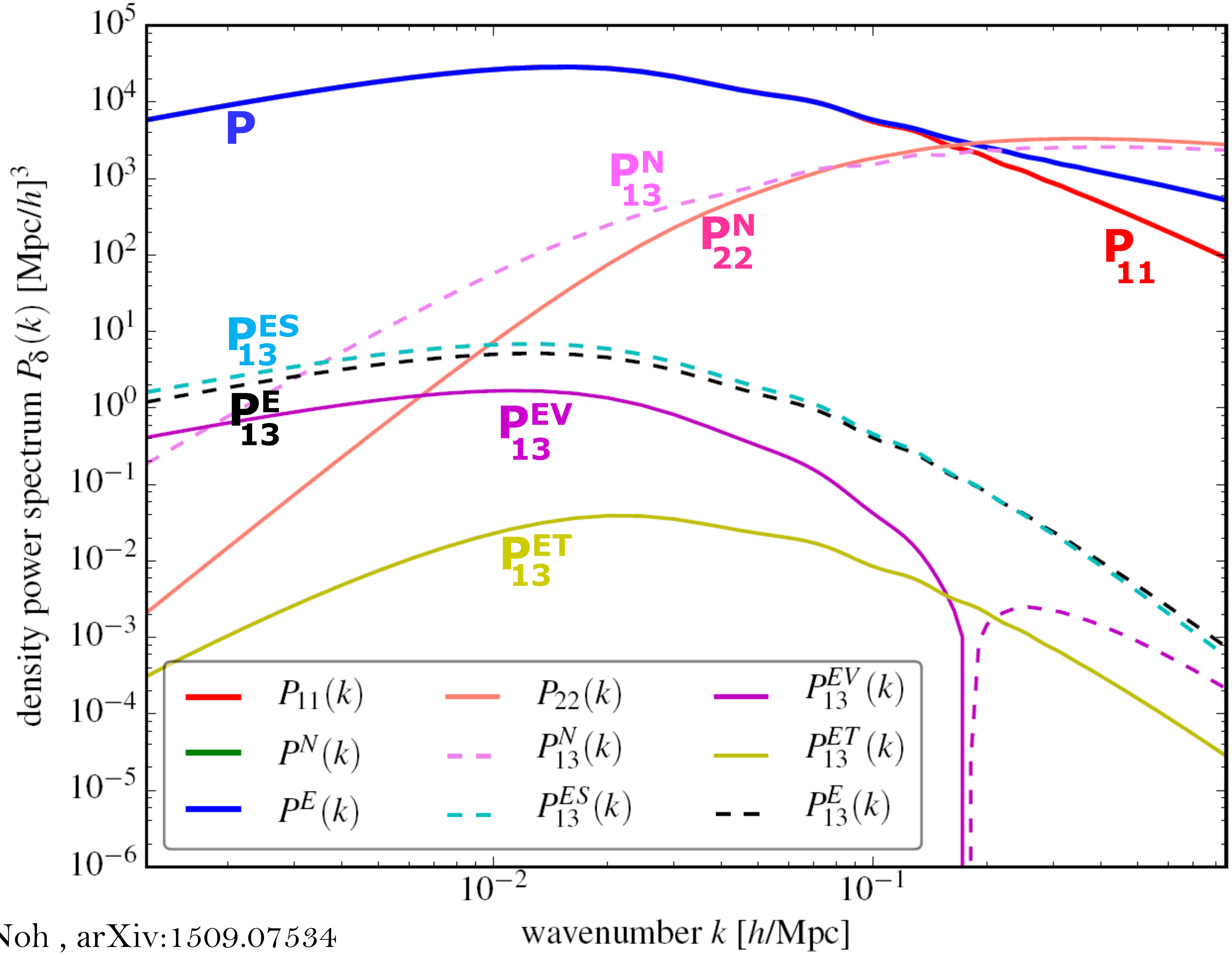
Tensor

$$a^2 N_{ij} \equiv -\frac{1}{c^2}\left\{u_{,ij}\nabla\cdot\mathbf{u} + \mathbf{u}\cdot\mathbf{u}_{,ij} - \frac{1}{3}\delta_{ij}\left[(\nabla\cdot\mathbf{u})^2 + \mathbf{u}\cdot(\Delta\mathbf{u})\right]\right\},$$

$$c^2\frac{\Delta}{a^2}\varphi = -4\pi G\varrho\delta + \frac{\dot{a}}{a}\frac{1}{a}\nabla\cdot\mathbf{u}.$$

