The Nature of Trapping Horizons in Collapses Forming Black Holes

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Outline

- Introduction: *Misner-Sharp & perfect fluids*
- Causal horizons in spherical symmetry: $R=2M$
  - **black hole horizons** (*ingoing/outgoing*)
  - **cosmological horizon**
- Causal nature of the horizons
- Results of simulations of **classical gravitational collapse**
- Conclusions

Background model

- The most general **spherical symmetric** form of the metric, to describe a deviation from the uniform background, can be written as,

$$ds^2 = -a^2\, dt^2 + b^2\, dr^2 + R^2 d\Omega^2$$

- This defines *a, b,* and *R* being functions of the comoving coordinate *r* the often called **cosmic time** *t* and involves a choice of the **time slicing** to keep the metric diagonal (gauge choice). The radius *R* is the **circumferential radial coordinate**.

- The **unperturbed solution,** describing an expanding homogeneous universe**,** is given by the FRW metric: $K = \pm 1, 0$ is the **curvature parameter**, $\tilde{a}(t)$ is the **scale factor**, and $R = \tilde{a}(t)\, r$ is the **circumferential radial coordinate**.

$$ds^2 = -dt^2 + \tilde{a}(t)[\frac{dr^2}{1 - Kr^2} + r^2 d\Omega^2]$$

- This slicing is very useful to impose initial conditions. The observer is comoving with the fluid (Preferred Slicing).

$$G_{\mu\nu} = 8\pi T_{\mu\nu} \qquad T_{\mu\nu} = (e+p)u_\mu u_\nu - p g_{\mu\nu}$$

COSMIC TIME

$$ds^2 = -a^2\,dt^2 + b^2\,dr^2 + R^2 d\Omega^2$$

$$D_t \equiv \frac{1}{a}\left(\frac{\partial}{\partial t}\right) \qquad D_r \equiv \frac{1}{b}\left(\frac{\partial}{\partial r}\right)$$

- Proper time / space derivative

$$U \equiv D_t R \qquad \Gamma \equiv D_r R$$

- 4-velocity & Lorentz factor

$$D_t U = -\left[\frac{\Gamma}{(e+p)} D_r p + \frac{M}{R^2} + 4\pi R p\right]$$

- Euler equation

$$D_t \rho = -\frac{\rho}{\Gamma R^2} D_r (R^2 U)$$

- Continuity equation

$$D_t e = \frac{e+p}{\rho} D_t \rho$$

$$D_t M = -4\pi R^2 p U$$

- Mass conservation

$$dM = aU\,dt + b\Gamma\,dr$$

$$D_r a = -\frac{a}{e+p} D_r p$$

- Lapse equation / pressure gradients

$$D_r M = 4\pi R^2 \Gamma e$$

$$\Gamma^2 = 1 + U^2 - \frac{2M}{R}$$

- Constraint equation

Equation of State

energy density: $e = \rho(1+\epsilon)$ pressure: $p = (\gamma - 1)\rho\epsilon$

ρ → rest mass density

ϵ → specific internal energy (velocity dispersion)

γ → adiabatic index - particle degree of freedom

- Barotropic fluid (no rest mass density): $p = we$ with $w \in [0,1]$
  - radiation dominated era: $w = 1/3$ RADIATION $(\gamma = 4/3)$
  - matter dominated era: $w = 0$ DUST $(\gamma = 1)$

- Polytropic fluid: $p = K(s)\rho^{\gamma}$ $(\gamma = 5/3,\ 4/3,\ 2)$
  - If the fluid is adiabatic (no entropy change): $K(s) = K$ (constant)

Calculation of Γ

- For a general perfect fluid, the G_{00} and G_{11} components of the Einstein field equations are

$$(G^0_0) \qquad 4\pi R^2 e R_r = (R + RU^2 - R\Gamma^2)_r/2$$

$$(G^1_1) \qquad 4\pi R^2 a p U = -(R + RU^2 - R\Gamma^2)_t/2$$

- It is convenient to define $M = (R + RU^2 - R\Gamma^2)/2$ to write

$$M = \int 4\pi R^2 e dR$$

$$D_t M = -4\pi R^2 p U$$

and

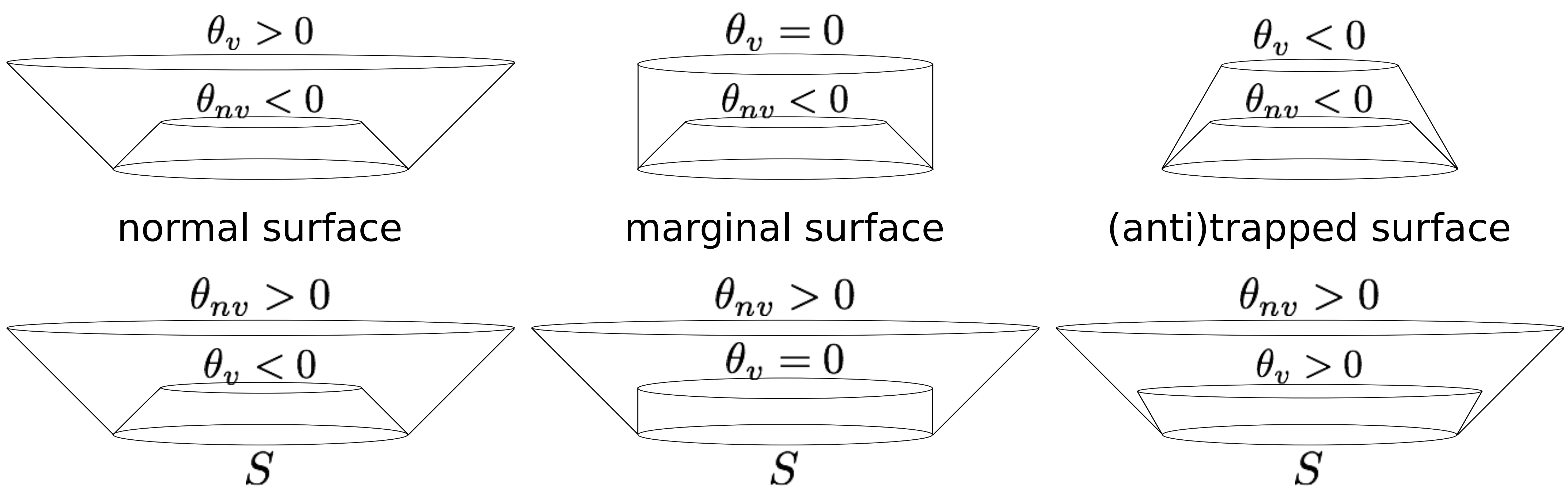
$$\Gamma^2 = 1 + U^2 - \frac{2M}{R}$$

The last is a **constraint equation** with M being the mass inside radius R. The second one describes adiabatic expansion or contraction of the fluid.

Trapping Horizons (Hayward)

A trapping horizon is the closure of a three-surface H foliated by marginally trapped surfaces ($\theta_v = 0$) on which $\theta_{nv} \neq 0$ and where the Lie derivative $\mathcal{L}_{nv}\theta_v \neq 0$.

- *outer horizon* if $\mathcal{L}_{nv}\theta_v < 0$.
- *inner horizon* if $\mathcal{L}_{nv}\theta_v > 0$.
- *future horizon* if $\theta_{nv} < 0$: $\theta_- < 0$ and $\theta_+ = 0$ (Black Holes)
- *past horizon* if $\theta_{nv} > 0$: $\theta_+ > 0$ and $\theta_- = 0$. (Expanding Universe)



Trapping horizon condition

The **Misner-Sharp Mass**:

$$M = \frac{R}{2}\left(1 - \nabla^a R \nabla_a R\right)$$

Expansion of **ingoing/outgoing** null-rays:

$$k^a, l^a = (1, \pm\frac{a}{b}, 0, 0)$$

$$\theta_{out} = h^{cd}\nabla_c k_d = \frac{2a}{R}(U + \Gamma)$$

$$\theta_{in} = h^{cd}\nabla_c l_d = \frac{2a}{R}(U - \Gamma)$$

For changes in R along a **worldline**:

$$dR = \left(\frac{\partial R}{\partial t} \pm \frac{a}{b}\frac{\partial R}{\partial r}\right) dt$$

$$\boxed{\theta_{\pm} = \frac{2}{R}(U \pm \Gamma) = \frac{2}{aR}\left.\frac{dR}{dt}\right|_{\pm} = 0 \quad \Rightarrow \quad \Gamma^2 = U^2}$$

Trapping Horizons within a moving medium

- A black hole **apparent horizon** (event horizon in the static case) is the asymptotic location of the **outermost trapped surface** for outgoing light-rays whereas the **cosmological horizon** is the **innermost trapped surface** for incoming light rays.

