Vittorio De Falco

Graduated in Mathematics at the University of Napoli

PhD student supervised by Prof. Maurizio Falanga

APPROXIMATION OF RELEVANT ELLIPTIC INTEGRALS IN THE SCHWARZSCHILD METRIC AND SOME ASTROPHYSICAL APPLICATIONS

Collaborators:

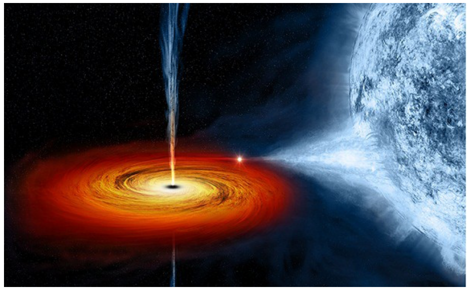
Maurizio Falanga

Luigi Stella

13-18 December 2015 **Geneva** **28th Texas Symposium on Relativistic Astrophysics**

ASTROPHYSICAL MOTIVATION

Discovery of X-ray emission coming from the accretion disks around the black holes



connect

Study how the matter around the black holes appears to an observer at infinity

first simulations

Luminet 1979

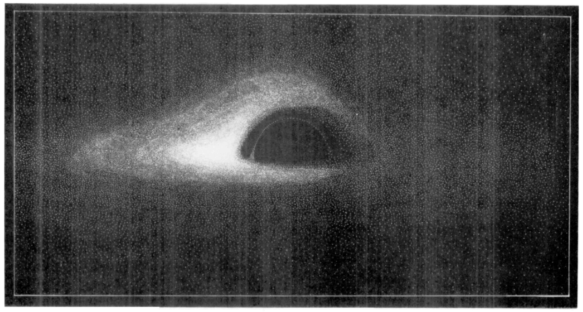
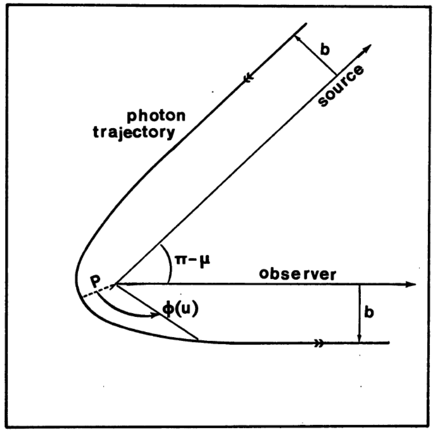
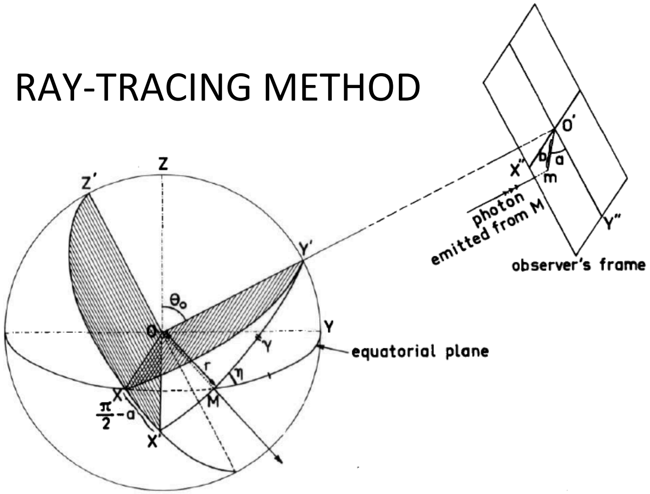