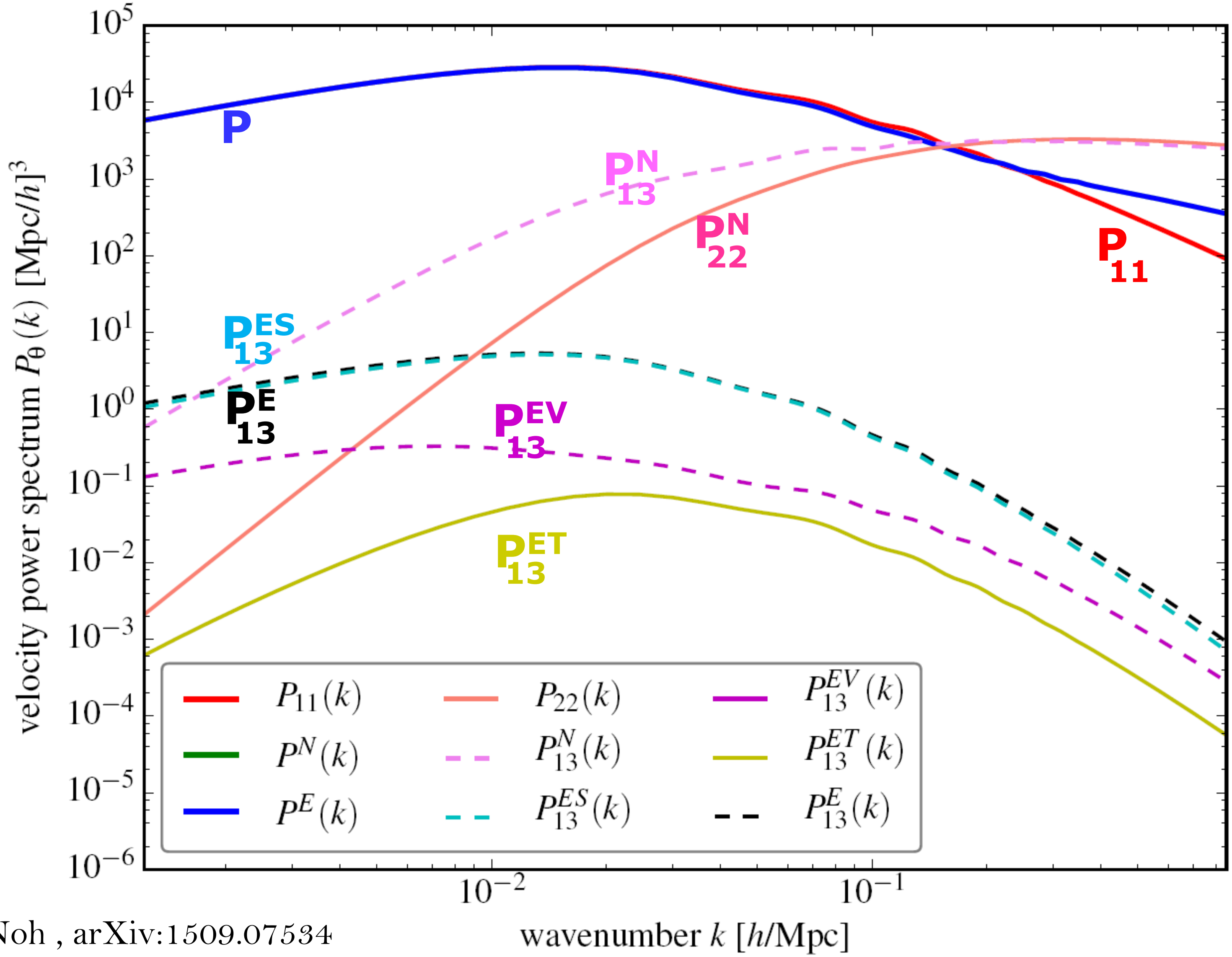
JH, Jeong & Noh , arXiv:1509.07534

Nonlinear Density Power-spectrum with vector and tensor contributions:



JH, Jeong & Noh , arXiv:1509.07534

Nonlinear Velocity Power-spectrum with vector and tensor contributions:



JH, Jeong & Noh , arXiv:1509.07534

Newtonian Limit

Chandrasekhar, ApJ (1965): **0PN, Minkowski**
JH, Noh & Puetzfeld, JCAP (2008): **cosmological**
Here: **as a limit of FNL&E PT**
JH & Noh, JCAP **04** (2013) 035

