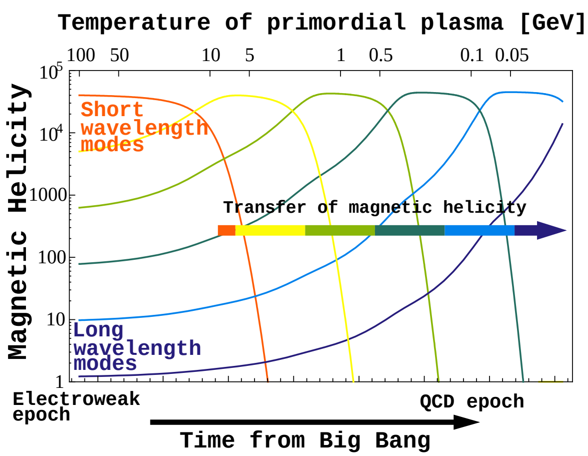
Relativistic chiral magnetohydrodynamics and evolution of cosmological magnetic fields



Oleg RUCHAYSKIY

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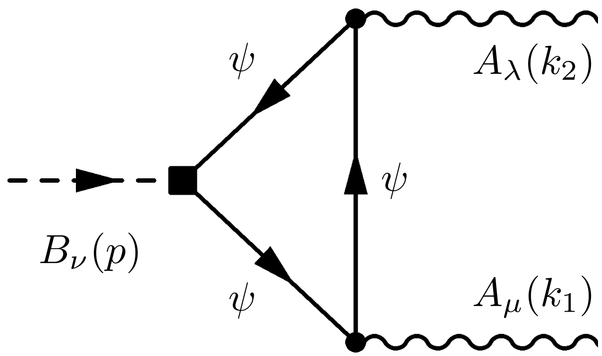


Phys. Rev. Lett. **108** (2012) 031301 [arXiv:1109.3350] (with A. Boyarsky and J. Fröhlich)

Phys. Rev. Lett. **109** (2012) 111602 [arXiv:1204.3604] (with A. Boyarsky and M. Shaposhnikov)

Phys. Rev. D (2015) [arXiv:1504.04854] (with A. Boyarsky and J. Fröhlich)

Main message

- Hot (dense) plasma of relativistic particles behaves very different from a laboratory plasmas
- In fact this is a rare situation when **quantum effects** affect dynamics at **macroscopic** (arbitrarily large) scales
- What I discuss in this talk **does not rely on any new physics** and is true fully within the Standard Model of particle physics

Axial symmetry

- Massless fermions can be left and right-chiral (left and right moving):

$$\text{Dirac equation:}\quad i\hbar\frac{\partial}{\partial t}\begin{pmatrix}\psi_L\\ \psi_R\end{pmatrix} = \begin{pmatrix} i\vec{\sigma}\cdot\nabla & \cancel{m}\to 0\\ \cancel{m}\to 0 & -i\vec{\sigma}\cdot\nabla\end{pmatrix}\begin{pmatrix}\psi_L\\ \psi_R\end{pmatrix}$$

  where $\gamma_5\psi_{R,L} = \pm\psi_{R,L}$ and $\gamma_5 = i\gamma_0\gamma_1\gamma_2\gamma_3 = \mathrm{diag}(-1,1)$.

- Number of left $N_L = \int d^3x\,\psi_L^\dagger\psi_L$ and right $N_R = \int d^3x\,\psi_R^\dagger\psi_R$ particles is conserved **independently**

- Electric charge: $Q = N_L + N_R$ – gauge symmetry

- **Axial charge**: $Q_5 = N_L - N_R$ – global symmetry

Axial anomaly

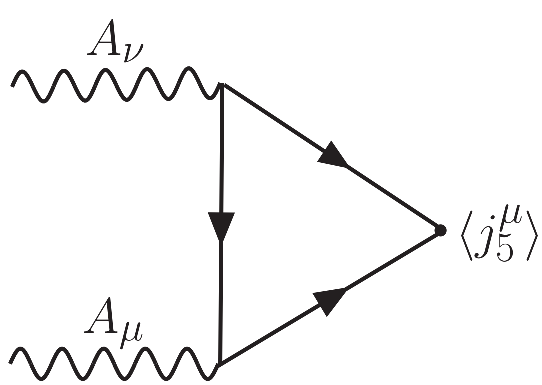
- Gauge interactions respects chirality ($D_\mu = \partial_\mu + eA_\mu$)...

$$i\hbar\frac{\partial}{\partial t}\begin{pmatrix}\psi_L\\ \psi_R\end{pmatrix} = \begin{pmatrix} i\vec{\sigma}\cdot(\nabla - \frac{e}{c}\vec{A}) & \cancel{m}\to 0 \\ \cancel{m}\to 0 & -i\vec{\sigma}\cdot(\nabla - \frac{e}{c}\vec{A})\end{pmatrix}\begin{pmatrix}\psi_L\\ \psi_R\end{pmatrix}$$

- ...but the difference of left and right-movers is **not conserved** once the quantum corrections are taken into account — **axial anomaly**

$$\frac{dQ_5}{dt} = \frac{d(N_L - N_R)}{dt} = \int d^3\vec{x}\Big(\partial_\mu j^5_\mu\Big) = \frac{\alpha}{\pi}\int d^3\,\vec{x}\vec{E}\cdot\vec{B}$$

$\alpha \equiv \frac{e^2}{4\pi}$ – fine-structure constant



QM interpretation: spectral flow

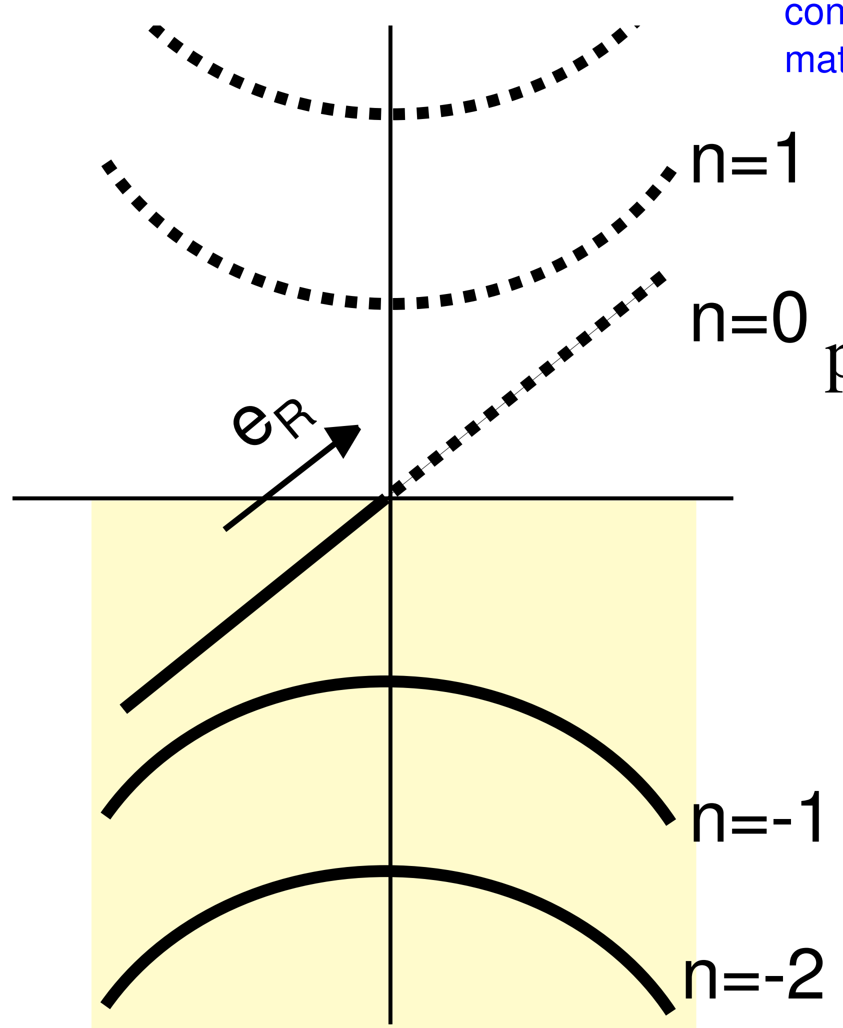
Landau levels: $E^2 = p_z^2 + e|B|(2n+1) + 2e\vec{B}\cdot\vec{s}$