Image of a black hole

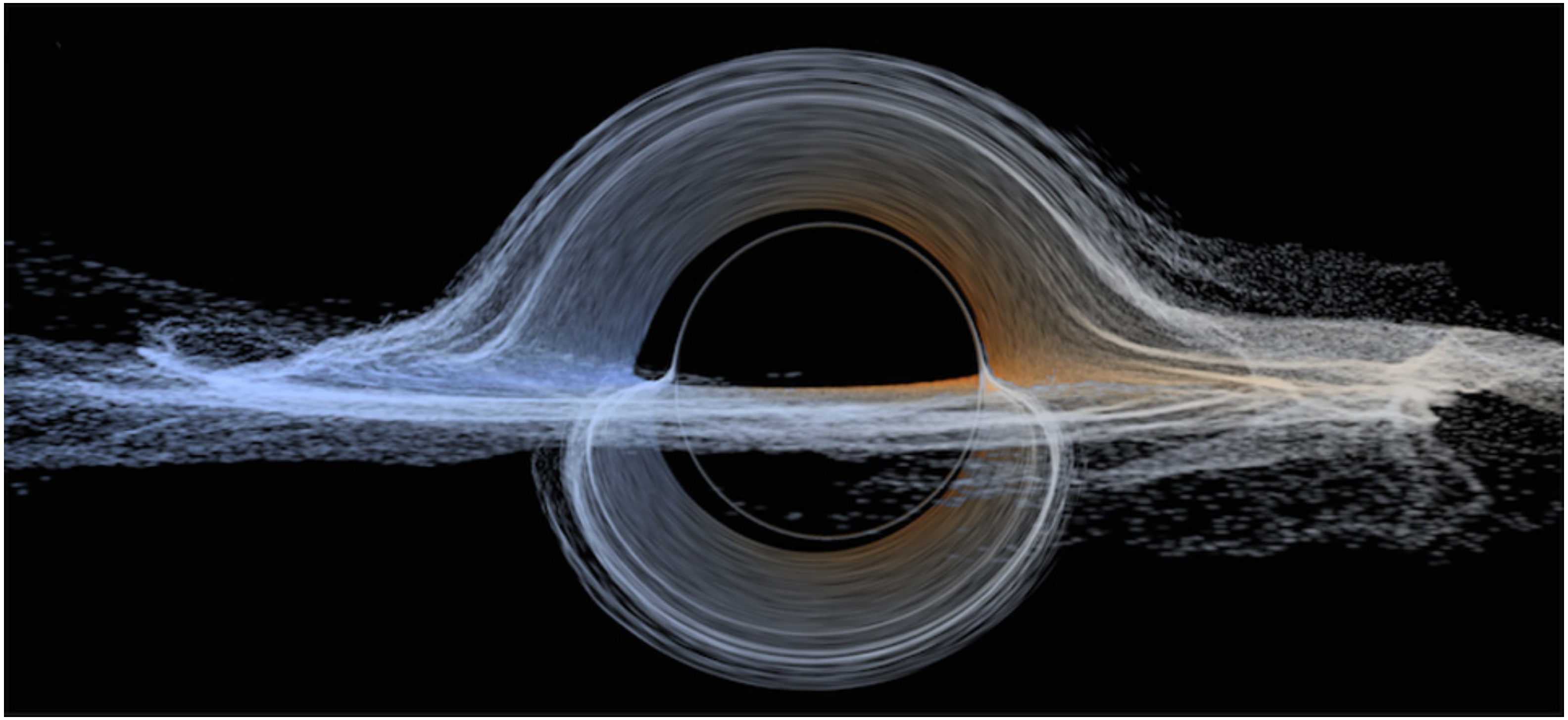


RAY-TRACING METHOD



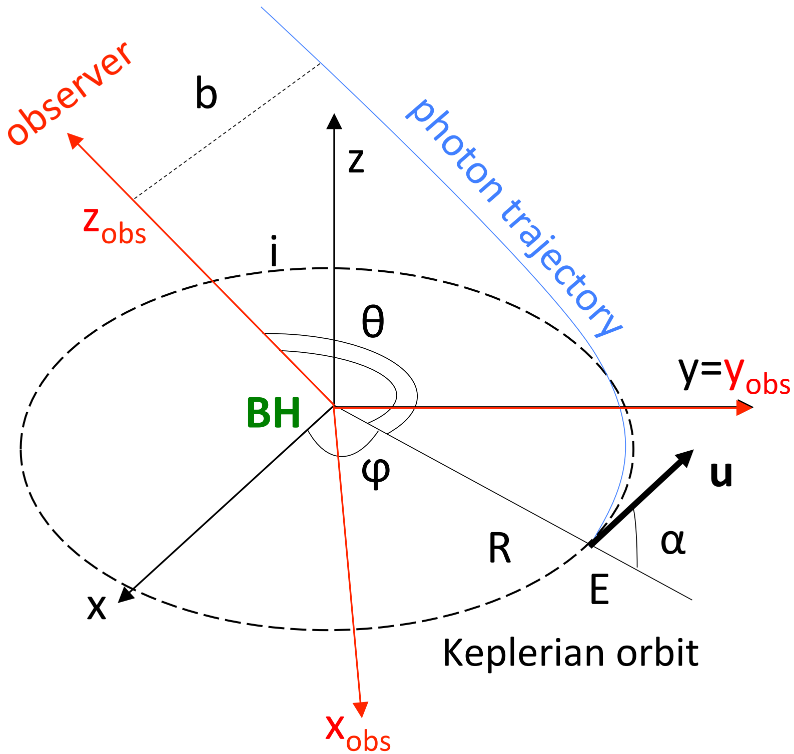
1 CHAPTER

INTRODUCTION



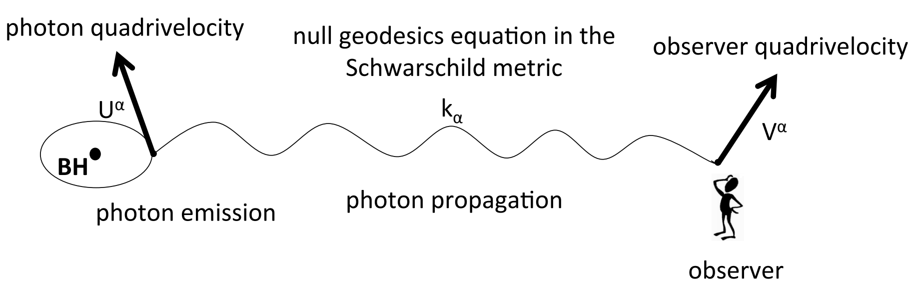
Numerical simulation of an accretion disk placed around a black hole

1.1 Geometrical structure



TARGETS

- *LIGHT BENDING (elliptic integral)*
- *TRAVEL TIME (elliptic integral)*
- *GRAVITATIONAL LENSING (elliptic integral)*
- *GRAVITATIONAL REDSHIFT (exact)*
- *FLUX (derived)*

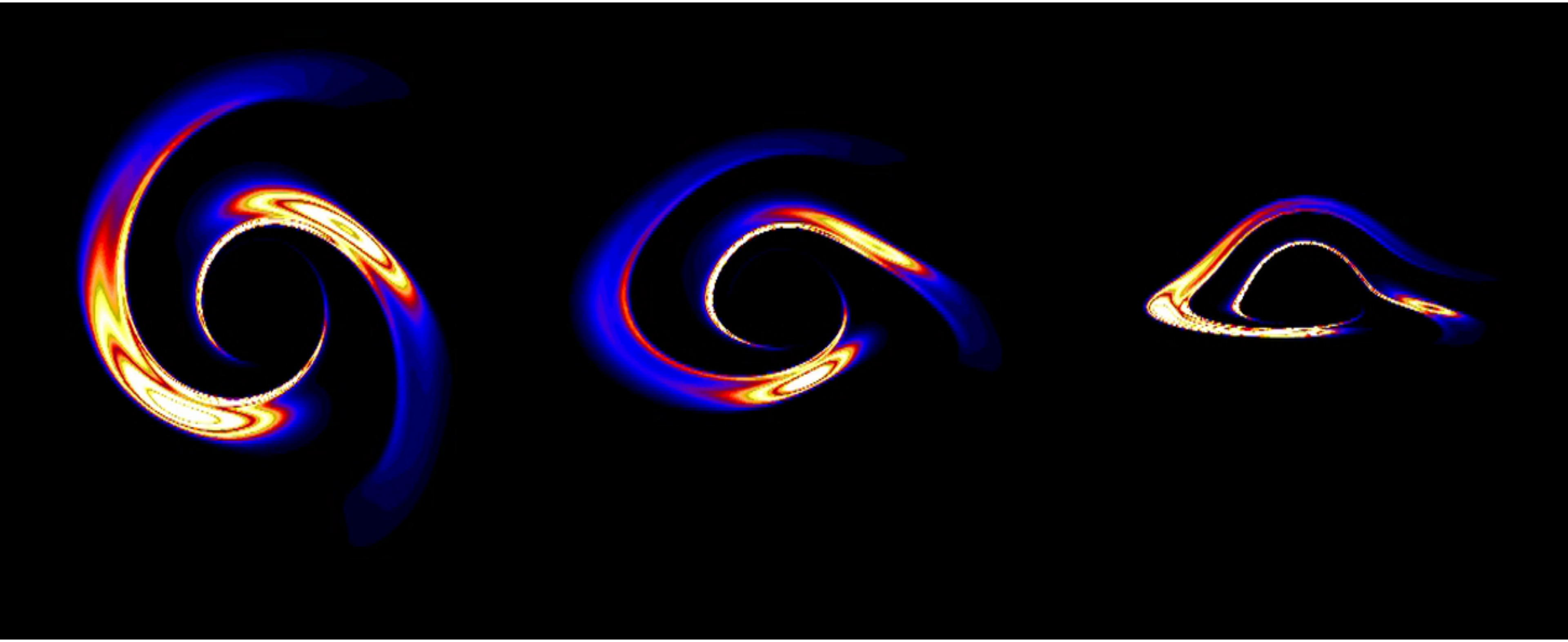


$$g = (1+z) = \frac{Eobs}{Eem} = \frac{(V^\alpha k_\alpha)_{obs}}{(U^\alpha k_\alpha)_{em}}$$

$$F = \int_{\nu_o} \int_R \int_\varphi I_{\nu_o} d\nu_o d\Omega$$

1.2 Simulations of the gravitational effects

Numerical simulations showing the gravitational effects mentioned before:

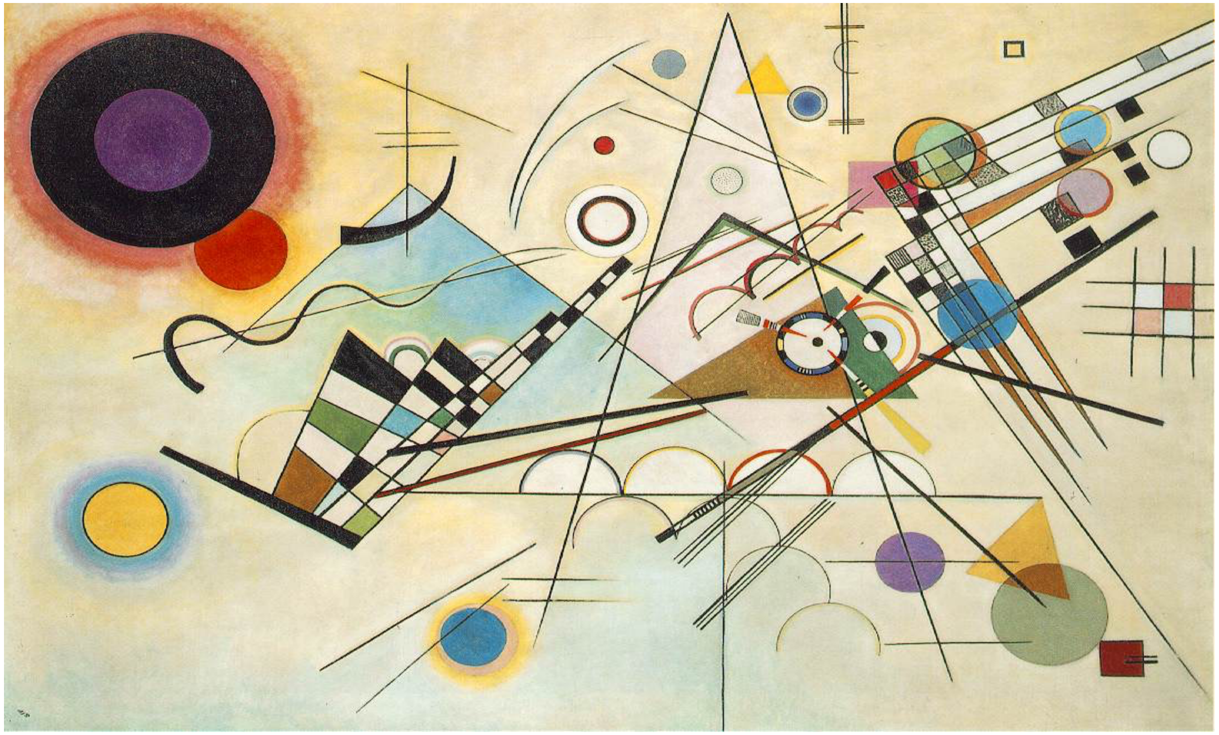


i=30° i=60° i=89°

The relativistic effects become stronger and stronger

2 CHAPTER

ELLIPTIC INTEGRALS



Kandisky - Composition VIII (1923)

2.1 Light bending

(*) Misner, Thorne & Wheeler - Gravitation

$$\theta = \int_R^\infty \frac{b}{r^2}\left[1 - \frac{b^2}{r^2}\left(1 - \frac{2M}{r}\right)\right]^{-\frac{1}{2}} dr$$

Substitution ad hoc

$$z = 1 - \cos\alpha$$

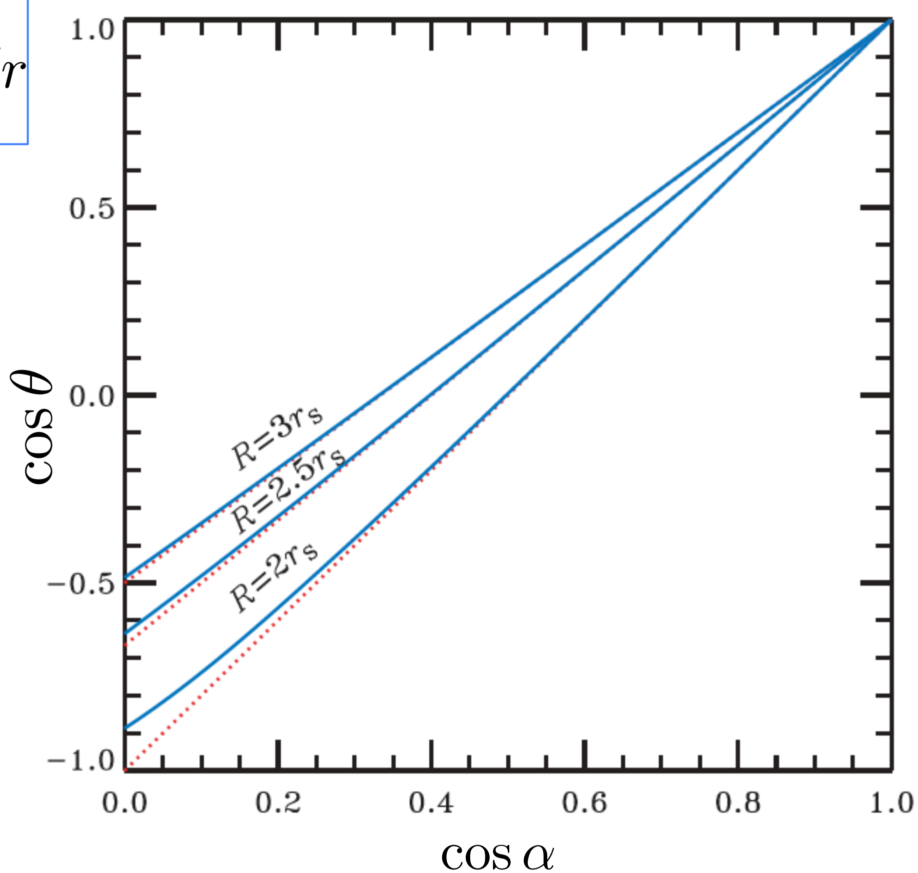
in

$$b = \frac{r\sin\alpha}{\sqrt{1 - \frac{2M}{r}}} \quad (*)$$

considering α small

Mathematical algebra

Beloborodov (2002)



Beloborodov (2002)

$$(1 - \cos\theta)(1 - u) = 1 - \cos\alpha$$ where $u = \frac{r_s}{r}$

2.2 Time delay

(*) Misner, Thorne & Wheeler - Gravitation

$$\Delta t = \int_R^\infty \frac{dr}{\left(1-\frac{2M}{r}\right)} \left\{ \left[1 - \frac{b^2}{r^2}\left(1-\frac{2M}{r}\right)\right]^{-\frac{1}{2}} - 1 \right\}$$