Infinite speed-of-light Limit in ZSG & UEG:

$$\alpha \ll 1, \quad \varphi \ll 1, \quad \frac{\widehat{v^k}\widehat{v_k}}{c^2} \ll 1, \quad \widetilde{p} \ll \widetilde{\varrho} c^2, \quad \frac{1}{c^2}\widetilde{\Pi} \ll 1, \quad \frac{c^2 k^2}{a^2 H^2} \gg 1,$$

Subhorizon limit

Covariant energy-conservation:

ZSG

$$\left[\frac{\partial}{\partial t} + \frac{1}{a(1+2\varphi)}\left(\mathcal{N}\widehat{v^k} + \frac{c}{a}\chi^k\right)\nabla_k\right]\widetilde{\varrho} + \left(\widetilde{\varrho} + \frac{\widetilde{p}}{c^2}\right)\left\{\mathcal{N}\left(3\frac{\dot{a}}{a} - \kappa\right) + \frac{(\mathcal{N}\widehat{v^k})_{,k}}{a(1+2\varphi)} + \frac{\mathcal{N}\widehat{v^k}\varphi_{,k}}{a(1+2\varphi)^2} + \frac{1}{\widehat{\gamma}}\left[\frac{\partial}{\partial t} + \frac{1}{a(1+2\varphi)}\left(\mathcal{N}\widehat{v^k} + \frac{c}{a}\chi^k\right)\nabla_k\right]\widehat{\gamma}\right\} = 0.$$

$$\alpha = -\frac{1}{c^2}U, \quad \varphi = \frac{1}{c^2}V, \quad \widehat{v^k} = \mathbf{v}, \qquad \mathcal{N} \equiv \sqrt{1 + 2\alpha + \frac{\chi^k \chi_k}{a^2(1+2\varphi)}},$$

ADM momentum-constraint:

$$\kappa = -\frac{12\pi G \varrho a}{c^2 \Delta}\nabla \cdot \left[(1+\delta)\,\mathbf{v}\right],$$

Tracefree ADM propagation:

$$\varphi = -\alpha.$$

$$\Rightarrow \quad \dot{\widetilde{\varrho}} + 3\frac{\dot{a}}{a}\widetilde{\varrho} + \frac{1}{a}\nabla \cdot (\widetilde{\varrho}\mathbf{v}) = 0$$

Equations in Newtonian limit:

Covariant energy-conservation: $$\dot{\widetilde{\varrho}} + 3\frac{\dot{a}}{a}\widetilde{\varrho} = -\frac{1}{a}\nabla\cdot(\widetilde{\varrho}\mathbf{v}),$$

Covariant momentum-conservation: $$\dot{\mathbf{v}} + \frac{\dot{a}}{a}\mathbf{v} + \frac{1}{a}\mathbf{v}\cdot\nabla\mathbf{v} = \frac{1}{a}\nabla U - \frac{1}{a\widetilde{\varrho}}\nabla\widetilde{p},$$

Trace of ADM propagation: $$\frac{\Delta}{a^2}U = -4\pi G\left(\widetilde{\varrho} - \varrho\right).$$

Post-Newtonian Approximation

Chandrasekhar, ApJ (1965): **1PN, Minkowski**
JH, Noh & Puetzfeld, JCAP (2008): **cosmological**
Here: **as a limit of FNL&E PT**
Noh & JH, JCAP **08** (2013) 040

1PN convention: (Chandrasekhar 1965)

$$ds^2 = -\left[1 - \frac{1}{c^2}2U + \frac{1}{c^4}\left(2U^2 - 4\Phi\right)\right]c^2dt^2 - \frac{1}{c^3}2aP_i c dt dx^i + a^2\left(1 + \frac{1}{c^2}2V\right)\gamma_{ij}dx^i dx^j,$$

$$\widetilde{\mu} \equiv \mu \equiv \varrho c^2\left(1 + \frac{1}{c^2}\Pi\right), \quad \widetilde{p} = p, \quad \widetilde{u}^i \equiv \frac{1}{c}\frac{1}{a}\overline{v}^i\widetilde{u}^0,$$

~Shear

Identification:

PT → φ, 1PN → V

$$\alpha = -\frac{1}{c^2}\left[U - \frac{1}{c^2}\left(U^2 - 2\Phi\right)\right], \quad \varphi = \frac{1}{c^2}V, \quad \kappa = -\frac{1}{c^2}\left(3\frac{\dot{a}}{a}U + 3\dot{V} + \frac{1}{a}P^k{}_{,k}\right),$$

$$\chi_i = \frac{1}{c^3}aP_i, \quad v_i = \frac{1}{c}\left\{\overline{v}_i + \frac{1}{c^2}\left[\overline{v}_i\left(\frac{1}{2}\overline{v}^2 + U + 2V\right) - P_i\right]\right\},$$

