Recovering MOND from an extended metric theory of gravity with a Palatini formalism

E. Barrientos[a] and S. Mendoza[b]

Instituto de Astronomía, Universidad Nacional Autónoma de México, AP 70-264, Distrito Federal 04510, México

[a]ebarrientos@astro.unam.mx
[b]sergio@astro.unam.mx

posgrado en ciencias físicas
unam

Abstract

In this work we construct a relativistic extension of the MODified Newtonian Dynamics (MOND) in the metric formalism $f(\chi)$ using the Palatini approach. We show that a simple power law: $f(\chi) = \chi^b$, with $b = 3/2$ corresponds to the non-relativistic form of MOND. Amongst the many approaches proposed to extend MOND to a relativistic regime, the Palatini metric formalism discussed here, yields second order field equations, which is a desirable (but not compulsory) requirement in a gravitational theory. We briefly discuss lensing applications of this proposal.

Introduction

Through dynamical observations of dwarf spheroidal galaxies (Hernandez *et al.*, 2010), globular clusters (Hernandez and Jiménez, 2012) and even wide open binaries (Hernandez *et al.*, 2012), it has became clear that Kepler's third law appears not to hold in its classical form on these systems. Instead Tully-Fisher relation substitutes Kepler's third law of motion.

The Tully-Fisher relation is an empirical relation between intrinsic luminosity of a galaxy (mass) and the width of its HI emission lines (rotation velocity). Briefly, it can be described as $V^4 \propto M$.

Milgrom (1983) noted that there is a constant acceleration $a_0 \sim 10^{-10}\mathrm{ms}^{-2}$ that appears "unexplained" in very different scaling relations. He hypothesized a modification of Newtonian dynamics below this constant acceleration a_0. For large accelerations $a >> a_0$ gravity is Newtonian. For small accelerations $a << a_0$ the modification would apply in the following form:

$$a = \frac{\sqrt{a_0 GM}}{r}, \tag{1}$$

In a recent work Bernal *et al.* (2011) developed a relativistic action based on dimensional grounds. This theory is built as a $f(\chi)$ function, where $\chi := RL_M^2$ where R is the Ricci's curvature scalar and L_M is a coupling constant to be fixed by empirical astronomical observations. In the weakest limit of this theory, acceleration is reduced to the MONDian one. Also, Mendoza *et al.* (2013) proved the compatibility of this theory with lensing observations.

In this work, we show the feasibility to recover the MONDian acceleration from an $f(\chi)$ theory in the Palatini formalism.

$f(\chi)$ in Palatini Formalism

The metric tensor and the connection are two different geometric objects.

Let us start with a dimensionally correct action motivated by the one built by Bernal *et al.* (2011)

$$\mathcal{S} = \frac{c^3}{16\pi G L_M^2}\int f(\chi)\sqrt{-g}d^4x + \frac{1}{c}\int \mathcal{L}_{\mathcal{M}}\sqrt{-g}d^4x, \tag{2}$$

where $f(\chi)$ is a dimensionless function given by: $\chi := L_M^2\mathcal{R}$.

L_M is a quantity with dimensions of length and $\mathcal{R}$ is not the traditional Ricci scalar R but is defined by: $\mathcal{R} := g^{\mu\nu}\mathcal{R}_{\mu\nu}$, with $g_{\mu\nu}$ being the metric tensor and $\mathcal{R}_{\mu\nu}$ the Ricci tensor defined exclusively in terms of the affine connection ${}^*\Gamma^{\alpha}{}_{\mu\nu}$.

The trace of the resulting equation from the variations of the action (2) with respect to the metric is given by:

$$L_M^2\frac{f(\chi)}{d\chi}\mathcal{R} - 2f(\chi) = \frac{8\pi G L_M^2}{c^4}T. \tag{3}$$

The variations with respect to the connection ${}^*\Gamma^{\alpha}{}_{\mu\nu}$ yields:

$$\nabla_\lambda\left(\sqrt{-h}h^{\mu\nu}\right) = 0, \tag{4}$$

where we have made the following conformal transformation: $h_{\mu\nu} := g_{\mu\nu}df(\chi)/d\chi$.

This last result is equivalent to the metric compatibility for the $h_{\mu\nu}$ metric and tells us that ${}^*\Gamma^{\alpha}{}_{\mu\nu}$ is the Levi-Civita connection of the $h_{\mu\nu}$ metric.

For this conformal transformation, $\mathcal{R}$ is related to the usual Ricci scalar R by:

$$\mathcal{R} = R - \frac{3}{f'}\Delta f' + \frac{3}{2f'^2}\nabla_\mu f'\nabla^\mu f'. \tag{5}$$

Equations (3) and (5) are the two fundamental equations of our theory. From (3) it is possible find $\mathcal{R}$ as a function of the matter, namely: $\mathcal{R} = \mathcal{R}(\mathcal{T})$ and then replace this result into (5) (f' is a function of $\mathcal{R}$ and therefore of T) in order to find $R = R(T)$, which is the solution we are looking for.

$f(\chi)$ as a power law

From now onwards we assume: $f(\chi) = \chi^b$. With this, the equation for $R = R(T)$ is:

$$R = \left[\frac{8\pi G L_M^{2(1-b)}T}{c^4(b-2)}\right]^{1/b} + \frac{3(b-1)}{bT}\left[\Delta T - \frac{(b+1)}{2bT}\nabla_\mu T\nabla^\mu T\right]. \tag{6}$$

This is the exact relation between the curvature R and the matter T .