- **Black Hole horizons :** $$\left(\frac{dR}{dt}\right)_{\text{out}} = a(U+\Gamma) = 0 \quad \rightarrow \quad \Gamma = -U$$

- **Cosmological horizon :** $$\left(\frac{dR}{dt}\right)_{\text{in}} = a(U-\Gamma) = 0 \quad \rightarrow \quad \Gamma = U$$

$$\Downarrow$$

$\Gamma^2 = U^2$ inserted into $\Gamma^2 = 1 + U^2 - \dfrac{2M}{R}$

$$\Downarrow$$

The horizon condition is independent of the slicing and holds also within a non-vacuum moving medium

$$R = 2M$$

Miller & Musco - arXiv:1412.8660

Causal Nature

$$\mathcal{L}_{out} = \mathcal{L}_k = k^a \partial_a = (\partial_t + \frac{a}{b}\partial_r) = a\,(D_t + D_r)$$

$$\mathcal{L}_{in} = \mathcal{L}_l = l^a \partial_a = (\partial_t - \frac{a}{b}\partial_r) = a\,(D_t - D_r)$$

Causal Nature of BH horizons: $\alpha = \frac{\mathcal{L}_{out}\theta_{out}}{\mathcal{L}_{in}\theta_{out}}$

Causal Nature of cosmological horizon: $\alpha = \frac{\mathcal{L}_{in}\theta_{in}}{\mathcal{L}_{out}\theta_{in}}$

$$\alpha = \frac{(D_t U \pm D_t \Gamma) \pm (D_r U \pm D_r \Gamma)}{(D_t U \pm D_t \Gamma) \mp (D_r U \pm D_r \Gamma)}$$

$> \mathbf{0}$: space-like

$= \mathbf{0} / \infty$: null

$< \mathbf{0}$: time-like

$$\boxed{\alpha = \frac{4\pi R_H^2(e+p)}{1-4\pi R_H^2(e-p)}}$$

$$R_H = 2M_H$$

$$\frac{1}{a}\frac{dR}{dt} = \frac{dR}{d\tau} = U + \Gamma v \quad + \quad d\left(\frac{R}{2M}\right)_H = 0$$

$\Downarrow$

3-velocity of the horizon with respect the matter: $v_H \equiv \left(\frac{b}{a}\frac{dr}{dt}\right)_H$

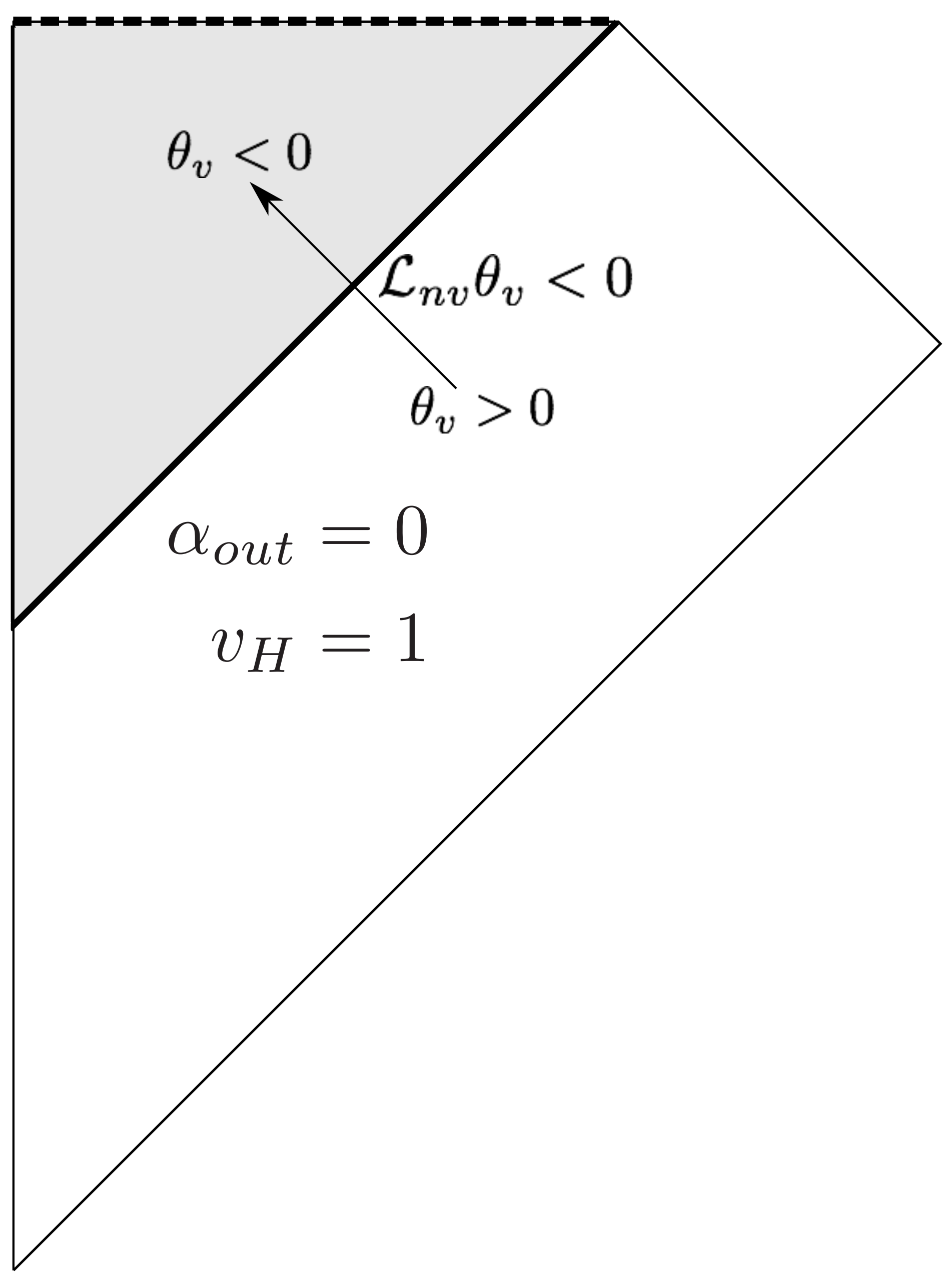
$$\boxed{v_H = \pm\frac{1+8\pi R_H^2 p}{1-8\pi R_H^2 e}} \Rightarrow \boxed{\alpha = \frac{v_H - 1}{v_H + 1}}$$