Substitution ad hoc

$$z = 1 - \cos\alpha$$

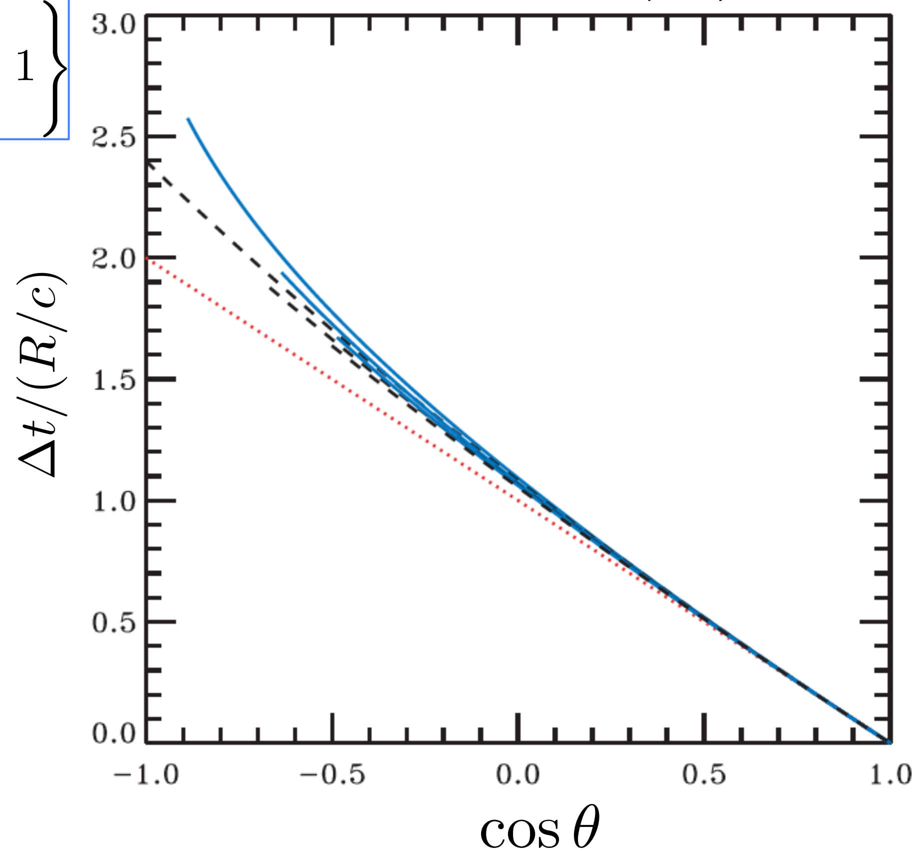
in

$$b = \frac{r\sin\alpha}{\sqrt{1-\frac{2M}{r}}} \quad (*)$$

considering α small

Mathematical algebra

Poutanen & Beloborodov (2006)



Poutanen & Beloborodov (2006)

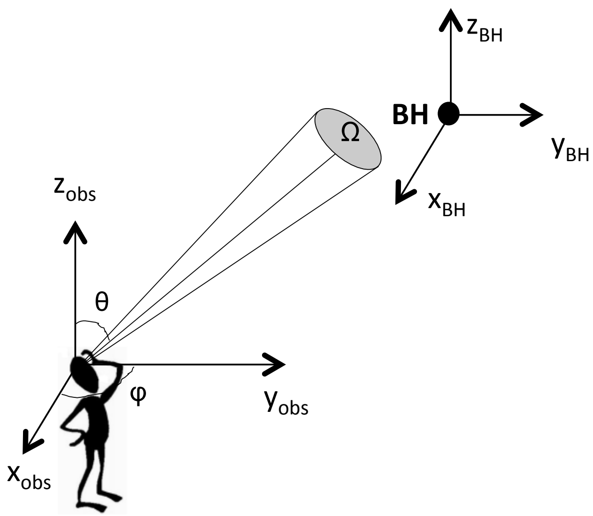
$$\frac{\Delta t}{R} = y\left[1 + \frac{uy}{8} + \frac{uy^2}{24} - \frac{u^2y^2}{112}\right]$$

where $u = \frac{r_s}{r}$ $\quad y = \frac{(1-\cos\alpha)}{(1-u)}$

2.3 Solid angle

(*) Misner, Thorne & Wheeler - Gravitation

$$d\Omega = \overbrace{\frac{\cos i}{D^2 \sin^2\theta \left(1 - \frac{2M}{R}\right)}}^{par1} \overbrace{\frac{\sin^2\alpha}{\cos\alpha}}^{par2} \overbrace{\left\{ \int_R^\infty \frac{dr}{r^2} \left[1 - \frac{b^2}{r^2}\left(1 - \frac{2M}{r}\right)\right]^{-\frac{3}{2}} \right\}}^{I}{}^{-1}$$

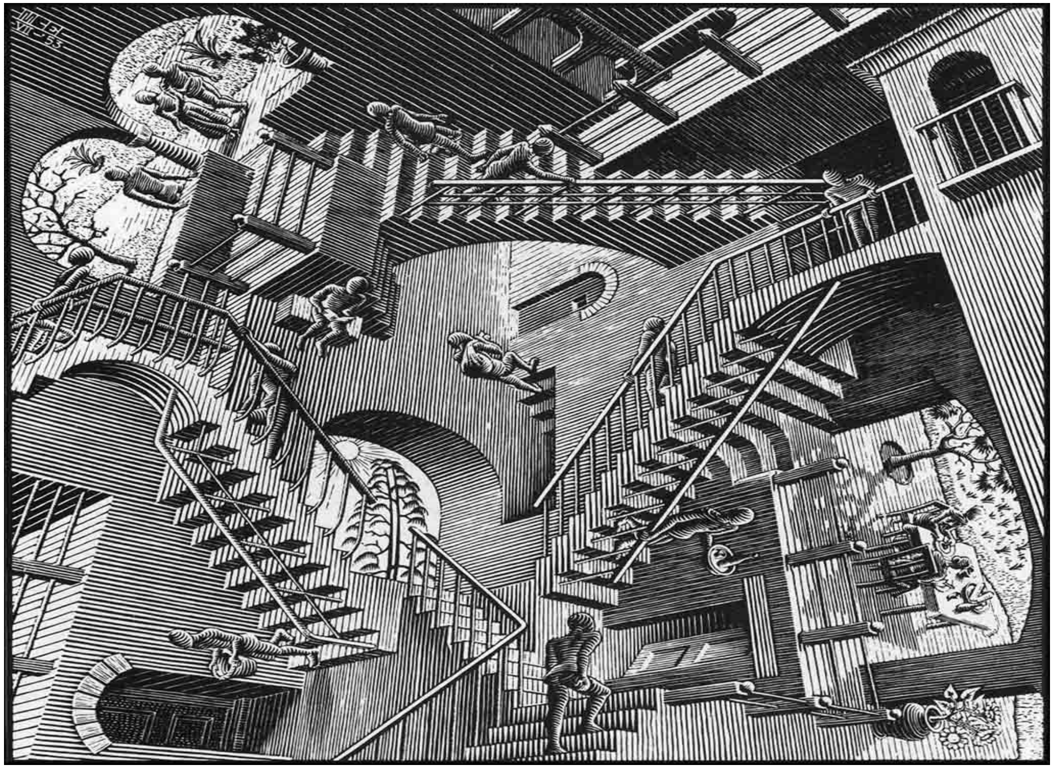


?

no approximated
equation found so far

3 CHAPTER

MATHEMATICAL METHOD



M. C. Escher - Relativity (1953)

3.1 Method

- Approximate the elliptic integrals with polynomial equations

$$\int f(x)dx \Longrightarrow P(x)$$

elliptic function *polynomial*

- Get rid of the square root present in the integral

general function

$$\sin\alpha = g(z) \Longrightarrow z = z(\alpha)$$

- We assume that α is small, so we have:

$$g(z) = \sin\alpha \approx \alpha \approx 0 \Longrightarrow$$ expand in *Taylor series* the integrand f(x)