To order of magnitude $\nabla \approx 1/r$, and so, $\mathcal{R} \approx R - 3/(2r^2)$. When $r \to \infty$, that is, in regions far enough from the matter distribution, where $a \lessapprox a_0$ it follows that: $\mathcal{R} \approx R$.

Since we are interested in the MONDian regime, this approach is useful to our purposes and equation (6) takes the following form:

$$R = \left[\frac{8\pi G L_M^{2(1-b)}T}{c^4(b-2)}\right]^{1/b}. \tag{7}$$

Weak field limit

Let us now perturbe spacerime about the flat Minlowsky metric up to second perturbation order. We take as base the work of Mendoza and Olmo (2014), in which they proved that, to be in accordance with the astronomical observations of lensing of individual, groups and clusters of galaxies together with the Tully-Fisher law, the metric coefficients are at second pertirvative order:

$${}^{(2)}g_{tt} = 1 + \frac{2\phi}{c^2}, \quad {}^{(2)}g_{rr} = -1 + \frac{2\phi}{c^2}, \quad {}^{(2)}g_{\theta\theta} = r^2\left(-1 + \frac{2\phi}{c^2}\right), \quad {}^{(2)}g_{\varphi\varphi} = r^2 sin^2\theta\left(-1 + \frac{2\phi}{c^2}\right) \tag{8}$$

which implies than the PPN parameter $\gamma = 1$.

Since the Tully-Fisher law describes the motion of non-relativistic dust particles, then the energy-momentum tensor trace is $T = \rho c^2$ with ρ the matter density. Then it follows that:

$$-\frac{2}{c^2}\nabla\cdot\mathbf{a} = \left[\frac{8\pi G L_M^{2(1-b)}\rho}{c^2(b-2)}\right]^{1/b}. \tag{9}$$

where the acceleration is defined by: $\mathbf{a} := \nabla\phi$.

Recovering MOND

Order of magnitude approach.

The flattening of rotation curves requires $a \approx r^{-1}$, which according to equation (9) means: $b = 3/2$.

Following (Bernal *et al.*, 2011), we define: $L_M := \zeta r_g^\alpha l_M^\beta$, where r_g and l_M are two "fundamental" lengths constructed with the quantities characterizing the problem:

$$r_g := \frac{GM}{c^2}, \quad l_M := \left(\frac{GM}{a_0}\right)^{1/2}. \tag{10}$$

The speed of light should not appear in a non-relativistic description of gravity, and so: $L_M \propto c$. With this restriction and the condition $\alpha + \beta = 1$, it follows that $\alpha = -1/2$, and $\beta = 3/2$.

Direct substitution of the values obtained for b and L_M, into equation (9) it follows that:

$$-2\nabla\cdot\mathbf{a} = (a_0 GM)^{1/2}\left[\frac{2\delta(r)}{\zeta r^2}\right]^{2/3}. \tag{11}$$

where we have used the fact that for a point-mass density $\rho = M\delta^3(r) = M\delta(r)/4\pi r^2$, where $\delta^3(r)$ is the 3-dimensional Dirac's delta function and $\delta(r)$ is the 1-dimensional one.

Full second order equation.

Let us consider Dirac's delta distribution as a normal function, so that we express the following: $[\delta(r)]^{2/3} = [\delta(r)]^{-1/3}\delta(r)$, and assume a power law for acceleration, i.e.: $\mathbf{a} = \lambda r^\sigma\mathbf{e_r}$.

Then, integrating in spherical coordinates, and using for $\delta(0)$ the following expression (Gsponer, 2008):

$$\delta(r=0) = \lim_{r\to 0}\frac{1}{2\pi r}, \tag{12}$$

equation (11) turns into:

$$-\frac{\lambda(\sigma+2)}{\sigma}r^\sigma\Big|_0^\infty = (a_0 GM)^{1/2}\left(\frac{\pi}{\zeta^2}\right)^{1/3}\frac{1}{r}\Big|_0. \tag{13}$$

Since ζ is a constant and λ is r-independent, necessarily $\sigma = -1$, as expected from the previous order of magnitude analysis.

In order to recover a MONDian acceleration, it is necessary that $\lambda = -(a_0 GM)^{1/2}$, and so a calibration for ζ is obtained: $\zeta = 2\sqrt{\pi}$.

Conclusions and perspectives

We obtained values for the parameters b, α, β and ζ of our theory. This implies that it is possible to build an $f(\chi) = \chi^{3/2}$ theory in the Palatini formalism which in its weak field limit of approximation reproduces a MONDian acceleration.

We choose to work in the frame of the Palatini formalism since it provides a deeper understanding of our proposal than the simple metric formalism since we do not restrict to a special kind of connection. While it is true that in GR, the Palatini formalism does not seems to bring something new, its use in areas where GR is not tested has been extended.

The next step to test our proposal is to see whether the Palatini proposal together with the results obtained by gravitational lensing in individual, groups and clusters of galaxies, i.e. with the PPN parameter $\gamma = 1$, reproduce the accelerated expansion of the universe observed in SNIa.

The main complaint that the $f(\chi)$ theory has had since it was proposed is about the presence of the mass M in the action, via L_M. Our next goal is to build a proposal which has no dependence on the mass M, but on the mass density ρ and make a connection with the energy-momentum tensor so that the proposal will be an $f(R,T)$ developed by Harko, Lobo, Nojiri, and Odintsov (2011).

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