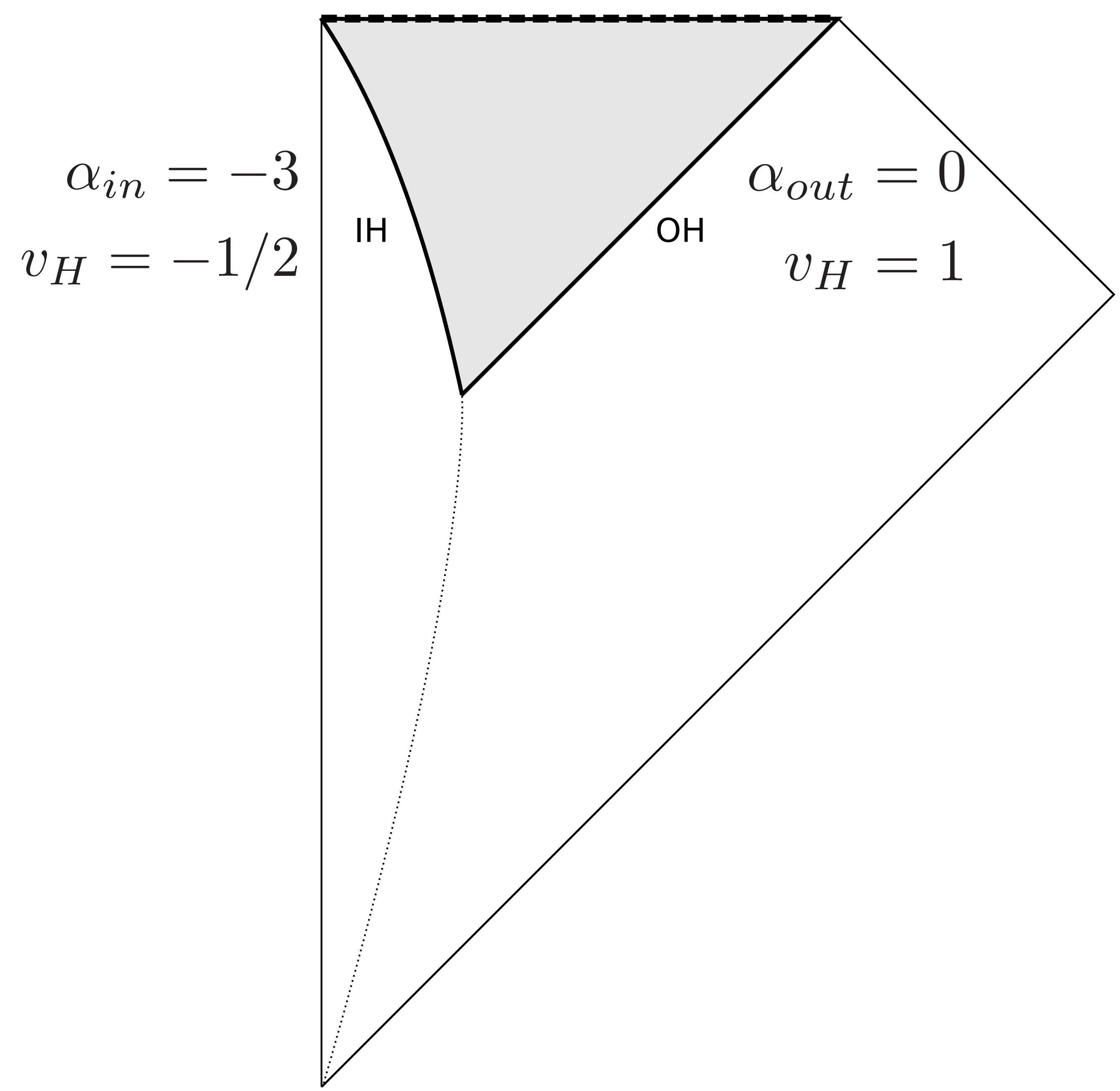
$|v_H| > 1$: space-like

$|v_H| = 1$: null

$|v_H| < 1$: time-like

Schwerschild BH / Oppenheimer-Snyder collapse