3.2 Light bending

$$\Psi = \int_R^\infty \frac{dr}{r^2}\left[\frac{1}{b^2} - \frac{1}{r^2}\left(1 - \frac{r_g}{r}\right)\right]^{-\frac{1}{2}} = b\int_R^\infty \frac{dr}{r^2}\left[1 - \frac{b^2}{r^2}\left(1 - \frac{uR}{r}\right)\right]^{-\frac{1}{2}} =$$

$$b = \frac{R\,\mathrm{sen}\,\vartheta}{\sqrt{1-\frac{r_g}{R}}}, \quad u = \frac{r_g}{R}, \quad \mathrm{sen}\,\vartheta = g(z) \qquad \Rightarrow \qquad b = \frac{R\,g(z)}{\sqrt{1-u}}$$

$$= b\int_R^\infty \frac{dr}{r^2}\left[1 - \underbrace{\frac{R^2 g^2(z)}{r^2(1-u)}\left(1 - \frac{uR}{r}\right)}_{y}\right]^{-\frac{1}{2}} \approx \qquad (1-y)^{-\frac{1}{2}} \simeq 1 + \frac{y}{2} + \frac{3}{8}y^2 + \frac{5}{16}y^3$$

$$\approx b\int_R^\infty \frac{dr}{r^2}\left[1 + \underbrace{\frac{R^2 g^2(z)}{2r^2(1-u)}\left(1 - \frac{uR}{r}\right)}_{\text{LINEARE}} + \underbrace{\frac{3}{8}\left[\frac{R^2}{r^2}\frac{g^2(z)}{(1-u)}\left(1 - \frac{uR}{r}\right)\right]^2}_{\text{QUADRATICO}} + \underbrace{\frac{5}{16}\left[\frac{R^2}{r^2}\frac{g^2(z)}{(1-u)}\left(1 - \frac{uR}{r}\right)\right]^3}_{\text{CUBICO}}\right] =$$

$$= b\int_R^\infty dr\left[\frac{1}{r^2} + \frac{R^2 g^2(z)}{2r^4(1-u)} - \frac{R^3 g^2(z)\,u}{2r^5(1-u)} + \frac{3}{8}\left[\frac{R^4}{r^6}\frac{g^4(z)}{(1-u)^2}\left(1 + \frac{u^2R^2}{r^2} - \frac{2uR}{r}\right)\right] + \frac{5}{16}\left[\frac{R^6}{r^8}\frac{g^6(z)}{(1-u)^3}\left(1 - \frac{u^3R^3}{r^3} - \frac{3uR}{r} + \frac{3u^2R^2}{r^2}\right)\right]\right] =$$

$$= b\int_R^\infty \left[\frac{1}{r^2} + \frac{R^2 g^2(z)}{2r^4(1-u)} - \frac{R^3 g^2(z)\,u}{2r^5(1-u)} + \frac{3}{8}\frac{R^4}{r^6}\frac{g^4(z)}{(1-u)^2} + \frac{3}{8}\frac{R^6}{r^8}\frac{g^4(z)\,u^2}{(1-u)^2} - \frac{3}{4}\frac{R^5}{r^7}\frac{g^4(z)}{(1-u)^2}u + \right.$$

$$\left. + \frac{5}{16}\frac{R^6}{r^8}\frac{g^6(z)}{(1-u)^3} - \frac{5}{16}\frac{R^9}{r^{11}}\frac{g^6(z)\,u^3}{(1-u)^3} - \frac{15}{16}\frac{R^7}{r^9}\frac{g^6(z)}{(1-u)^3}u + \frac{15}{16}\frac{R^8}{r^{10}}\frac{g^6(z)}{(1-u)^3}u^2\right] dr =$$

$$= \frac{b}{R}\left[1 + \frac{g^2(z)}{6(1-u)} - \frac{g^2(z)\,u}{8(1-u)} + \frac{3}{40}\frac{g^4(z)}{(1-u)^2} + \frac{3}{56}\frac{g^4(z)\,u^2}{(1-u)^2} - \frac{1}{8}\frac{g^4(z)}{(1-u)^2}u + \right.$$

$$\left. + \frac{5}{112}\frac{g^6(z)}{(1-u)^3} - \frac{1}{32}\frac{g^6(z)}{(1-u)^3}u^3 - \frac{15}{128}\frac{g^6(z)}{(1-u)^3}u + \frac{5}{48}\frac{g^6(z)}{(1-u)^3}u^2\right] =$$

- Having even powers of g(z) and researching a complete polynomial function, we choose:

$$g(z) = \sqrt{Az^2 + Bz}$$

$$= \sqrt{\frac{Az^2+Bz}{1-\mu}} \Bigg[1 + \frac{Az^2+Bz}{6(1-\mu)} - \frac{(Az^2+Bz)\mu}{8(1-\mu)} + \frac{3}{40}\frac{(Az^2+Bz)^2}{(1-\mu)^2} + \frac{3}{56}\frac{(Az^2+Bz)^2\mu^2}{(1-\mu)^2} - \frac{1}{8}\frac{(Az^2+Bz)^2\mu}{(1-\mu)^2} +$$

$$+ \frac{5}{112}\frac{(Az^2+Bz)^3}{(1-\mu)^3} - \frac{1}{32}\frac{(Az^2+Bz)^3}{(1-\mu)^3}\mu^3 - \frac{15}{128}\frac{(Az^2+Bz)^3}{(1-\mu)^3}\mu + \frac{5}{48}\frac{(Az^2+Bz)^3}{(1-\mu)^3}\mu^2 \Bigg] \approx$$

TERZO ORDINE IN z

$$\approx \sqrt{\frac{Az^2+Bz}{1-\mu}} \Bigg[1 + \frac{Az^2+Bz}{6(1-\mu)} - \frac{(Az^2+Bz)\mu}{8(1-\mu)} + \frac{3}{40}\frac{(B^2z^2+2ABz^3)}{(1-\mu)^2} + \frac{3}{56}\frac{(B^2z^2+2ABz^3)\mu^2}{(1-\mu)^2} -$$

$$- \frac{1}{8}\frac{(B^2z^2+2ABz^3)\mu}{(1-\mu)^2} + \frac{5}{112}\frac{B^3z^3}{(1-\mu)^3} - \frac{1}{32}\frac{B^3z^3}{(1-\mu)^3}\mu^3 - \frac{15}{128}\frac{B^3z^3\mu}{(1-\mu)^3} + \frac{5}{48}\frac{B^3z^3\mu^2}{(1-\mu)^3} \Bigg] =$$

$$= \sqrt{\frac{Az^2+Bz}{1-\mu}} \Bigg[1 + \left(\frac{B}{6(1-\mu)} - \frac{B\mu}{8(1-\mu)} \right) z + \left(\frac{A}{6(1-\mu)} - \frac{A\mu}{8(1-\mu)} + \frac{3B^2}{40(1-\mu)^2} + \frac{3B^2\mu^2}{56(1-\mu)^2} - \frac{1}{8}\frac{B^2\mu}{(1-\mu)^2} \right) z^2 +$$

$$+ \left(\ldots\ldots \right) z^3 \Bigg]$$

- To find the approximation we compare the approximation with the original form in a particular computing convenience case. We choose u=0 and R=1, so we have:

$$\Psi = b\int_R^{\infty}\frac{dr}{r^2}\left[1-\frac{R^2\sin^2\alpha}{r^2(1-u)}\left(1-\frac{uR}{r}\right)\right]^{-\frac{1}{2}} \overset{u=0,\,R=1}{=} b\int_1^{\infty}\frac{dr}{r^2}\left[1-\frac{\sin^2\alpha}{r^2}\right]^{-\frac{1}{2}} = b\int_1^{\infty}\frac{r\,dr}{r^2\sqrt{r^2-\sin^2\alpha}} \overset{x^2=r^2-\sin^2\alpha,\; x\,dx=r\,dr}{=}$$

$$= b\int_{\cos\alpha}^{\infty}\frac{x\,dx}{x\,(x^2+\sin^2\alpha)} = \frac{\cancel{b}}{\sin\alpha}\int_{\cos\alpha}^{\infty}\frac{dx}{\sin\alpha\left[\frac{x^2}{\sin^2\alpha}+1\right]} = \left[\operatorname{arctg}\frac{x}{\sin\alpha}\right]_{\cos\alpha}^{\infty} = \frac{\pi}{2}-\operatorname{arctg}\operatorname{cotg}\alpha = \alpha$$

$$\operatorname{cotg}\alpha = \operatorname{tg}\left(\frac{\pi}{2}-\alpha\right), \qquad \operatorname{arctg}\left(\frac{1}{x}\right)+\operatorname{arctg}x=\frac{\pi}{2}$$

- We have to get rid of the square root with an even trigonometric function, so we have:

(GIUSTIFICARE)

3° ORDINE IN Z IN PARENTESI

$$1-\cos\psi \approx \frac{\psi^2}{2}-\frac{\psi^4}{24} \approx \left[\frac{Az^2+Bz}{2(1-u)}\right]\left[1+C^2z^2+2Cz+2Dz^2+2CDz^3\right] -$$

NON PROPRIO ESATTO MA RESTA 1

$$-\frac{1}{24}\left[\frac{B^2z^2+2ABz^3}{(1-u)^2}\right]\left[1+C^2z^2+2Cz+2Dz^2+2CDz^3\right] \approx$$

APPROSSIMAZIONE AL 3° ORDINE IN Z

$$\approx \frac{Bz}{2(1-u)}\left[1+2Cz+(C^2+2D)z^2\right]+\frac{Az^2}{2(1-u)}\left[1+2Cz\right]-$$

$$-\frac{1}{12}\frac{ABz^3}{(1-u)^2}-\frac{1}{24}\frac{B^2z^2}{(1-u)^2}\left[1+2Cz\right] =$$

sostituisco il valore di C $\quad C = \frac{B}{6(1-u)} - \frac{Bu}{8(1-u)}$

$$= \frac{Bz}{2(1-u)} + \left[\frac{BC}{(1-u)} + \frac{A}{2(1-u)} - \frac{B^2}{24(1-u)^2} \right] z^2 + O(x^3) =$$

$$= \frac{Bz}{2(1-u)} + \left[\frac{B^2}{6(1-u)^2} - \frac{B^2 u}{8(1-u)} + \frac{A}{2(1-u)} - \frac{B^2}{24(1-u)^2} \right] z^2 + O(x^3) =$$

$$= \frac{Bz}{2(1-u)} + O(x^3) \qquad \text{where} \quad A = -\left(\frac{B}{2}\right)^2$$

- Testing in the particular case (u=0 and R=1) we have: $1 - \cos\alpha = \dfrac{Bz}{2}$

- Therefore choosing B = 2 and A = -1, it implies that z = 1-cosα. The final approximation is:

$$(1 - \cos\theta)(1 - u) = 1 - \cos\alpha$$

3.3 Time delay

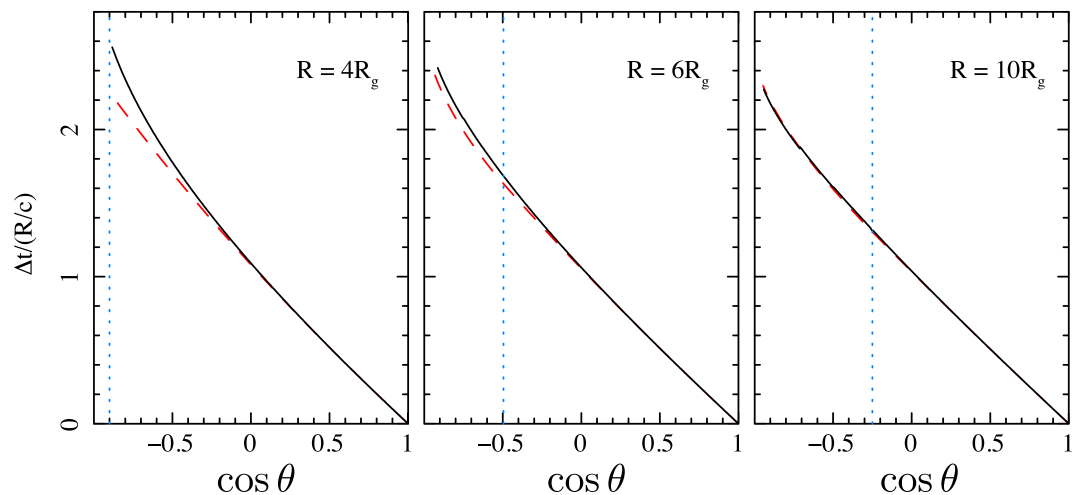
$$\Delta t = \int_R^\infty \frac{dr}{(1-\frac{2M}{r})}\left\{\left[1-\frac{b^2}{r^2}\left(1-\frac{2M}{r}\right)\right]^{-\frac{1}{2}}-1\right\}$$

Substitution obtained rigorously with the mathematical method

$$z = 1 - \cos\alpha$$

Performing the same calculations made in the case of the light bending

De Falco & Falanga (2015)



$$\frac{\Delta t}{R} = y\left[1+\frac{uy}{8}+\frac{uy^2}{24}-\frac{u^2y^2}{112}\right]$$ where $u = \frac{r_s}{r}$ $\quad y = \frac{(1-\cos\alpha)}{(1-u)}$