G.E. Volovik, cond-mat/9802091

- Only particles with $(\vec{B}\cdot\vec{s} < 0)$ have **massless** branches:

$$E = \begin{cases} -p_z & \text{left branch} \quad (\vec{p}\cdot\vec{s}) > 0 \\ p_z & \text{right branch} \quad (\vec{p}\cdot\vec{s}) < 0 \end{cases}$$

- Electric field $\vec{E} = E\hat{z}$ **creates right particle** (because $p_z(t) = p_z(0) + eEt$)
- Electric field destroys left particles
- **Total number** does not change
- **Difference** of left minus right appears – **chiral anomaly!**



Axial anomaly $\leftrightarrow$ change of helicity

- If we introduce **magnetic helicity**

$$\mathcal{H} = \int d^3x\, \vec{A} \cdot \vec{B}$$

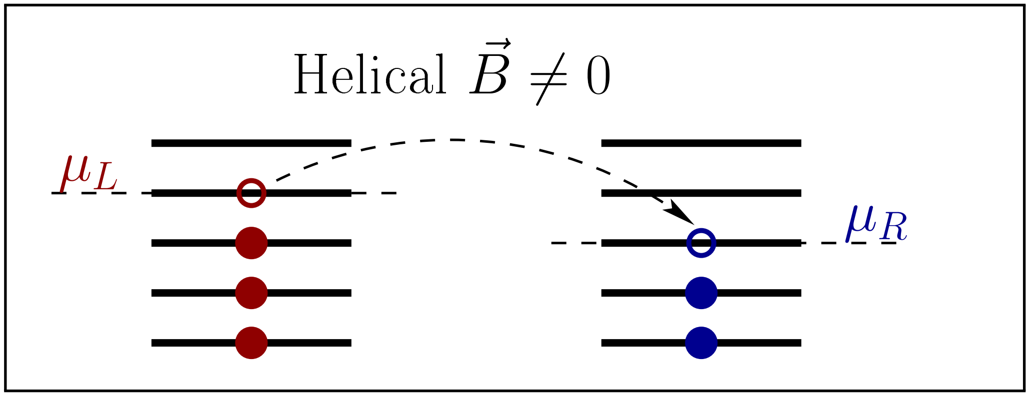
- ...then

$$\int d^3x\, \vec{E} \cdot \vec{B} = -\frac{1}{2}\frac{d\mathcal{H}(t)}{dt}$$

- Axial anomaly couples change of **left-right asymmetry** to the change of magnetic helicity:

$$\boxed{\frac{d}{dt}\left((N_L - N_R) + \frac{\alpha}{\pi}\mathcal{H}\right) = 0}$$

Axial anomaly at finite densities



- μ_L – Fermi level for left fermions
- μ_R – Fermi level for right fermions

- Change helicity of $\vec{B}$-field $\Rightarrow$ due to axial anomaly of $N_{L,R}$ changes:

$$\delta N_{L,R} = \int_{t_i}^{t_f} dt\, \dot{N}_{L,R}(t) = \mp \int dt \frac{\alpha}{\pi} \int dV\, E \cdot B$$

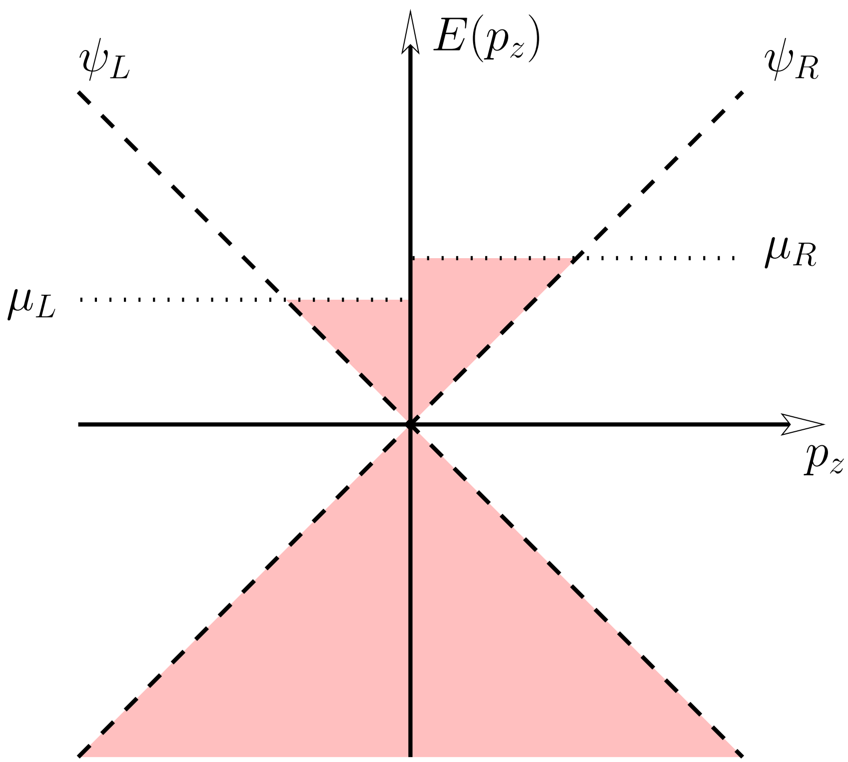
Nielsen & Ninomiya (1983);

Rubakov (1986)

- The energy: $\delta \mathcal{E} = \delta N_L \mu_L + \delta N_R \mu_R = \dfrac{\alpha(\mu_L - \mu_R)}{2\pi} \displaystyle\int dV\, A \cdot B$

- Free energy of magnetic fields in plasma: $\delta \mathcal{F} = \int dV\, \frac{\alpha(\mu_L - \mu_R)}{2\pi} A \cdot B$

Current along the magnetic field



Along the direction of the magnetic field $\vec{B}$:

- Left particles are moving one direction ($\vec{p} \downarrow\downarrow \vec{B}$)
- Right particles are moving in the other direction ($\vec{p} \uparrow\uparrow \vec{B}$)
- The number of left and right particles is different ($\mu_L \neq \mu_R$)
- $\Rightarrow$ The current proportional to the disbalance flows:

$$\vec{j} = \frac{e^2}{2\pi^2}(\mu_L - \mu_R)\vec{B}$$

Vilenkin'78;

Redlich & Wijewardhana (1985);

Rubakov'86;

Cheianov, Fröhlich, Alexeev'98;

Fröhlich & Pedrini'00,'02;

Fukushima, Kharzeev, Warringa'08

Son & Surowka'09

- So, the Maxwell equations contain current, **proportional to μ_5**

Kharzeev'11

$$\mu_5 = \mu_L - \mu_R$$

- MHD turns into **chiral MHD**:

Vilenkin (1978)

Fröhlich & Pedrini (2000–2001)

Joyce & Shaposhnikov (1997)

$$\begin{aligned} \mathrm{curl}\,\vec{E} &= -\frac{\partial \vec{B}}{\partial t} \\ \mathrm{curl}\,\vec{B} &= \sigma \vec{E} + \frac{2\alpha}{\pi}\mu_5 \vec{B} \end{aligned}$$

$\leftarrow$Chiral magnetic effect

- **In addition**, $\mu_5(t)$ should be allowed to **become dynamical**:

$$\frac{d(N_L - N_R)}{dt} \propto \frac{d\mu_5(t)}{dt} \propto \frac{\alpha}{\pi}\int d^3x\,\vec{E}\cdot\vec{B}$$

Dynamics of μ_5

Boyarsky, Fröhlich, **O.R.**, PRL (2012)

$$\begin{aligned}
\mathrm{curl}\,\vec{E} &= -\frac{\partial \vec{B}}{\partial t} \\
\mathrm{curl}\,\vec{B} &= \sigma \vec{E} + \frac{2\alpha}{\pi}\mu_5 \vec{B} \quad \leftarrow \text{Chiral magnetic effect} \\
\frac{\partial \mu_5}{\partial t} &\propto \frac{2\alpha}{\pi}\int d^3x\, \vec{E}\cdot\vec{B} - \Gamma_{\text{flip}}\mu_5 \quad \leftarrow \text{chirality-flipping due to finite mass of fermions}
\end{aligned}$$

- Without B chirality flipping reactions drive $\mu_5 \to 0$ ($\mu_5 = \mu_0 e^{-\Gamma_{\text{flip}} t}$)
- Without μ_5 finite conductivity drives $B \to 0$ ($B_k = B_0 e^{-\frac{k^2 t}{\sigma}}$). This is called **magnetic diffusion**

Instability

- Maxwell equations with μ_5 are **unstable**:

$$\frac{\partial B}{\partial t} = \underbrace{\frac{1}{\sigma}\nabla^2 B}_{\text{magnetic diffusion}} + \underbrace{\frac{\alpha\mu_5}{\pi}\,\mathrm{curl}\,B}_{\text{instability}}$$

- Circular polarized waves $B_k^{\pm}$

$$\frac{\partial B_k^{\pm}}{\partial t} = -\frac{k^2}{\sigma}B_k^{\pm} \pm \frac{\alpha\mu_5}{\pi}B_k^{\pm}$$

- Exponential growth for $k < \frac{\alpha}{\pi}\mu_5$ (for one of the circular polarizations depending on the sign of μ_5) — **generation of helical magnetic fields**

$$\boxed{B_{\pm} = B_0 \exp\left(-\frac{k^2}{\sigma}t \pm \frac{\alpha}{\pi}\frac{k\mu_5}{\sigma}t\right)}$$

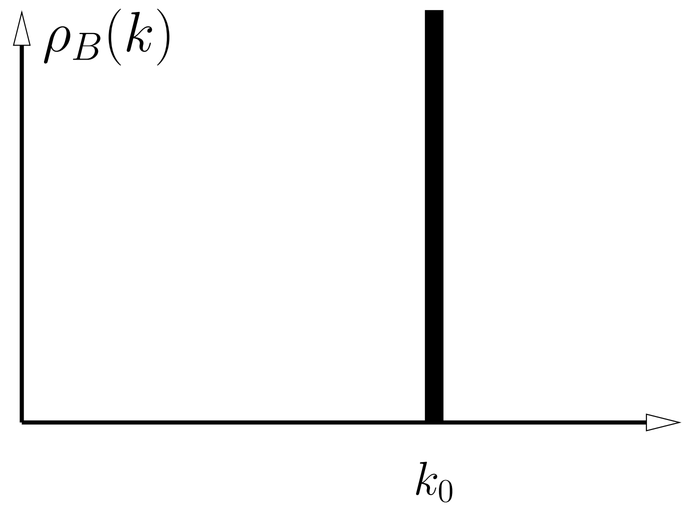
Attractor solution

- Consider **sharply peaked at k_0 maximally helical field**

$$\frac{d\mu_5}{dt} = -\rho_B\Big(\mu_5 - \bar\mu_5\Big) - \cancel{\Gamma_{\text{flip}}\mu_5}$$

$$\frac{d\rho_B}{dt} = \frac{\rho_B}{t_\sigma}\left(\frac{\mu_5}{(\bar\mu_5)} - 1\right)$$