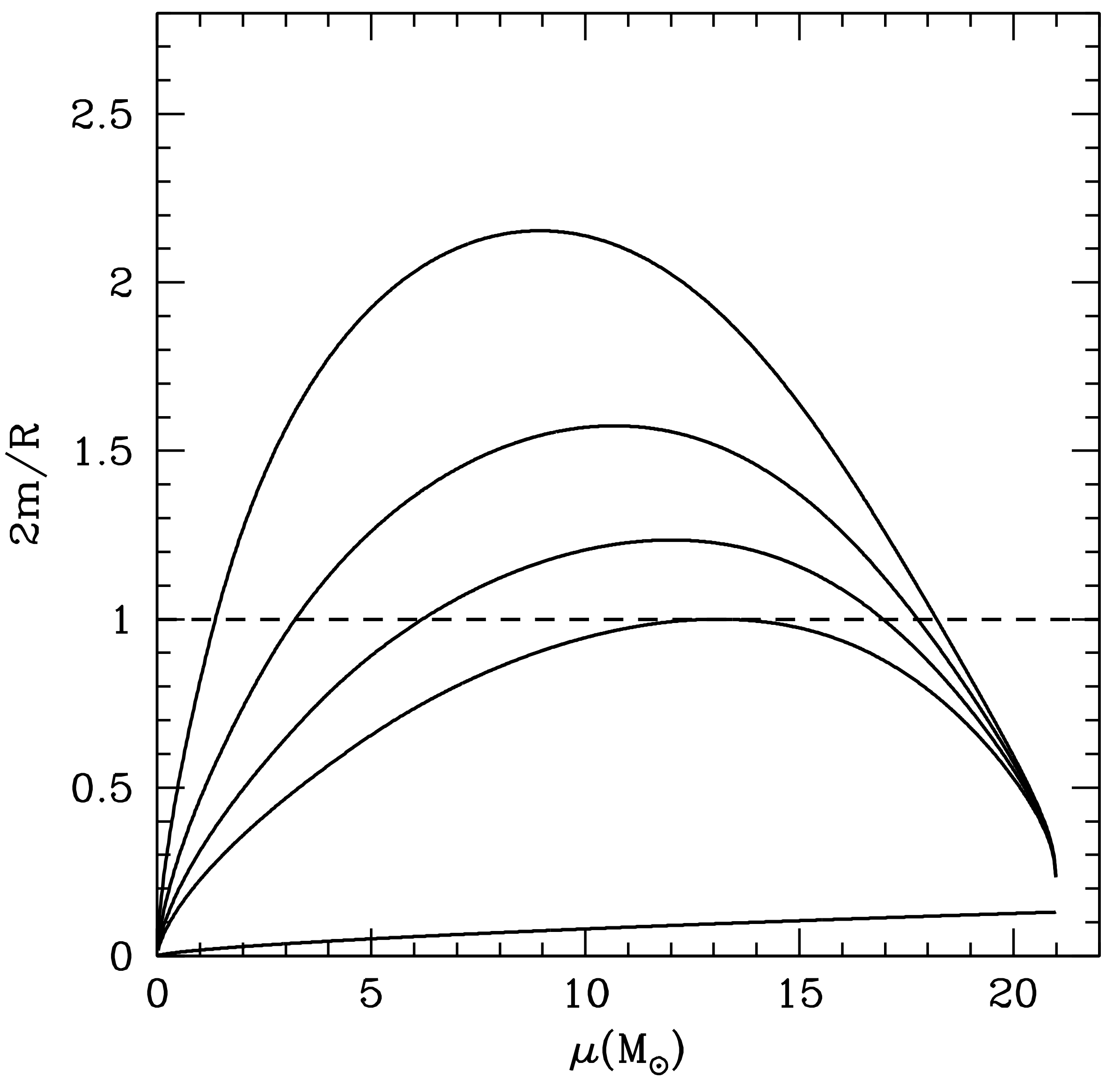


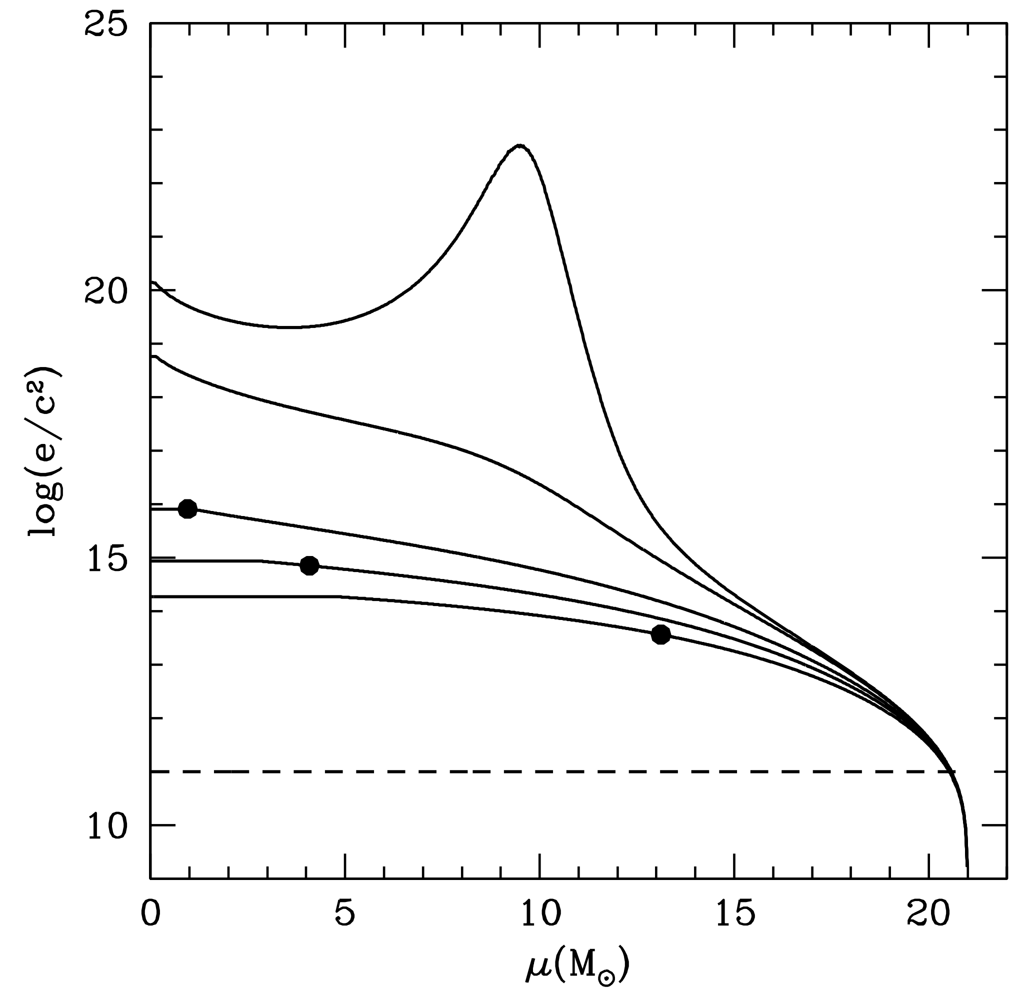


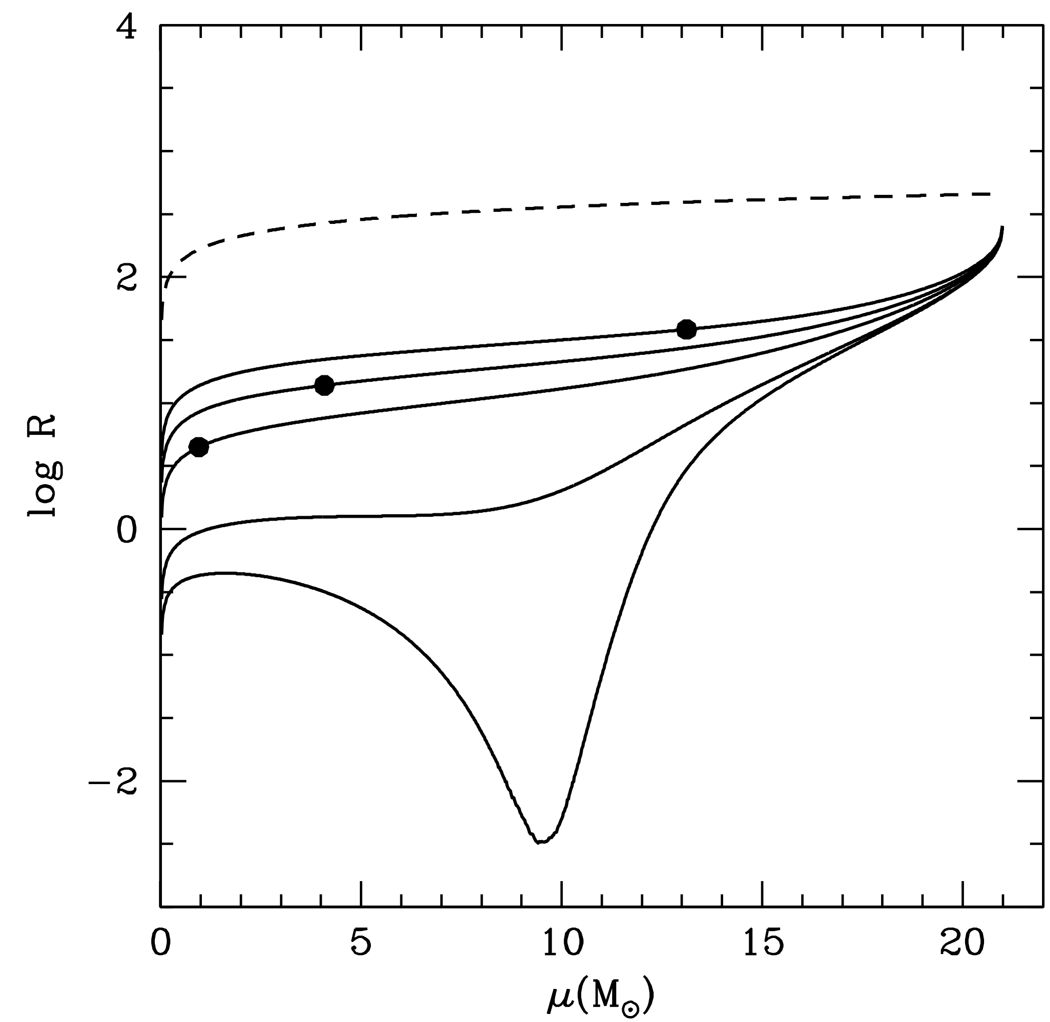
May & White (1966)