3.4 Solid angle

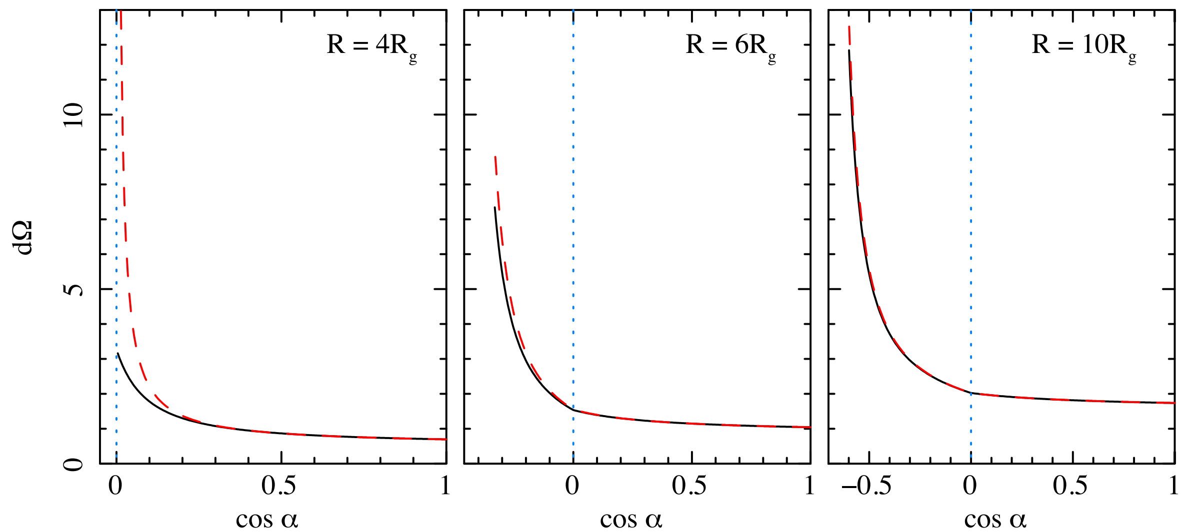
$$d\Omega = \frac{\cos i}{D^2 \sin^2\theta \left(1 - \frac{2M}{R}\right)} \frac{\sin^2\alpha}{\cos\alpha} \left\{ \int_R^\infty \frac{dr}{r^2} \left[1 - \frac{b^2}{r^2}\left(1 - \frac{2M}{r}\right)\right]^{-\frac{3}{2}} \right\}^{-1}$$

Substitution obtained rigorously with the mathematical method

$$z = 1 - \cos\alpha$$

Performing the same calculations made in the case of the light bending

De Falco & Falanga (2015)



$$d\Omega = cons1 \cdot R \cdot \left[2z + (1 - 2C)\, z^2 + \left(1 - C + 2C^2 - 2D\right) z^3\right]$$

where $C = \frac{4 - 3u}{1 - u}$ $\quad D = \frac{39u^2 - 91u + 56}{56(1-u)^2}$

4 CHAPTER

APPLICATIONS

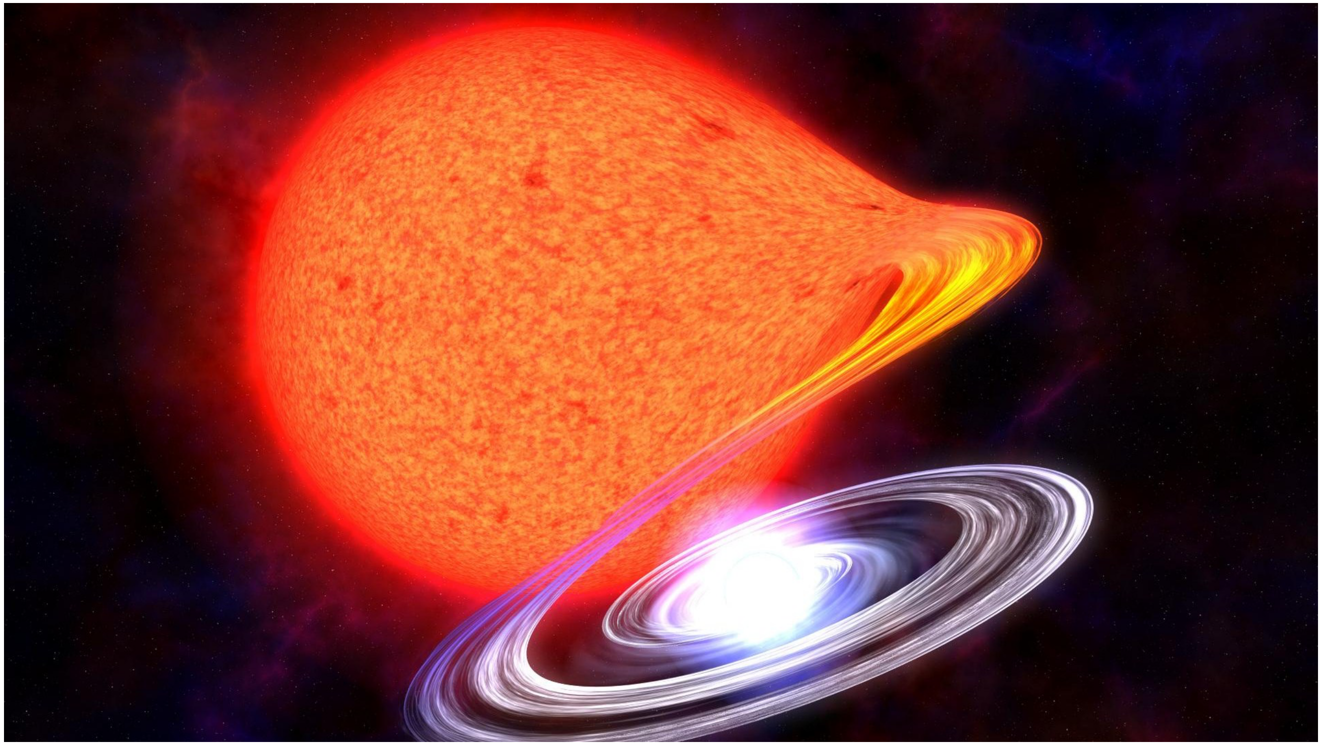
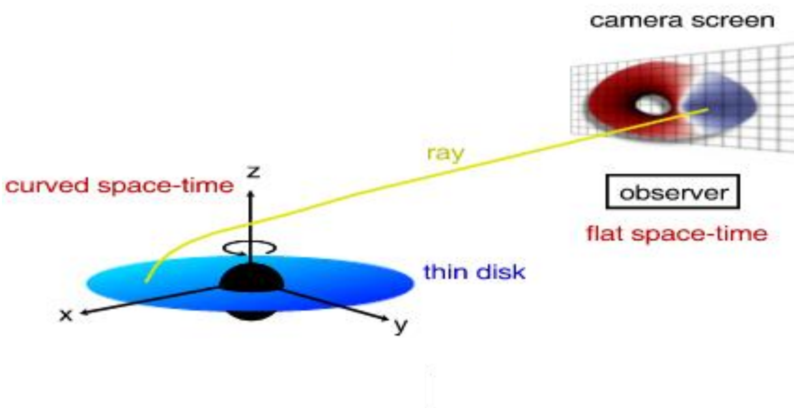