where $t_\sigma = \frac{2\sigma}{k_0^2}$ and $\rho_B \gg \Gamma_{\text{flip}}$

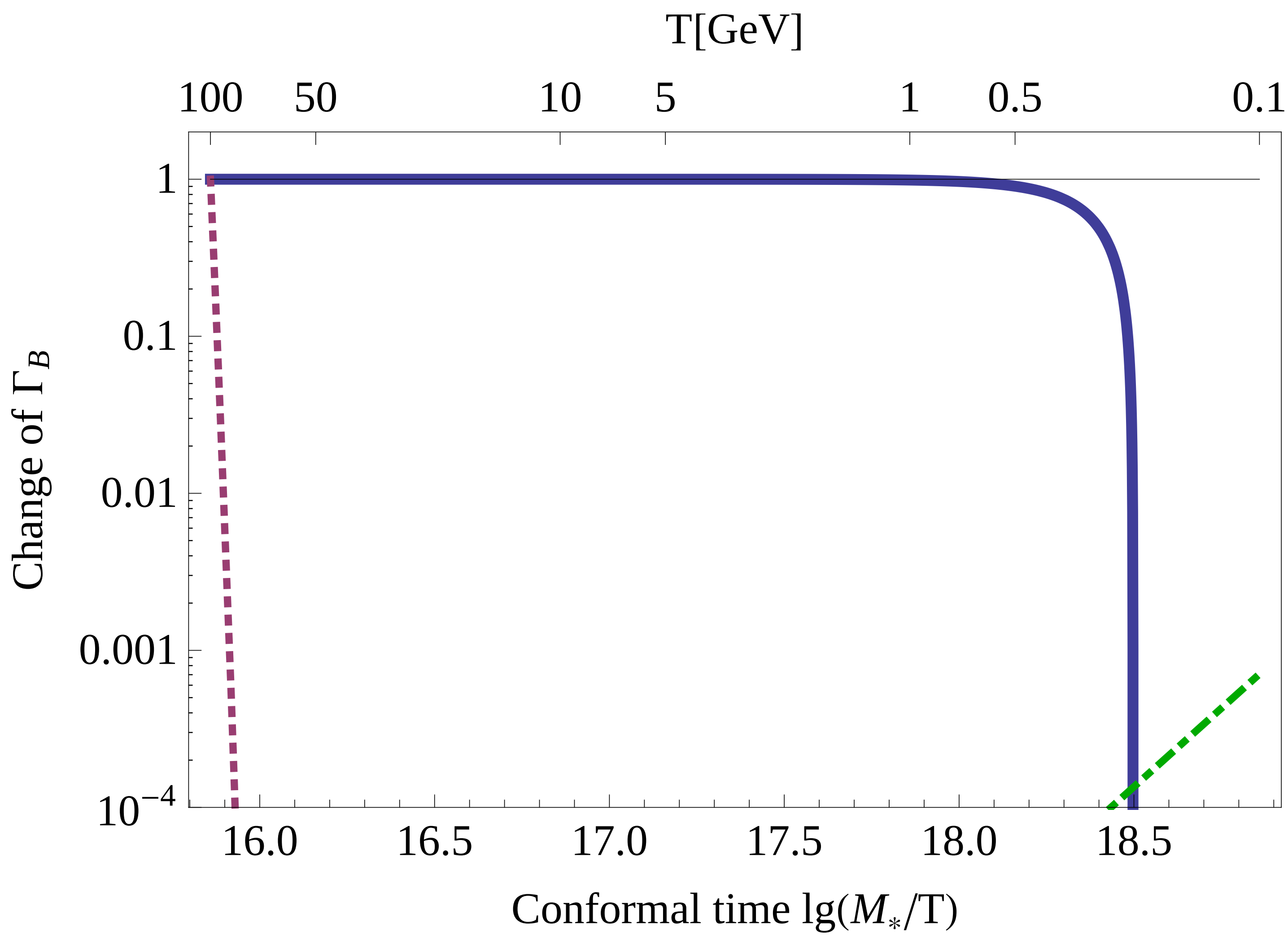


- Large ρ_B drives μ_5 to an **attractor solution** $\boldsymbol{\bar\mu_5 = \frac{2\pi k_0}{\alpha}}$

- Electric conductivity of the plasma is **finite** but **magnetic diffusion is compensated by the presence of μ_5**

O.R. with A. Boyarsky, J. Fröhlich PRL 2012 **[1109.3350]**

Evolution of magnetic energy density

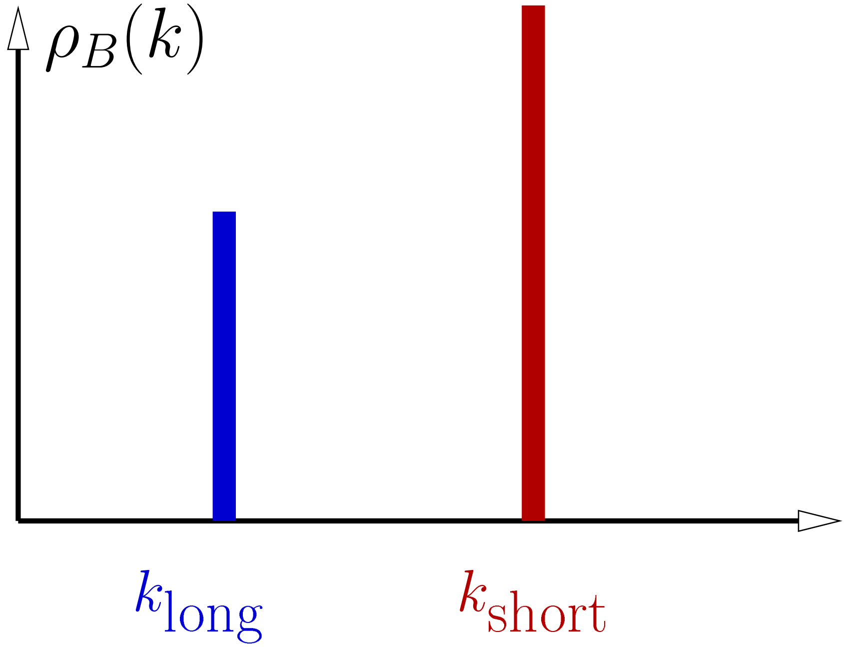


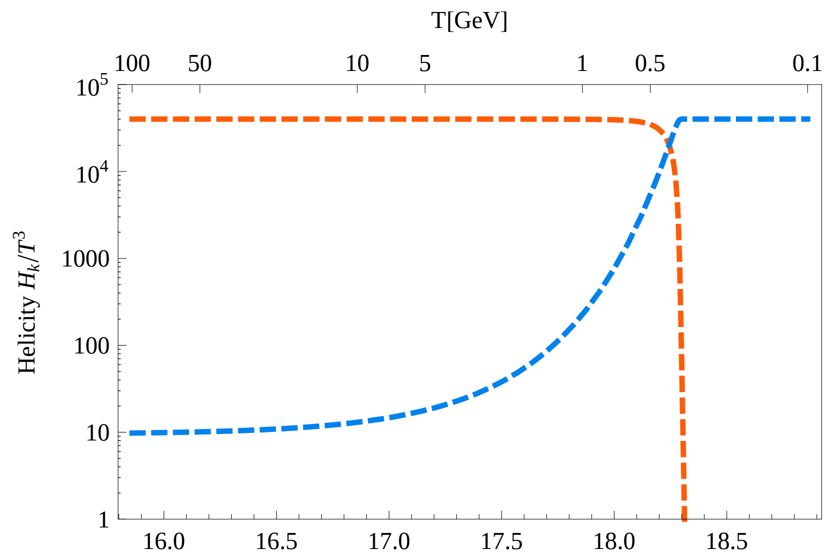
PRL (2012) [1109.3350]

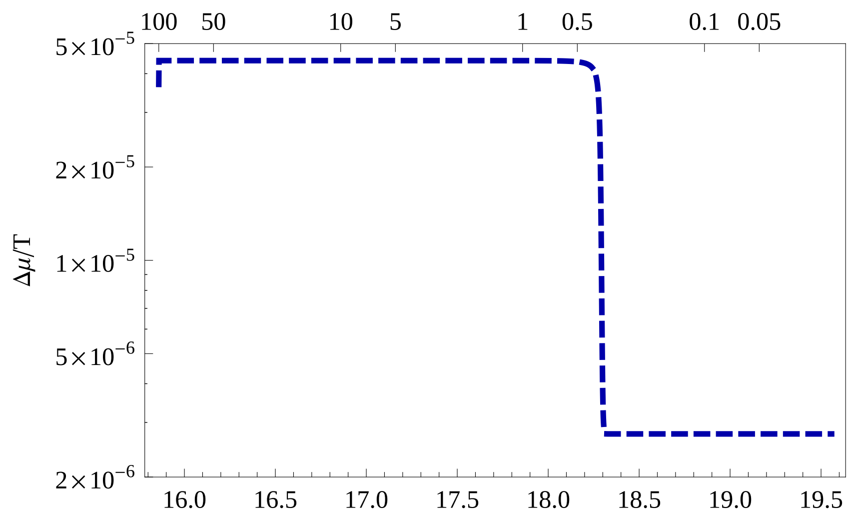
See also Tashiro et al. [1206.5549]