Homogenous initial density

$$p = K\rho^{\gamma} \quad (\gamma = 5/3, K = 0.05)$$



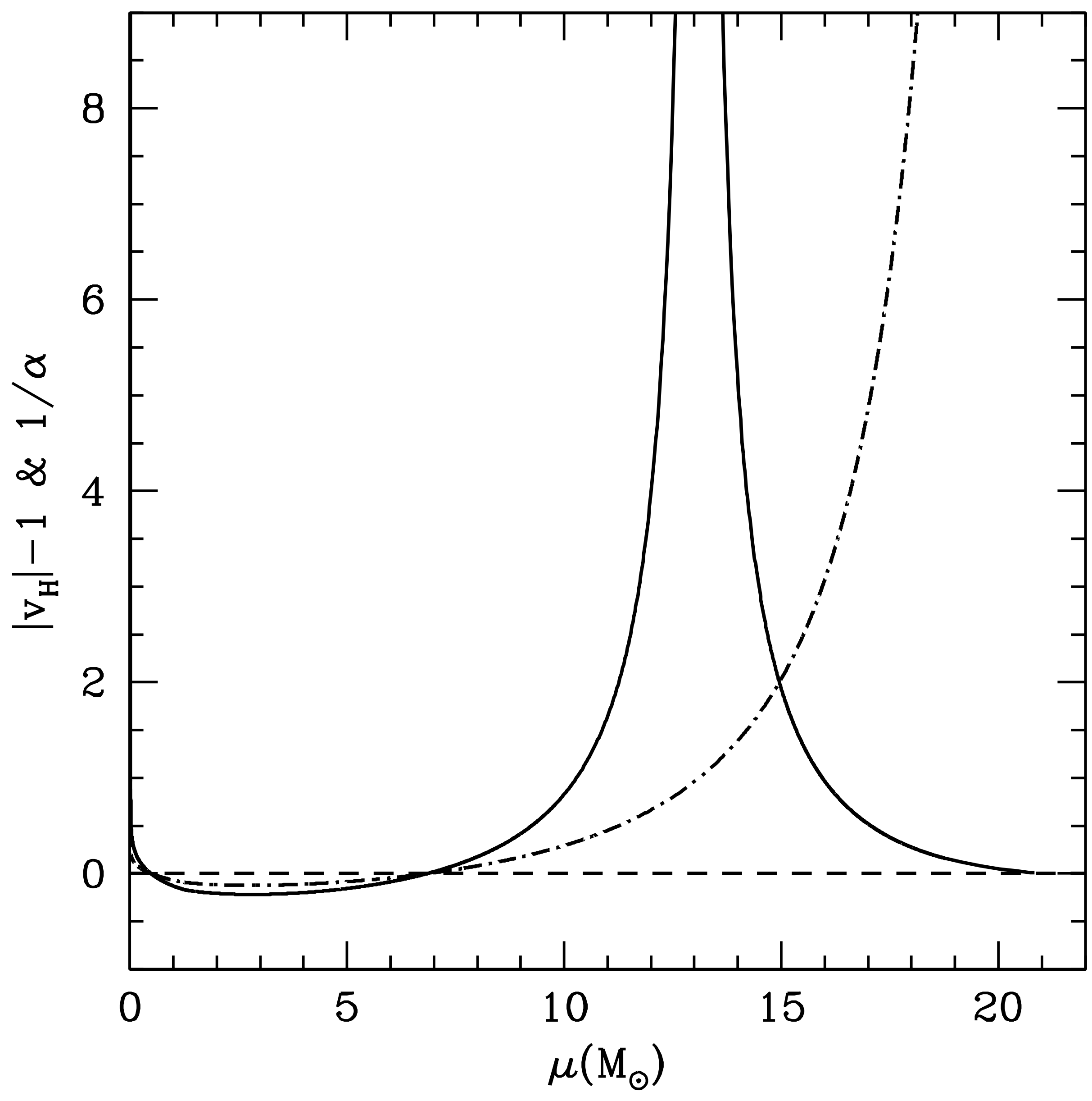


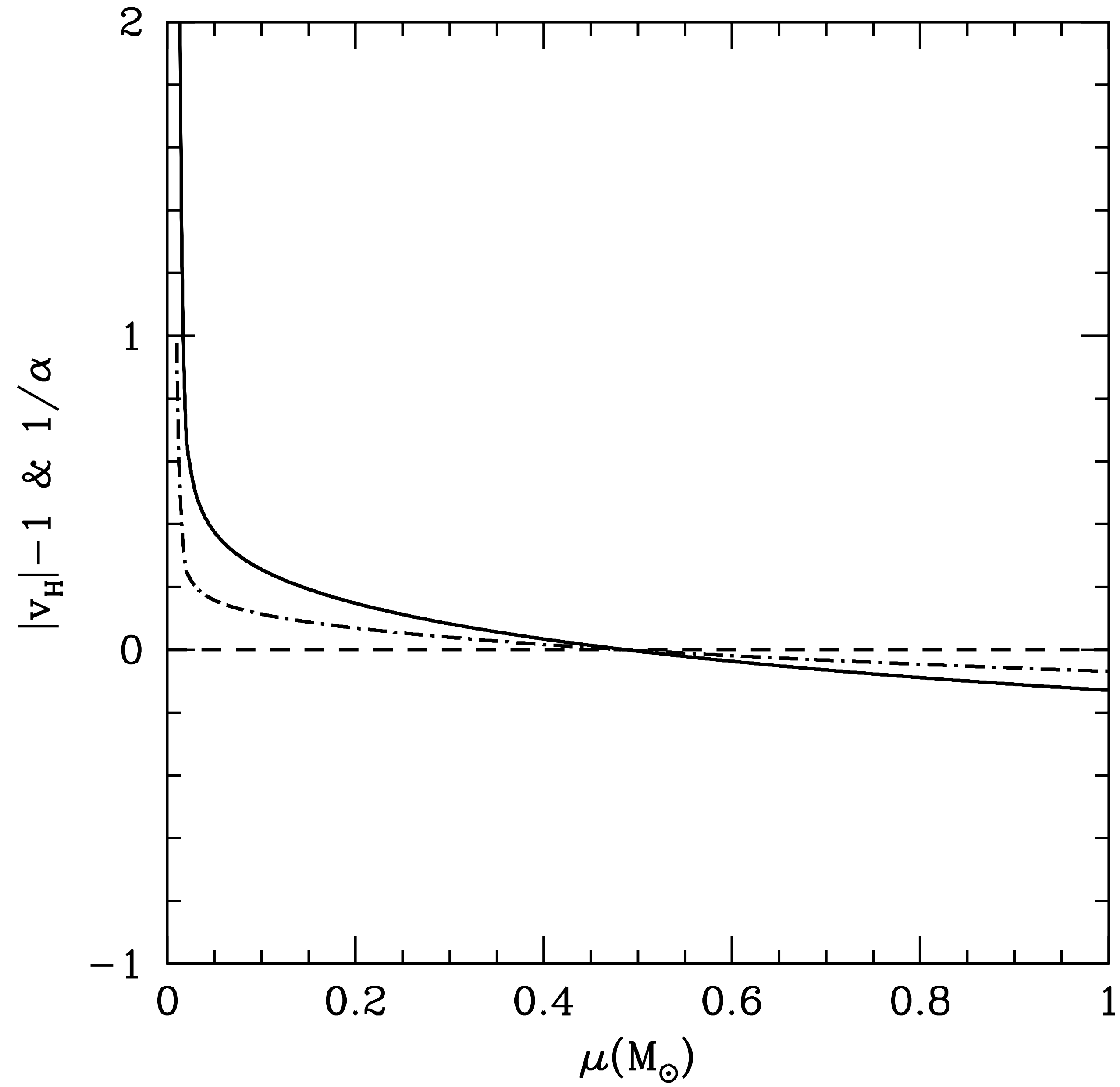


May & White (1966): Causal Nature

$$\alpha = \frac{4\pi R_H^2(e+p)}{1-4\pi R_H^2(e-p)}$$