→ **1PN equations,**
without taking temporal gauge

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Basic 1PN Equations:

Tracefree ADM propagation: $V = U.$

Covariant energy-conservation:

$$\frac{1}{a^3}\left(a^3\widetilde{\varrho}\right)^{\cdot} + \frac{1}{a}\left(\widetilde{\varrho}\overline{v}^i\right)_{,i} = -\frac{1}{c^2}\left[\widetilde{\varrho}\left(\frac{\partial}{\partial t} + \frac{1}{a}\overline{\mathbf{v}}\cdot\nabla\right)\left(\frac{1}{2}\overline{v}^2 + 3U + \widetilde{\Pi}\right) + \left(3\frac{\dot{a}}{a} + \frac{1}{a}\nabla\cdot\overline{\mathbf{v}}\right)\widetilde{p}\right],$$

Covariant momentum-conservation:

$$\frac{1}{a}\left(a\overline{v}_i\right)^{\cdot} + \frac{1}{a}\overline{v}_{i,k}\overline{v}^k - \frac{1}{a}U_{,i} + \frac{1}{a}\frac{\widetilde{p}_{,i}}{\widetilde{\varrho}} = \frac{1}{c^2}\left[\frac{1}{a}\overline{v}^2U_{,i} + \frac{2}{a}\left(\Phi - U^2\right)_{,i} + \frac{1}{a}\left(aP_i\right)^{\cdot} + \frac{1}{a}\overline{v}^k\left(P_{i,k} - P_{k,i}\right)\right.$$

fourth-order in perturbation

$$\left. + \frac{1}{a}\left(\overline{v}^2 + 4U + \widetilde{\Pi} + \frac{\widetilde{p}}{\widetilde{\varrho}}\right)\frac{\widetilde{p}_{,i}}{\widetilde{\varrho}} - \overline{v}_i\left(\frac{\partial}{\partial t} + \frac{1}{a}\overline{\mathbf{v}}\cdot\nabla\right)\left(\frac{1}{2}\overline{v}^2 + 3U\right) - \overline{v}_i\frac{1}{\widetilde{\varrho}}\left(\frac{\partial}{\partial t} + \frac{1}{a}\overline{\mathbf{v}}\cdot\nabla\right)\widetilde{p}\right],$$

Trace of ADM propagation:

$$\frac{\Delta}{a^2}U + 4\pi G\left(\widetilde{\varrho} - \varrho\right) = -\frac{1}{c^2}\left\{\frac{1}{a^2}\left[2\Delta\Phi - 2U\Delta U + \left(aP^i{}_{,i}\right)^{\cdot}\right] + 3\ddot{U} + 9\frac{\dot{a}}{a}\dot{U} + 6\frac{\ddot{a}}{a}U\right.$$

$$\left. + 8\pi G\left[\widetilde{\varrho}\overline{v}^2 + \frac{1}{2}\left(\widetilde{\varrho}\widetilde{\Pi} - \varrho\Pi\right) + \frac{3}{2}\left(\widetilde{p} - p\right)\right]\right\},$$

ADM momentum-constraint:

$$0 = \frac{1}{a^2}\left(P^k{}_{,ki} - \Delta P_i\right) - 16\pi G\widetilde{\varrho}\overline{v}_i + \frac{4}{a}\left(\dot{U} + \frac{\dot{a}}{a}U\right)_{,i},$$

$$\overline{v}_i = \left(1 - \frac{3}{c^2}U\right)\widehat{v}_i + \frac{1}{c^2}P_i, \quad \widehat{v}_i = \left(1 - \frac{v^2}{2c^2}\right)v_i,$$

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Gauge conditions:

$$\frac{1}{a} P^i{}_{|i} + n\dot{U} + m\frac{\dot{a}}{a} U = 0,$$

Harmonic gauge : (Weinberg 1972)	$n = 4,\ m = \text{arbitrary},$
Chandrasekhar′s gauge :	$n = 3,\ m = \text{arbitrary},$
Uniform−expansion gauge :	$n = 3 = m,$
Transverse−shear gauge :	$n = 0 = m.$

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Propagation speed issue

Under the general gauge:

Trace of ADM propagation:

$$\frac{\Delta}{a^2} U + 4\pi G \left(\varrho - \varrho_b \right)$$
$$+ \frac{1}{c^2} \Bigg\{ 2 \frac{\Delta}{a^2} \Phi - (n-3) \ddot{U} - (2n + m - 9) \frac{\dot{a}}{a} \dot{U} + \left[(6-m) \frac{\ddot{a}}{a} - m \frac{\dot{a}^2}{a^2} \right] U$$
$$+ 8\pi G \left[\varrho v^2 + \tfrac{1}{2} \left(\varrho \Pi - \varrho_b \Pi_b \right) + U \left(\varrho - \varrho_b \right) + \tfrac{3}{2} \left(p - p_b \right) \right] \Bigg\} = 0.$$

Propagation speed $\frac{c}{\sqrt{n-3}}$ **is gauge dependent!**

of the potential

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Resolution using Weyl tensor:

$$E_{ij} = -\frac{1}{c^2}\left[\frac{1}{2}(U+V)_{,i|j} - \frac{\Delta}{6}(U+V)\gamma_{ij}\right] + L^{-2}\mathcal{O}^{-4},$$

$$H_{ij} = \frac{1}{c^3}\frac{1}{2a}\eta_{(i}{}^{kl}\left\{\left[\frac{1}{2}\left(P^m{}_{|ml} - \Delta P_l\right) + \frac{1}{3}v_l\Delta(U+V)\right]\gamma_{kj)} + P_{l|kj)} - v_l(U+V)_{,k|j)}\right\} + L^{-2}\mathcal{O}^{-5},$$

$\dot{E}_{ij}$, $\dot{H}_{ij}$ equations:

$$\ddot{E}_{ij} + 3\frac{\dot{a}}{a}\dot{E}_{ij} + \left(\frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2}\right)E_{ij} = \frac{c^2}{a^2}\left[\Delta E_{ij} - E^k_{(i|j)k} - \frac{1}{2}E_{k(i}{}^{|k}{}_{j)} + \frac{1}{2}\left(E^{kl}{}_{|kl} - \Delta E^k_k\right)\gamma_{ij} + \frac{1}{2}E^k_{k|ij}\right] + \frac{1}{a^2}\left\{\left[(V+2U)_{,(i}E^k_{j)}\right]_{|k} - \left[(V+2U)^{,k}E_{ij}\right]_{|k} + \frac{1}{2}\left[(V+2U)^{,k}E_{k(i}\right]_{|j)} - \frac{1}{2}\left[(V+2U)^{,l}E^k_l\right]_{|k}\gamma_{ij}\right\} + \frac{4\pi G}{c^2}\left(-\Delta\Pi_{ij} + \Pi^k_{(i|j)k}\right), \qquad (225)$$

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With Relativistic Pressure

Infinite speed-of-light limit, except for pressure

JH & Noh, JCAP **10** (2013) 054

Case with Relativistic Pressure in ZSG:

$$\alpha \ll 1, \quad \varphi \ll 1, \quad \frac{\widehat{v}^k \widehat{v}_k}{c^2} \ll 1, \quad \widetilde{p} \ll \widetilde{\varrho} c^2, \quad \frac{1}{c^2} \widetilde{\Pi} \ll 1, \quad \frac{c^2 k^2}{a^2 H^2} \gg 1,$$

Covariant energy-conservation:

$$\dot{\widetilde{\varrho}} + 3\frac{\dot{a}}{a}\left(\widetilde{\varrho} + \frac{\widetilde{p}}{c^2}\right) + \frac{1}{a}\nabla \cdot \left[\left(\widetilde{\varrho} + \frac{\widetilde{p}}{c^2}\right)\mathbf{v}\right] = \frac{1}{c^2}\frac{2}{a}\mathbf{v} \cdot \nabla \widetilde{p},$$

Covariant momentum-conservation:

$$\dot{\mathbf{v}} + \frac{\dot{a}}{a}\mathbf{v} + \frac{1}{a}\mathbf{v} \cdot \nabla \mathbf{v} - \frac{1}{a}\nabla U = -\frac{1}{\widetilde{\varrho} + \widetilde{p}/c^2}\left(\frac{1}{a}\nabla \widetilde{p} + \frac{\dot{\widetilde{p}}}{c^2}\mathbf{v}\right),$$

Trace of ADM propagation:

$$\frac{\Delta}{a^2} U = -4\pi G \left(\widetilde{\varrho} - \varrho\right),$$

No pressure!

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