X-axis – log of time

Two modes

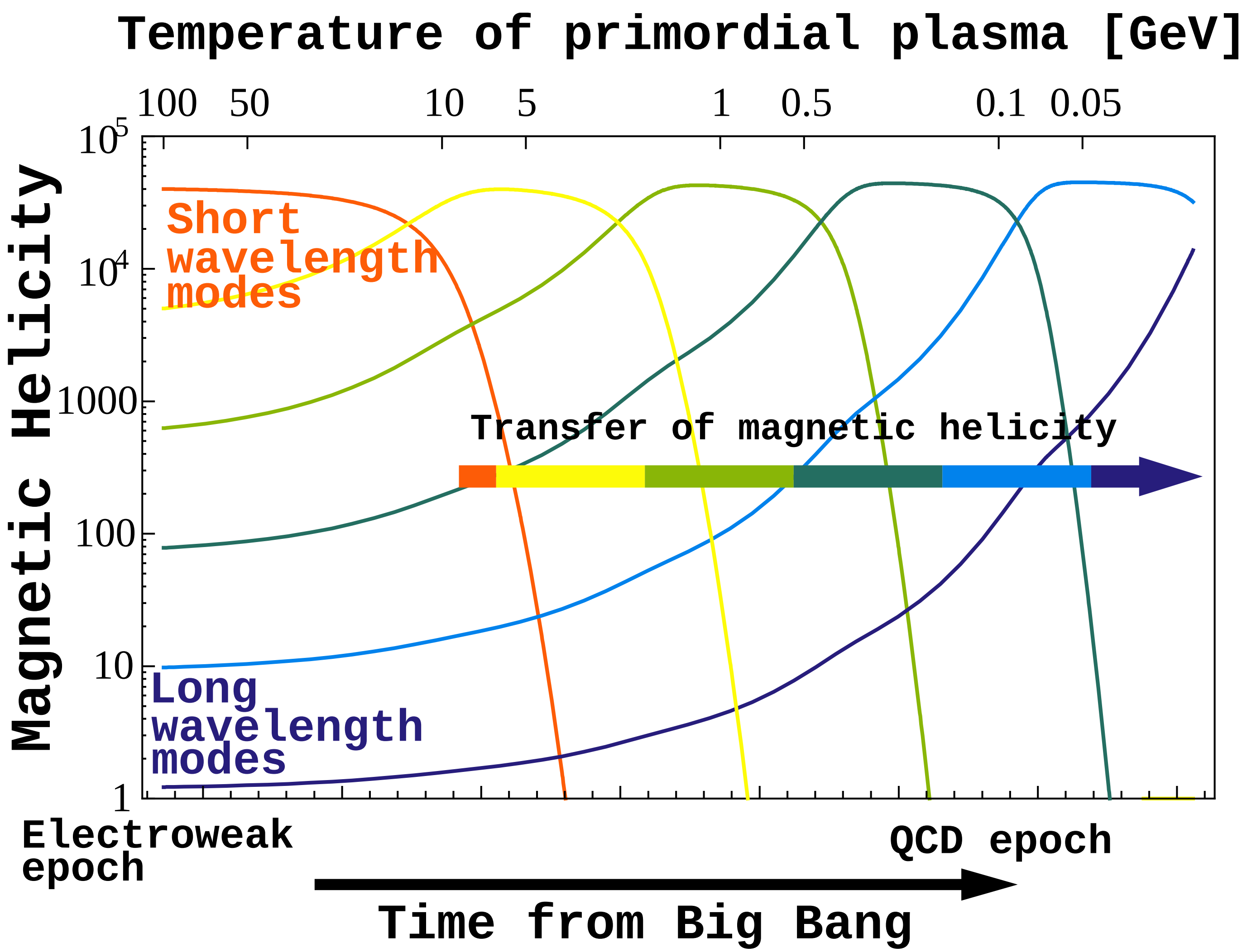






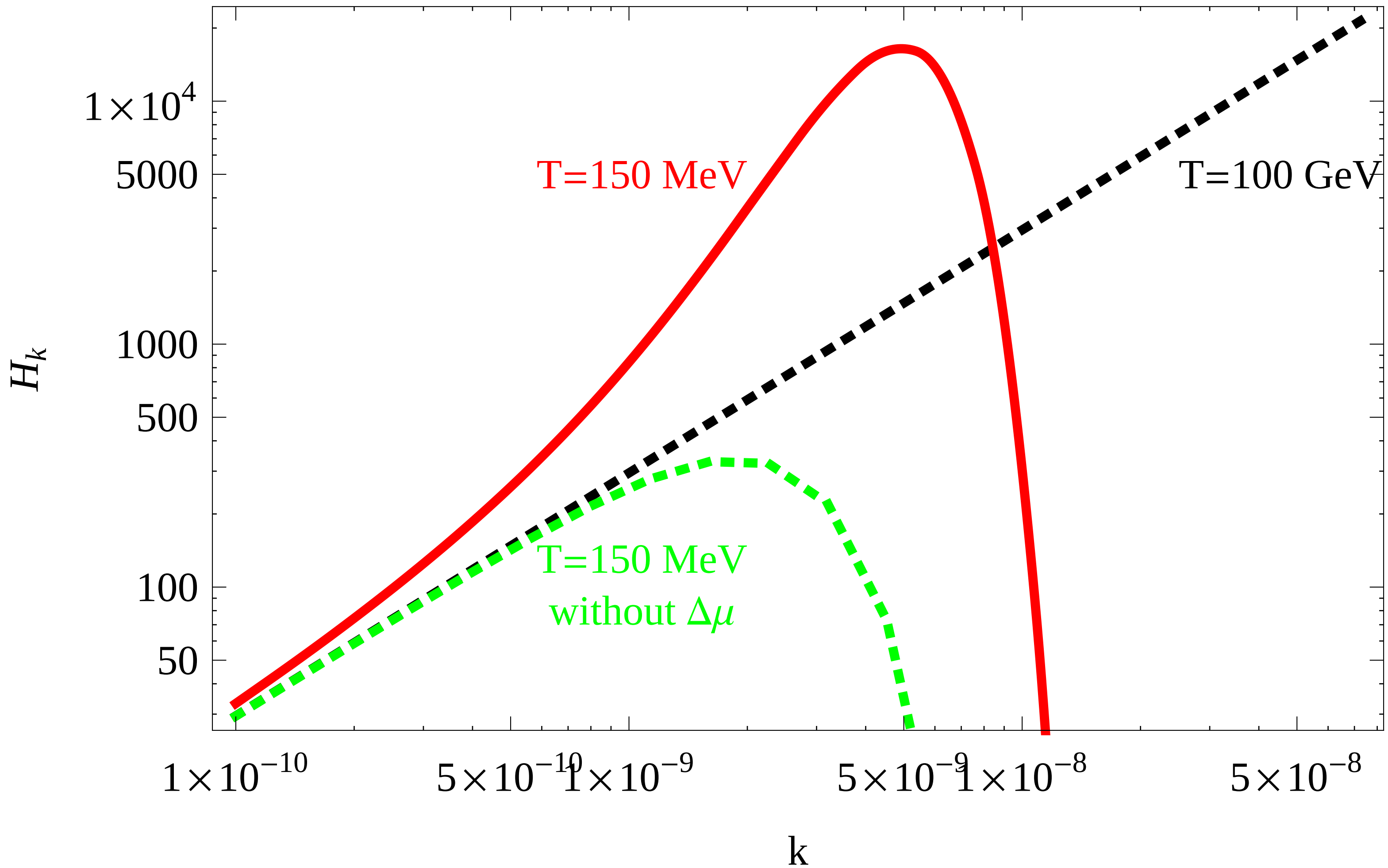
- In case of two modes the helicity gets transferred from the shorter one to the longer one
- Chemical potential follows the wave-number of the mode with higher helicity $\mu = \frac{2\pi k}{\alpha}$