$$v_H = \frac{1+8\pi R_H^2 p}{1-8\pi R_H^2 e}$$

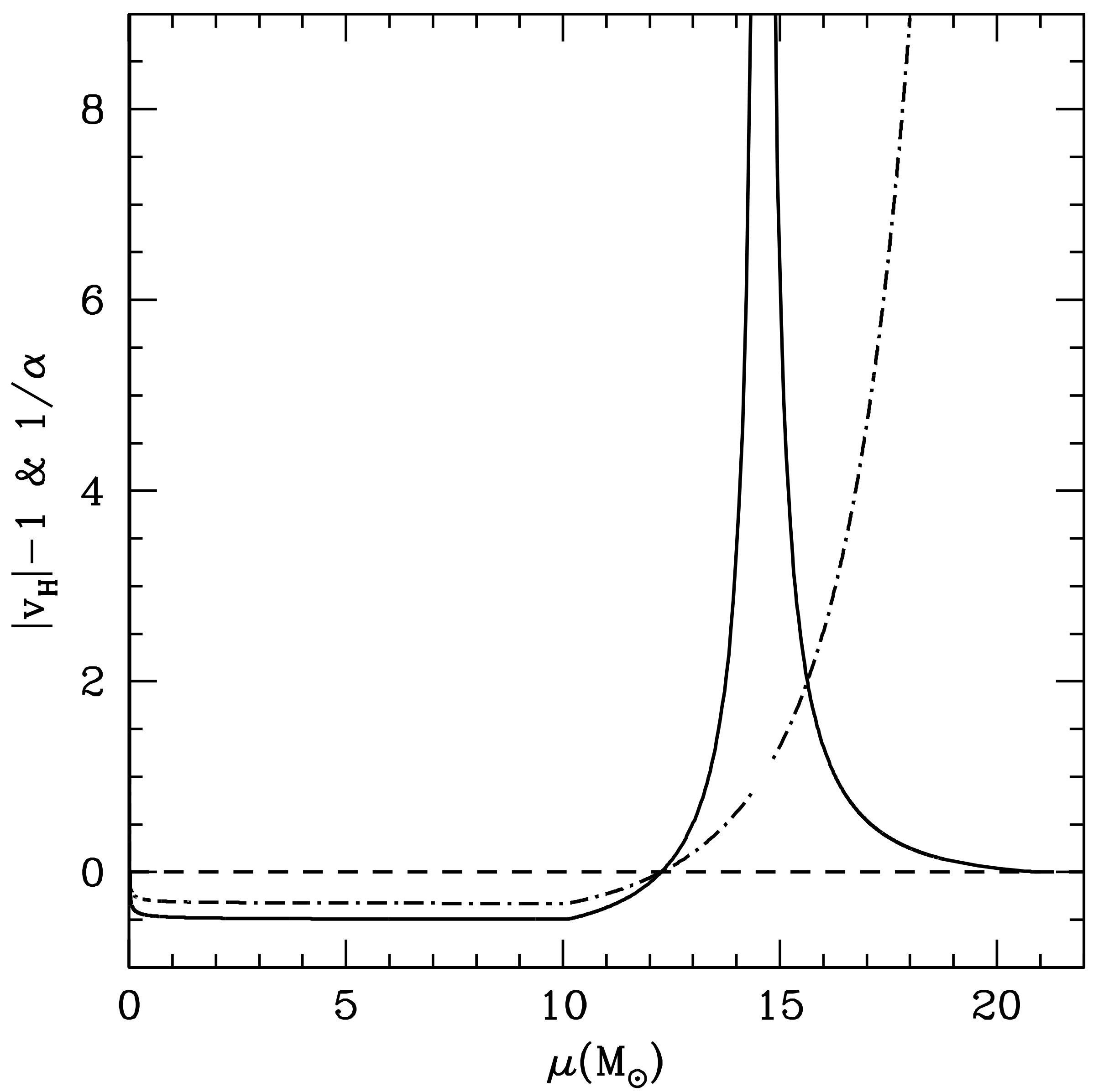


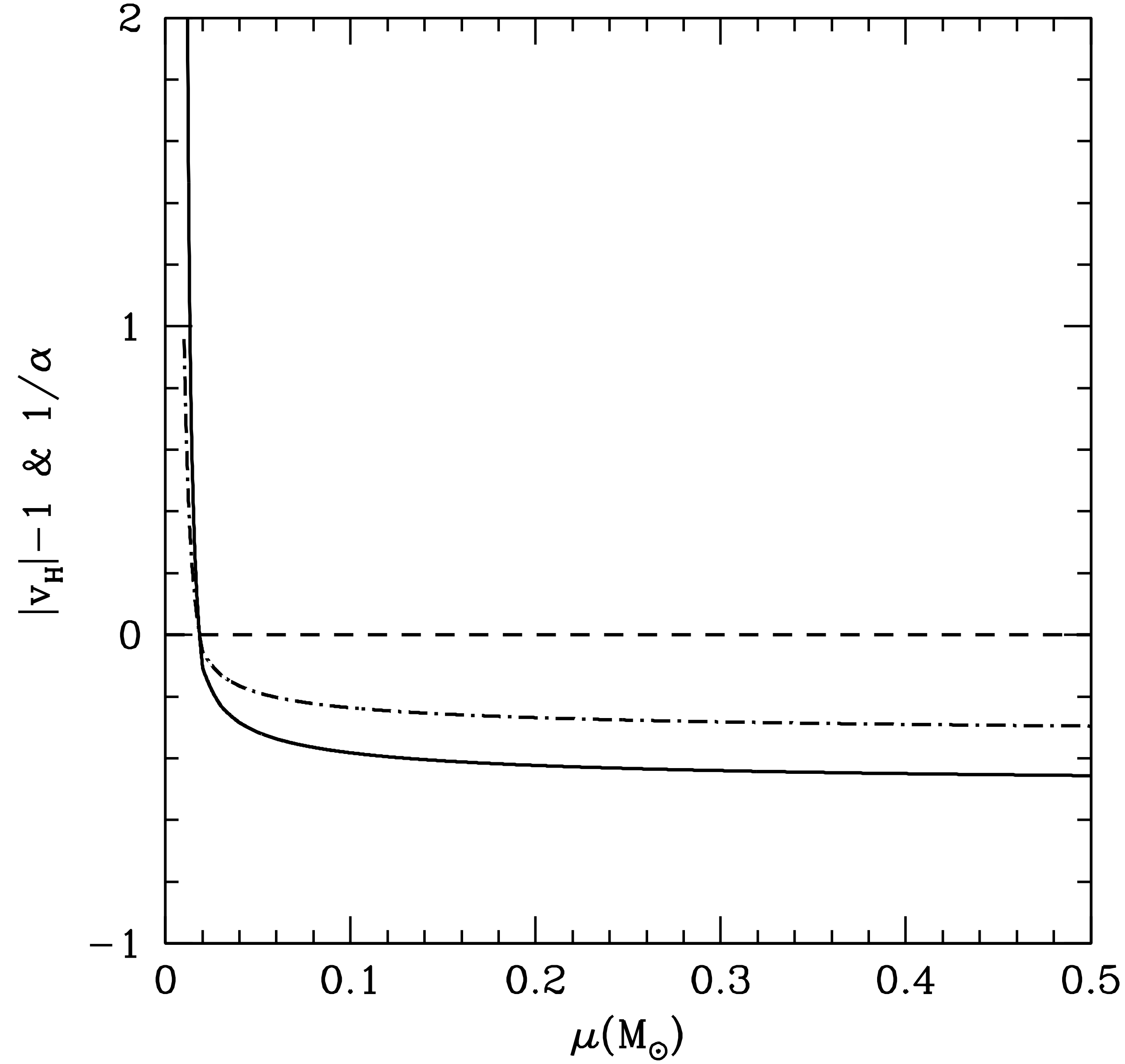


$$p = K\rho^{\gamma} \quad (\gamma = 4/3, \text{ HOM I.C.})$$

$$\alpha = \frac{4\pi R_H^2(e+p)}{1 - 4\pi R_H^2(e-p)}$$

$$v_H = \frac{1 + 8\pi R_H^2 p}{1 - 8\pi R_H^2 e}$$