Image of an accretion disk's formation in a binary system

4.1 Iron line profile

- Flux formula $$F = \int_R \int_\varphi g^4 d\Omega$$



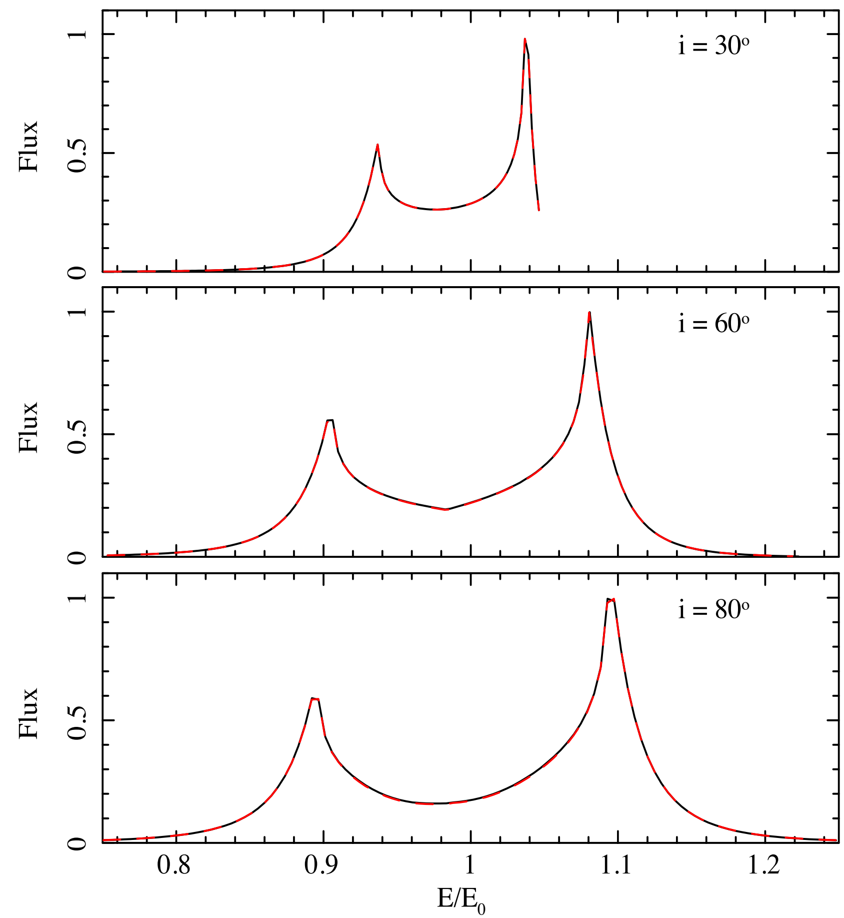
- Numerical code

40 millions of points

Original: more than 60 min

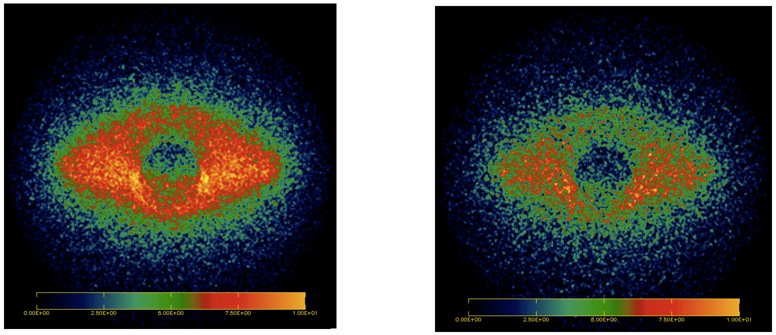
Approximate: less than 1 min

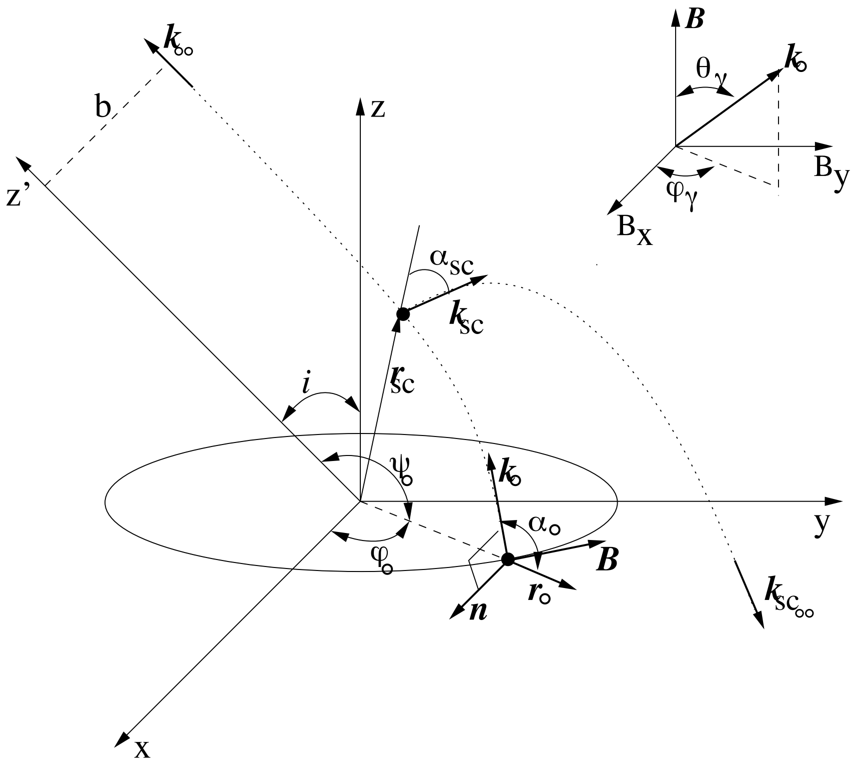
De Falco & Falanga (2015)



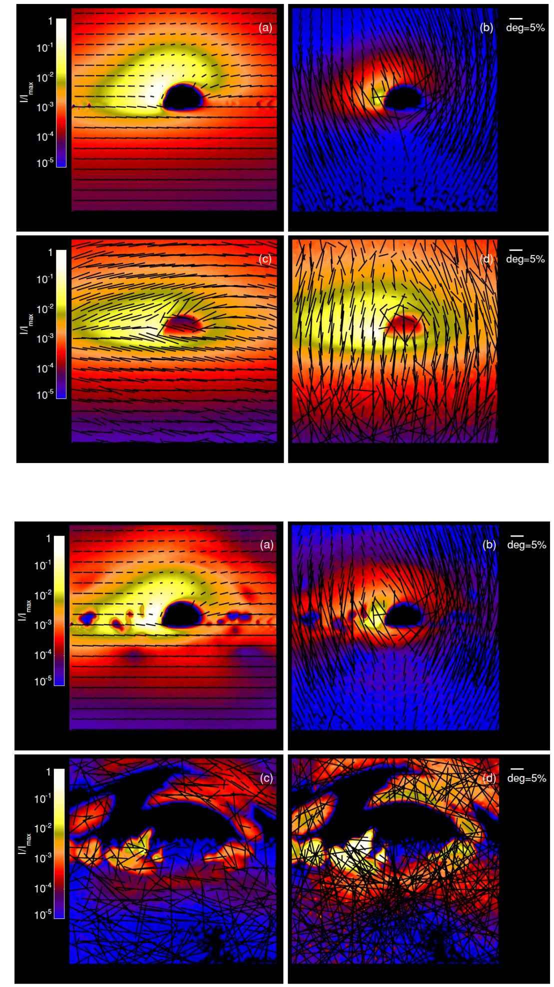
4.2 Polarization

Melia, Falanga & Goldwurm 2011





Schnittman & Krolik 2010



5 CHAPTER

CONCLUSIONS

✓ PRESENT WORKS

- ☒ PUBLISH THIS WORK
- ☒ OPTIMIZE CODES TO CALCULATE THE FLUX AND THE POLARIZATION

✗ FUTURE PROJECTS

- ☐ EXTENSION TO THE KERR METRIC
- ☐ DEVELOP FAST NUMERICAL CODES IN THE KERR METRIC

THANK YOU FOR YOUR ATTENTION!