Evolution of helicity spectrum



Process continues while $\Gamma_B \gg \Gamma_{\text{flip}}$ (recall that $\Gamma_B \propto \rho_B$)

Evolution of helicity spectrum



Inverse cascade without turbulence!

Inhomogeneous dynamics of μ_5

- Back-reaction of magnetic fields on chemical potential (via axial anomaly) will make μ_5 inhomogeneous (i.e. $\mu_5(x)$)

Boyarsky et al., [1504.04854]

$$\begin{aligned}
\mathrm{curl}\,\vec{E} &= -\frac{\partial \vec{B}}{\partial t} \\
\mathrm{curl}\,\vec{B} &= \sigma \vec{E} + \frac{2\alpha}{\pi}\mu_5(\vec{\boldsymbol{x}}, t)\vec{B} + \textbf{gradient terms?} \\
\frac{\partial \mu_5(\vec{\boldsymbol{x}}, t)}{\partial t} + \mathcal{O}(\nabla \mu_5) &\propto \frac{2\alpha}{\pi}\vec{E}(\vec{\boldsymbol{x}}, t)\cdot \vec{B}(\vec{\boldsymbol{x}}, t) - \Gamma_{\text{flip}}\mu_5
\end{aligned}$$

- Can there by terms, proportional $\vec{E}$?

- Recall: $\vec{j}_{\text{CME}}$ is a $\mathcal{T}$-odd current, and $\vec{E}$ is $\mathcal{T}$-even

Fröhlich & Pedrini (2000) (2002)

- Let us introduce a pseudo-scalar field such that $\boxed{\dot{\theta} = \mu_5}$

See also Nicolis, Son, Dubovsky, et al. (2011–2013)

Inhomogeneous dynamics of μ_5

- θ is $\mathcal{T}$-odd and therefore one can now write currents like $\nabla\theta \times \vec{E}$:

$$\vec{j} = \dot{\theta}\vec{B} + \nabla\theta \times \vec{E}$$

- We started from a thermodynamical system with **only** Standard Model particles (electrons, photons).

- It turns out that if the system has chiral anomaly, it is not described by the usual magneto-hydrodynamics, rather it is described by "chiral" (or axion) hydrodynamics with an **extra degree of freedom**

- Axion obeys a second order in time differential equation – **diffusion-type** equation for μ_5, as it should be

Full system of chiral MHD equations

See talk by Jennifer Schober

Chiral anomaly : $\frac{D\mu_5}{\partial t} = D_5\,\Delta\mu_5 + \Lambda^{-2}\vec{E}\cdot\vec{B},$

Axion $\leftrightarrow$ chemical potential : $\frac{\partial\Theta}{\partial t} + \vec{U}\cdot\boldsymbol{\nabla}\Theta = \frac{\alpha}{\pi}\mu_5,$

Maxwell's eq. with CME : $\boldsymbol{\nabla}\times\vec{B} = \frac{4\pi}{c}\vec{J} + \frac{\vec{B}}{c}\frac{\partial\Theta}{\partial t} + (\boldsymbol{\nabla}\Theta)\times\vec{E},$

Bianchi identity : $\boldsymbol{\nabla}\times\vec{E} = -\frac{1}{c}\frac{\partial\vec{B}}{\partial t},$

Ohm's law : $\vec{J} = \sigma\left(\vec{E} + \frac{1}{c}\vec{U}\times\vec{B}\right),$

Continuity eqn. : $\frac{D\rho}{dt} = 0,$

Navier-Stokes eqn. : $\rho\frac{D\vec{U}}{dt} = -c_s^2\nabla p + \vec{F} + \frac{1}{c}\vec{J}\times\vec{B}$

$$\frac{Df}{dt} \equiv \dot{f} + \vec{U}\cdot\nabla f$$

In short

$$\boxed{\frac{d}{dt}\left((N_L - N_R) + \frac{\alpha}{\pi}\mathcal{H}\right) = 0}$$

Change in magnetic helicity is coupled to the chiral chemical potential $\mu_5(t)$

Change of magnetic helicity $\Longrightarrow$ Change of $\mu_L - \mu_R$

– **Change in magnetic helicity** excites μ_5 and then evolution of **already existing** magnetic fields occurs differently

– Presence of μ_5 **excites helical magnetic fields**

Change of $\mu_L - \mu_R$ $\Longrightarrow$ Change of magnetic helicity

Generation of magnetic fields

- The presence of **chiral imbalance** excites magnetic fields (dynamo effect)
- What is the origin of **chiral imbalance**?
- All charged fermions are massive
- Mass breaks left/right symmetry

$$(i\gamma^\mu\partial_\mu - m)\psi = \begin{pmatrix} -m & i(\partial_t + \vec{\sigma}\cdot\vec{\nabla}) \\ i(\partial_t - \vec{\sigma}\cdot\vec{\nabla}) & -m \end{pmatrix}\begin{pmatrix}\psi_L\\ \psi_R\end{pmatrix} = 0$$

- In the early Universe when particles have $\langle E\rangle \sim T \gg m$ – can we think about μ_5 as approximately conserved? – **No!**
- Chirality-flipping reaction: $e_L + \gamma \to e_R + \gamma$