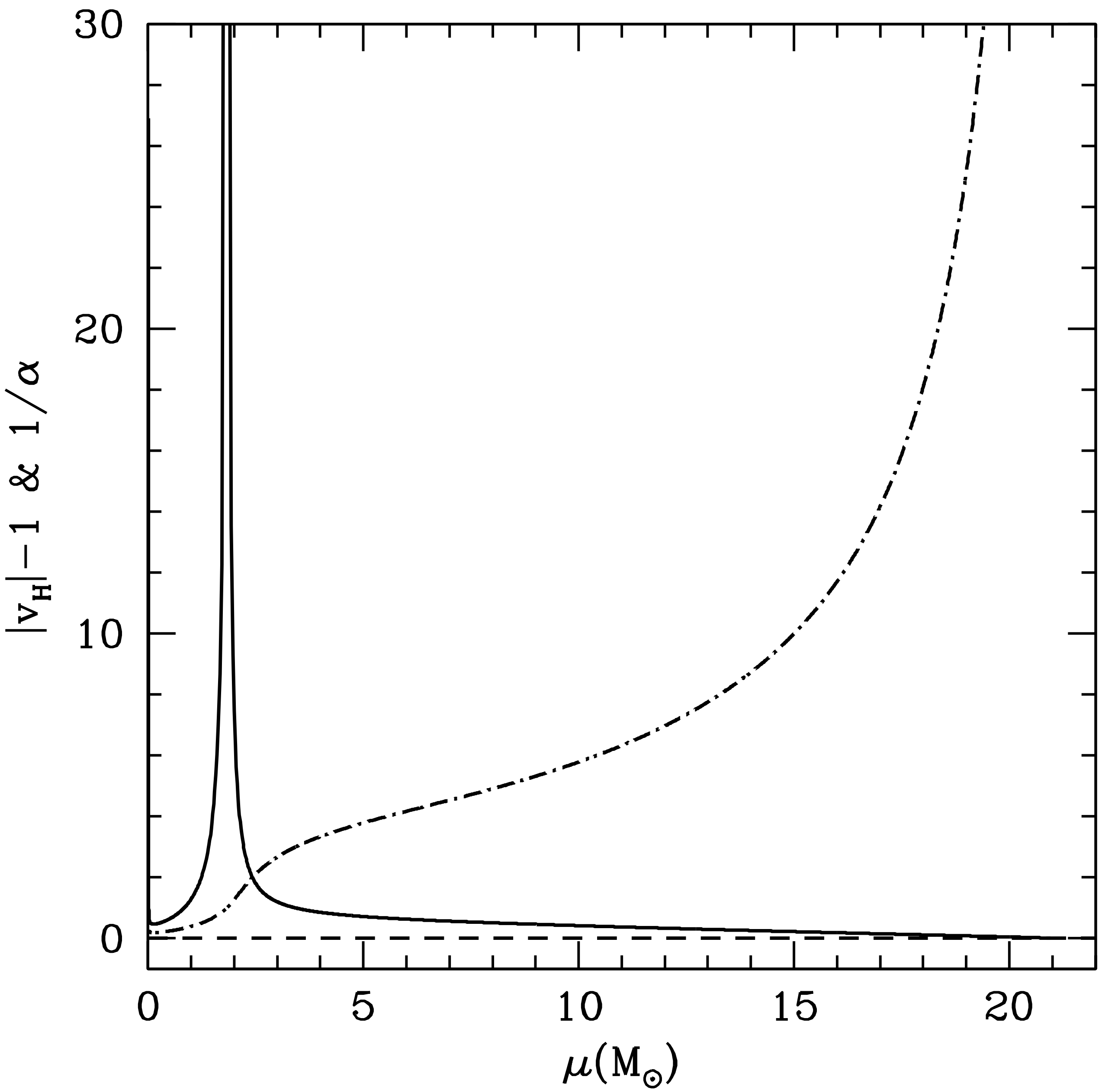


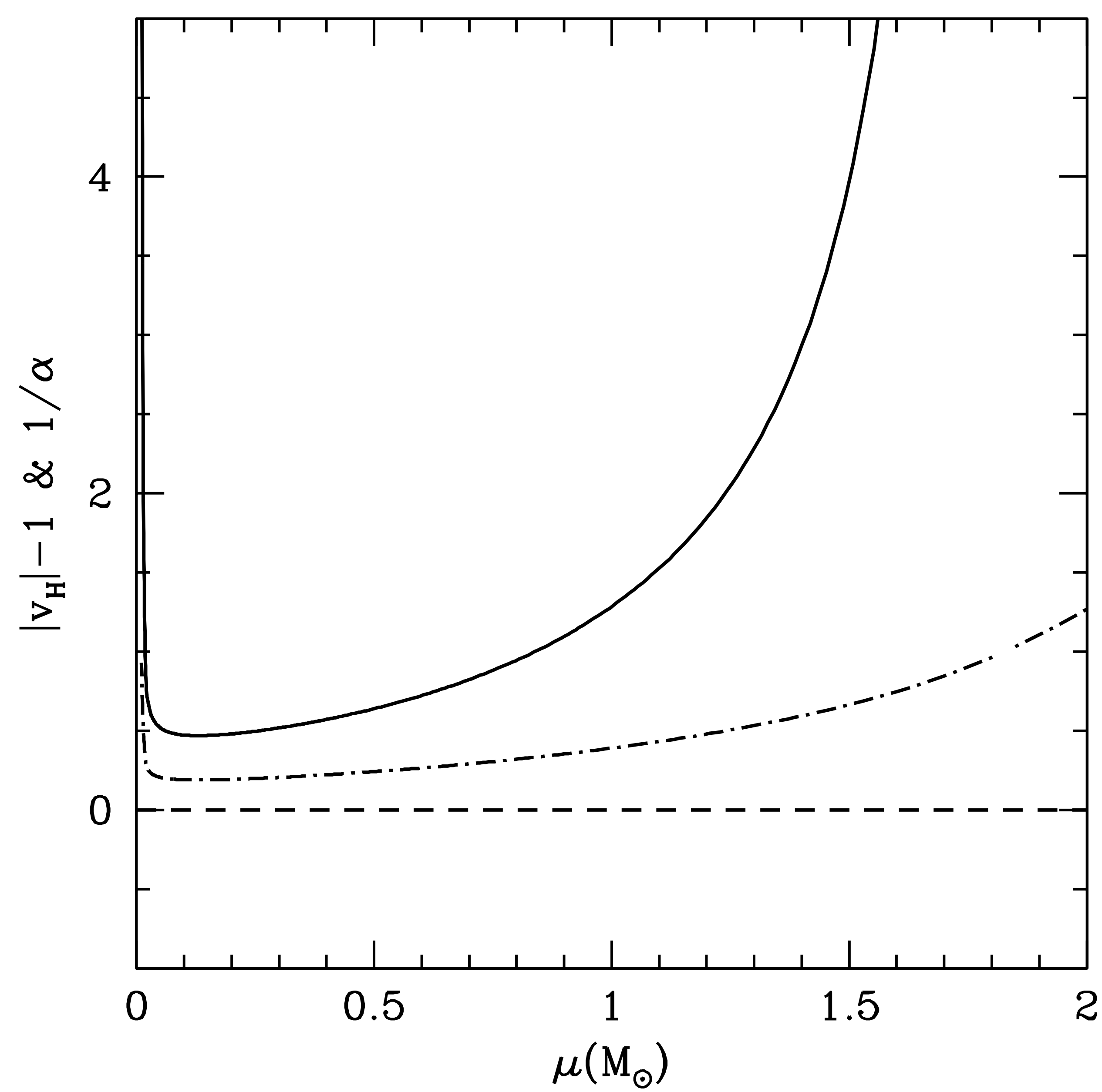


$$p = K\rho^{\gamma} \quad (\gamma = 5/3, \text{TOV I.C.})$$

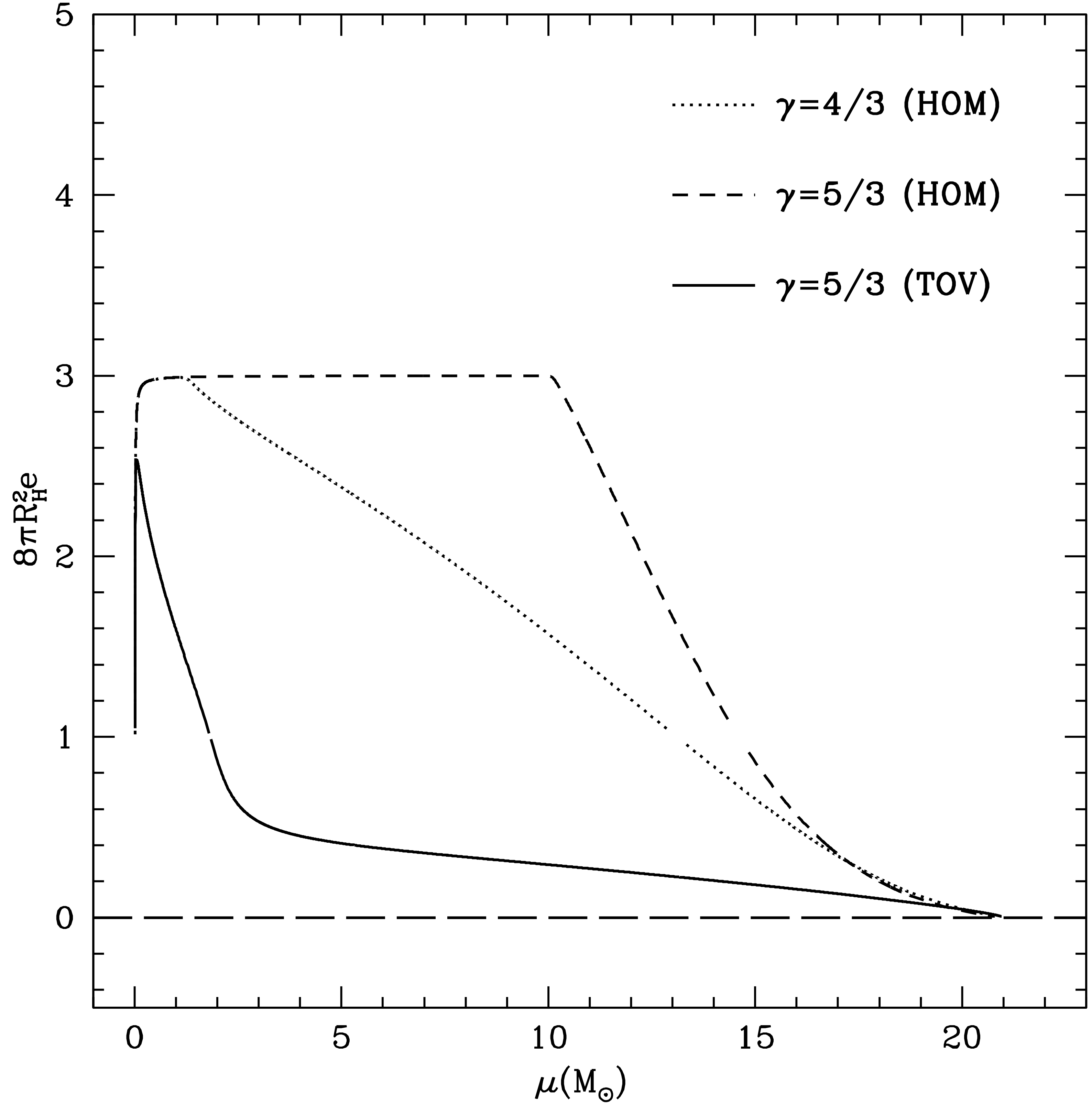
$$\alpha = \frac{4\pi R_H^2 (e+p)}{1 - 4\pi R_H^2 (e-p)}$$