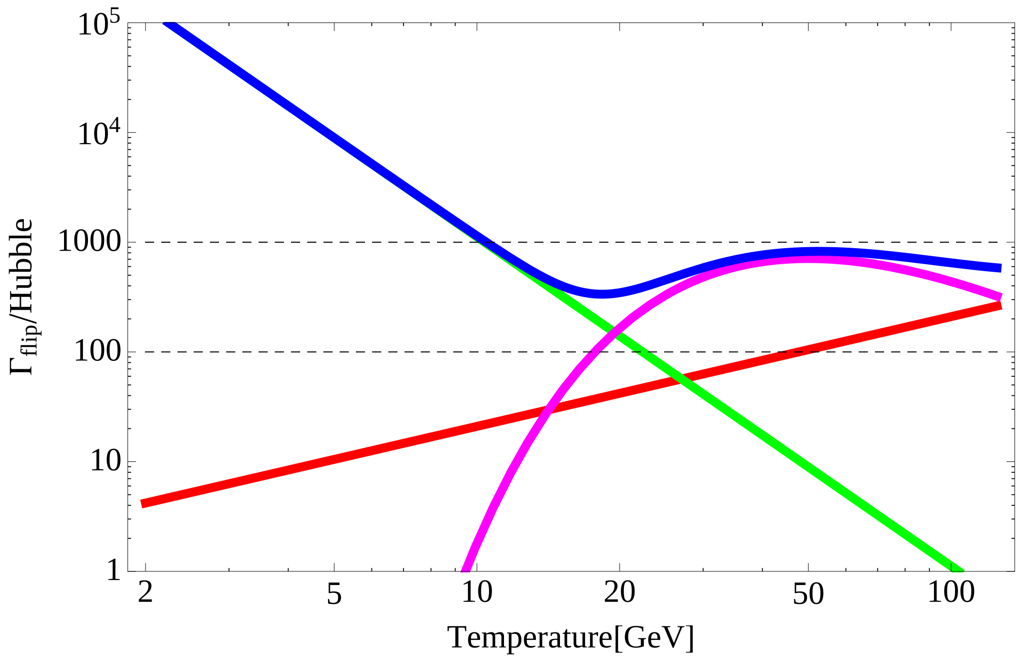
- Although $T \gg m$ and these reactions are suppressed as $(m/T)^2$ as compared to chirality-preserving reactions after long time they will wash out μ_5:

$$\frac{d\mu_5}{dt} = -\Gamma_f \mu_5$$

- Starting from $T \sim 80$ TeV chirality flipping processes are in equilibrium ($\Gamma_{\text{flip}}(T) \gg H(T) = T^2/M_*$) Cambell et al. (1992)

- Although $\left(\frac{m_e}{80\ \text{TeV}}\right)^2 \sim 10^{-17}$ chirality flipping reactions are in thermal equilibrium for $T < 80$ TeV and drive $\mu_L - \mu_R$ to zero **exponentially fast** (suppression of at least e^{-1000} over one Hubble time)

Generation of magnetic fields



Symmetric phase:

- $\Gamma_{\textbf{high-temp}} \sim \frac{80\ \text{TeV}}{M_*} T$

Broken phase:

- $\Gamma_{\text{EM}} \propto \alpha^2 T \left(\frac{m_e}{3T}\right)^2$
- $\Gamma_{\text{W}} \propto G_F^2 T^5 \left(\frac{m_e}{3T}\right)^2$

- If a **source** creates an asymmetry in a left sector $\Rightarrow$ its transition to a right sector is not immediate ($e^{-\Gamma_{\text{flip}} t}$) but instead can be long and be accompanied by the generation of magnetic fields (after which the whole system slowly relaxes to the expected minimum as $1/t^a$)

Summary

- System with chiral anomaly when put at non-zero temperature/finite density **necessarily** contains an effective degree of freedom that couples to the change of magnetic helicity
- This additional **IR degree of freedom** is the difference of chemical potentials of left and right particles
- If such a degree of freedom gets excited – this leads to an instability and exponential growth of the helical magnetic field. This instability is **always present**, however time of development may be long
- Evolution of relativistic systems with magnetic fields **are not described** by the standard MHD equations (as was previously believed) but rather by chiral MHD

Thank you for your attention!

More realistic analysis?

– Local $\mu_5(t, x)$

– Velocities

- The non-linear system of equations:

$$\begin{cases} \mathrm{curl}\,\vec{B} = \sigma\vec{E} + \frac{\alpha}{\pi}\Big(\dot{\theta}_5\vec{B} + \nabla\theta_5 \times \vec{E}\Big) \\ \Lambda^2\ddot{\theta}_5 + \text{gradient terms} = \frac{2\alpha}{\pi}\vec{E}\cdot\vec{B} + \text{chirality flip}, \\ \vec{B}\cdot\nabla\dot{\theta}_5 = 0, \\ \mu_5 = \dot{\theta}_5 \end{cases}$$

- Tracking solution?

- Take $\vec{B}(x) = B(t)\big(\sin(kz), \cos(kz), 0\big)$. In this case $\vec{E}\cdot\vec{B}$ is **constant in space**

- $\Rightarrow \mu_5 = \dot{\theta}_5$ depends **on time only**. The gradient terms are not important. But the equation is still non-linear!

Analysis of non-linear equations

$\vec{B}(x) = B(t)\big(\sin(kz), \cos(kz), 0\big)$

Boyarsky, Fröhlich, Ruchayskiy [1504.04854]

- The solution at any time is given by

$$\begin{cases} \text{amplitude } B(t) = \dfrac{C_1}{\sqrt{1 + C_2 \exp\left(\frac{2(k^2-\gamma^2)}{\sigma} t\right)}}, \\ \mu_5(t) = \mu_5^0 - \dfrac{\alpha}{2\pi} \dfrac{B^2(t) - B_0^2}{k\Lambda^2} \end{cases}$$

where $\gamma^2 = \frac{\alpha}{\pi} k \mu_5^0 + \frac{\alpha^2 B_0^2}{2\pi^2 \Lambda^2}$.

- If $\gamma > k$ we get a non-trivial static solution at $t \to \infty$

$$B_\infty^2 = B_0^2 \left[1 + \left(\frac{2k^2}{\beta_0^2}\right)\left(\frac{\mu_5^0}{\mu_5^\infty} - 1\right)\right]$$

where $\mu_5^\infty \equiv \mu_5(t \to \infty) = \frac{\pi k}{\alpha}$

Analysis of non-linear equations

- Do we see energy transfer from short scale to longer scales? — **Yes**

- Linearized analysis: perturb force-free configuration with the wavenumber k and the amplitude B_∞ ($\vec{B} = B_\infty(\sin(kz), \cos(kz), 0)$) by longer mode q

- Homogeneous case: every $q < k$ is unstable (energy transfer to longer wavelength)

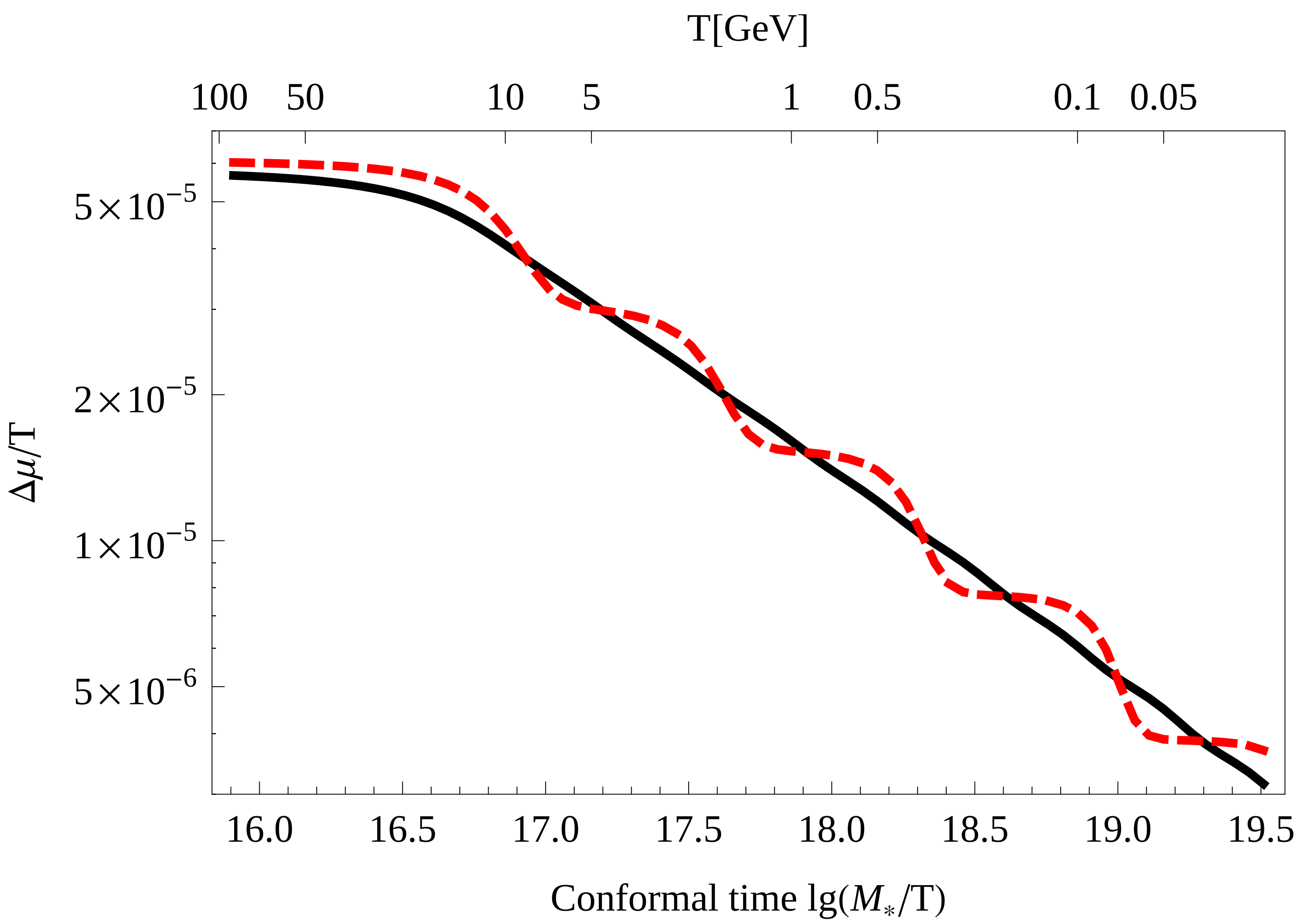
- Inhomogeneous case: unstable q:

$$\frac{k - \sqrt{k^2 - 4\beta^2}}{2} < q < \frac{k + \sqrt{k^2 - 4\beta^2}}{2}, \qquad \text{where } \beta^2 = \frac{\alpha^2 B^2}{\pi^2 T^2}$$

- $\Rightarrow$ Longer wavelength modes start to grow exponentially!

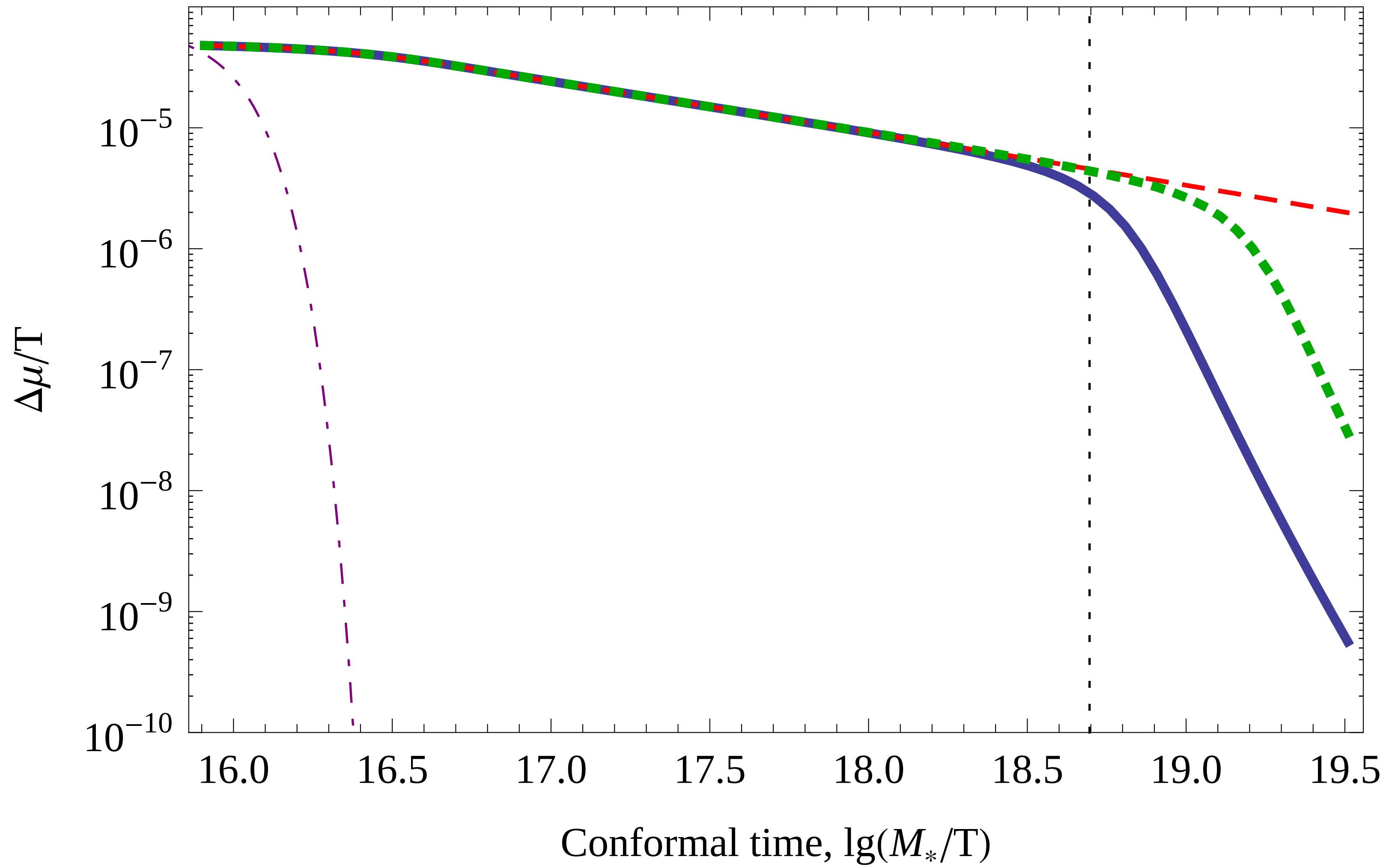
Boyarsky, Fröhlich, Ruchayskiy [1504.04854]

Evolution of chemical potential



Process continues while $\rho_B \gg \Gamma_{\text{flip}}$

Evolution of chemical potential



Continuous initial spectrum with $\mathcal{H}_k \propto k$ and fraction of magnetic energy density $\mathbf{5 \times 10^{-5}}$ (blue) or $\mathbf{5 \times 10^{-4}}$ (green). Red – evolution without flip