$$v_H = \frac{1 + 8\pi R_H^2 p}{1 - 8\pi R_H^2 e}$$

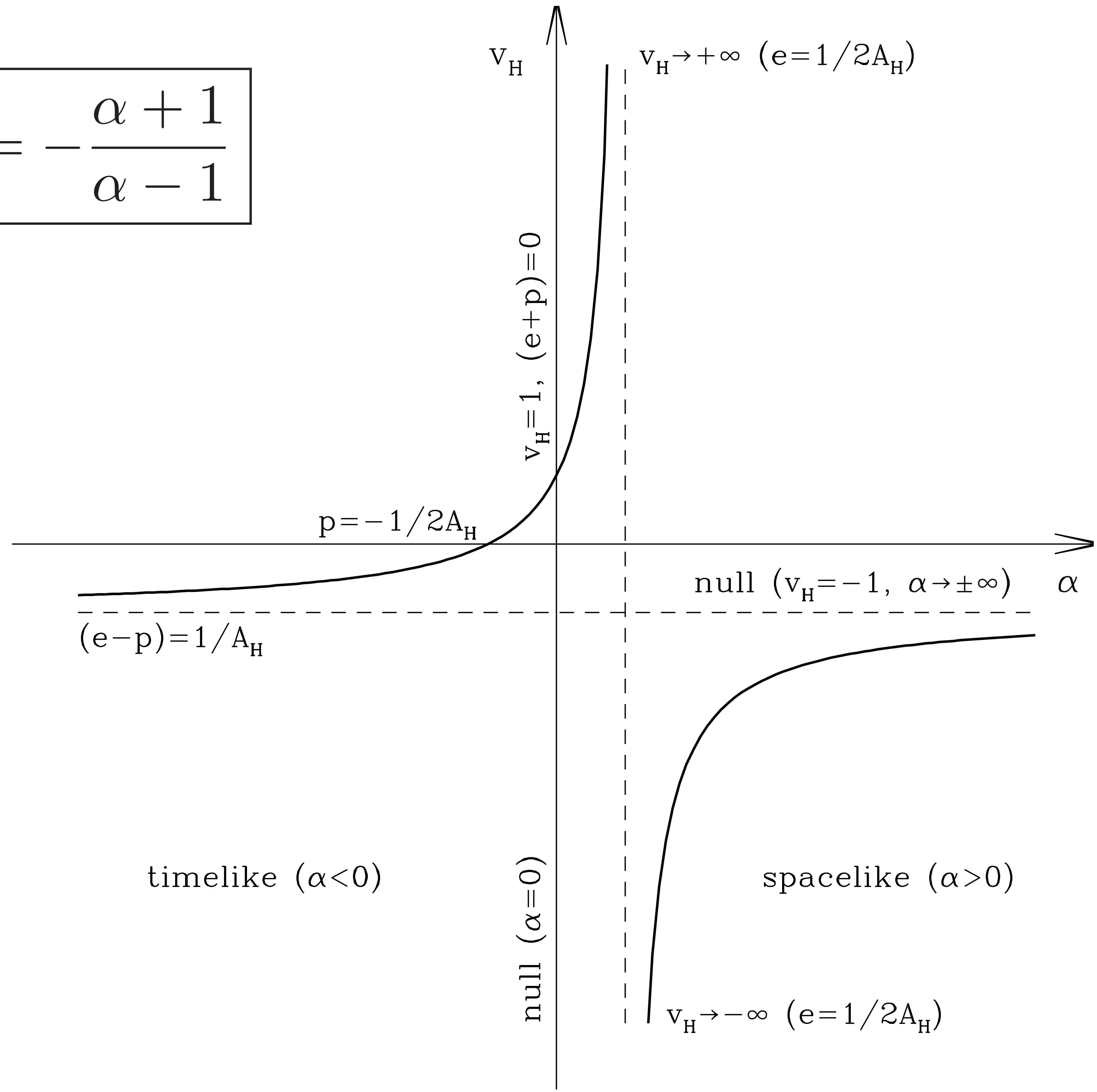




Causal Nature Summary



$$v_H = -\frac{\alpha + 1}{\alpha - 1}$$



Conclusions & Future perpectives

- With the Misner-Sharp equations (cosmic time slicing) we have studied the causal nature of trapping horizons appearing in gravitational collapse for polytropic stars forming black hole using a spherically symmetric Lagrangian numerical code.

- Within the classical regime of GR we have observed space-like (outer) outgoing horizon and space-like/time-like (outer/inner) ingoing horizon depending on the choice of the equations of state and initial conditions.

- The conditions of **horizon formation** and **disappearance** are independent of the initial conditions:

$$\alpha = 1 \, , \; v_H = \pm\infty$$

- The formalism developed seems to show the possibility of incorporating quantum effects within the classical formulation of the GR-hydro equations modifying the equation of state according to quantum gravity (*Rovelli's Talk*).

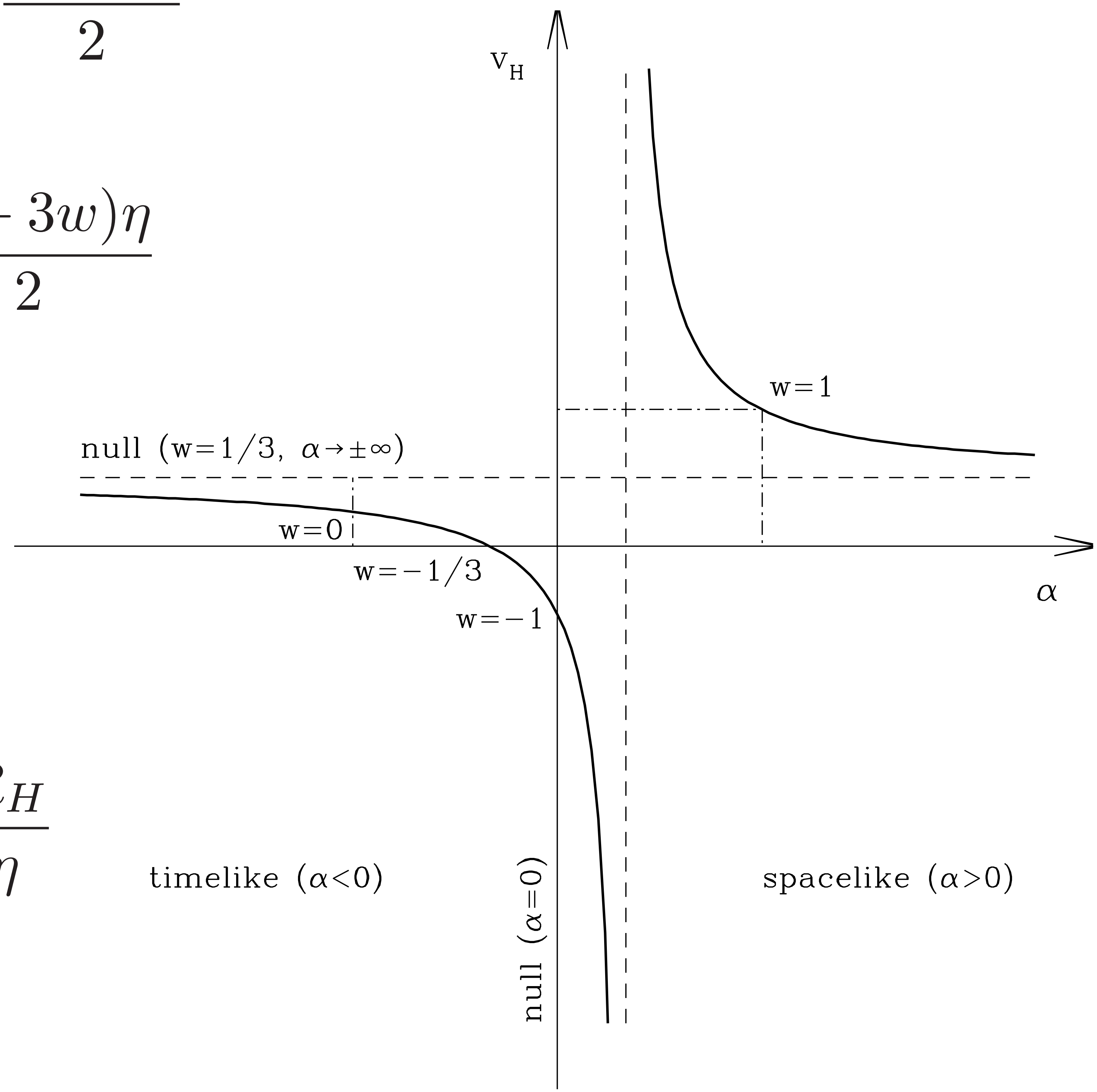
Can we get a bounce instead of a singularity?

$$\alpha = \frac{w+1}{w-1/3}\ , \quad v_H = \frac{1+3w}{2}$$

$$R_H = \frac{3(1+w)t}{2} = \frac{(1+3w)\eta}{2}$$

$$\boxed{v_H = \frac{\alpha+1}{\alpha-1}}$$

$$v_{tot} = 1 + v_H = 1 + \frac{dR_H}{d\eta}$$



Black Hole Thermodynamics

Hayward-Kodama surface gravity: $\kappa = \frac{1}{2\sqrt{-\gamma}}\partial_i\left[\sqrt{-\gamma}\gamma^{ij}\nabla_j R\right]$

Using Misner-Sharp equations: $\kappa = \frac{1}{2R}\left[\frac{2M}{R} - 4\pi R^2(e-p)\right]$

$$R = 2M \;\Rightarrow\; \kappa = \frac{1}{4M_H} - 4\pi M_H(e-p)$$

$$\Downarrow$$

$$dM_H = \frac{\kappa}{8\pi}dA_H + \frac{(e-p)}{2}dV_H$$

$$\alpha = 2\pi R_H \frac{e+p}{\kappa}$$

Table 2: casual nature of R_H for BHs

e, p, R_H	v_H	κ	α	Type
		outgoing horizon		
$(e+p) > 0$	$v_H > 1$	$\kappa > 0$	$\alpha > 0$	space-like
$(e+p) = 0$	$v_H = 1$	$\kappa > 0$	$\alpha = 0$	null
$(e+p) < 0$	$0 < v_H < 1$	$\kappa > 0$	$\alpha < 0$	time-like
		static horizon		
$8\pi R_H^2 p = -1$	$v_H = 0$	$\kappa > 0$	$\alpha = -1$	time-like
		ingoing horizon		
$(e+p) > 0$	$-1 < v_H < 0$	$\kappa < 0$	$\alpha < 0$	time-like
$4\pi R_H^2(e-p) = 1$	$v_H = -1$	$\kappa = 0$	$\alpha = -\infty$	null
$(e+p) > 0$	$v_H < -1$	$\kappa > 0$	$\alpha > 0$	space-like

$$v_H = \frac{1 + 8\pi R_H^2 p}{1 - 8\pi R_H^2 e} = \frac{\kappa + 2\pi R_H(e+p)}{\kappa - 2\pi R_H(e+p)} = \frac{1+\alpha}{1-\alpha}$$

$$\alpha = \frac{4\pi R_H^2(e+p)}{1 - 4\pi R_H^2(e-p)} = 2\pi R_H \frac{e+p}{\kappa} = \frac{v_H - 1}{v_H + 1}$$