On homogeneous and isotropic Universe

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Notation

$(\mathbb{M}, g)$ - geodesically complete (pseudo)Riemannian manifold (space-time)

$g_{\alpha\beta}(t,x), \quad \operatorname{sign} g_{\alpha\beta} = (+---)$ - Lorentzian signature metric

$\{x^\alpha\} = \{t, x^\mu\}, \quad \alpha = 0,1,2,3; \quad \mu = 1,2,3$ - coordinates

Theorem. Let a four dimensional space-time be the topological product $\mathbb{M} = \mathbb{R} \times \mathbb{S}$ where $t \in \mathbb{R}$ is a time coordinate $x \in \mathbb{S}$ is a three dimensional space of constant curvature. We suppose that sufficiently smooth homogeneous and isotropic metric of Lorentzian signature is given on it. Then there is such coordinate system t, x^μ in some neighbourhood of each point where metric has the form

$$ds^2 = dt^2 + a^2 \hat{g}_{\mu\nu} dx^\mu dx^\nu, \tag{1}$$

- the Friedmann metric

where $a(t) > 0$ is an arbitrary function (scale factor) and $\hat{g}_{\mu\nu}(x)$ is a negative definite three dimensional constant curvature metric on $\mathbb{S}$ depending only on space coordinates.

Brief History

A. Friedmann. *Zs.Phys.* 10(1922)377; ibid. 21(1924)326.

The first papers were metric (1) was used in cosmology. No theorem.

G. Lemaitre. *Ann. Soc. Sci. (Bruxelles)* 47A(1927)49; ibid. A53(1933)51.

Metric (1) was used in several cosmological models. No theorem.

H. P. Robertson. *Proc. Nat. Acad. Sci.* 15(1929)822;
Rev. Mod. Phys. 5(1933)62; *Ap. J*. 82(1935)284.

In the first two papers, the theorem was formulated but not proved.
Instead, he referred to: D. Hilbert. *Mathematische Annalen,* 15(1924)1*.*

The first part of the proof is given in a general case.

G. Fubini. *Annali di Matematica, Pura Appl.[3],* 9 (1904) 33.

The second part of the proof is given in one way.

In the third paper, metric (1) was obtained in a different way by considering a set of observers with given properties.

R. C. Tolman. *Proc. Nat. Acad. Sci*. 16(1930) 320, ibid. 409, ibid. 511;

Metric (1) was obtained from different assumptions. In particular he assumed spherical symmetry and used Einstein's equations.

A. G. Walker. *Proc. London Math. Soc. Ser.2* 42(1936) 90.

The theorem was proved in one way.

For proof, see, for example, S. Weinberg. *Gravitation and cosmology.*1972

Constant curvature space $\mathbb{S}$

$$\hat{K} = -1, 0, 1$$ - Gaussian curvature

Spherical coordinates on $\mathbb{S}$:

$$\hat{g}_{\mu\nu} dx^\mu dx^\nu = \begin{cases} d\chi^2 + \sin^2\chi\left(d\vartheta^2 + \sin^2\vartheta d\varphi^2\right), & \hat{K} = 1, \\ d\chi^2 + \chi^2\left(d\vartheta^2 + \sin^2\vartheta d\varphi^2\right), & \hat{K} = 0, \\ d\chi^2 + \mathrm{sh}^2\chi\left(d\vartheta^2 + \sin^2\vartheta d\varphi^2\right), & \hat{K} = -1 \end{cases}$$

$$ds^2 = dt^2 + a^2 \hat{g}_{\mu\nu} dx^\mu dx^\nu$$

Stereographic coordinates on $\mathbb{S}$:
$$\hat{g}_{\mu\nu} dx^\mu dx^\nu = \frac{\eta_{\mu\nu} dx^\mu dx^\nu}{\left(1 + b_0 x^2\right)^2}$$

where: $\eta_{\mu\nu} := \mathrm{diag}(---)$

$$x^2 := \eta_{\mu\nu} x^\mu x^\nu \leq 0$$

$$\hat{K} := -12 b_0$$

$$x \in \mathbb{R}^3, \quad \hat{K} = 1,$$

$$x \in \mathbb{R}^3, \quad \hat{K} = 0,$$

$$x \in \mathbb{B}_r^3, \quad \hat{K} = -1,$$

Coordinate transformation

$$g_{\alpha\beta} = \begin{pmatrix} 1 & 0 \\ 0 & \dfrac{a^2 \eta_{\mu\nu}}{\left(1 + b_0 x^2\right)^2} \end{pmatrix}$$

- Friedmann metric in stereographic coordinates

Coordinate transformation $x^\mu \mapsto \dfrac{x^\mu}{a}$

$$g_{\alpha\beta} = \begin{pmatrix} 1 + \dfrac{\dot{b}^2 x^2}{4b^2\left(1 + bx^2\right)^2} & \dfrac{\dot{b} x_\nu}{2b\left(1 + bx^2\right)^2} \\ \dfrac{\dot{b} x_\mu}{2b\left(1 + bx^2\right)^2} & \dfrac{\eta_{\mu\nu}}{\left(1 + b\ x^2\right)^2} \end{pmatrix}$$

- Friedmann metric after coordinate transformation

where $b(t) := \dfrac{b_0}{a^2(t)}$ - function on time

$\dot{b} := \dfrac{db}{dt}$ - time derivative

Homogeneous and isotropic metric ?

$$g_{\alpha\beta} = \begin{pmatrix} 1 & 0 \\ 0 & \dfrac{a^2 \eta_{\mu\nu}}{\left(1+bx^2\right)^2} \end{pmatrix}$$

$a(t) > 0$

$b(t)$

- two arbitrary functions of time. We cannot remove one of these functions by coordinate transformation without appearing off diagonal terms.

Changing topology of space in time: $b(t) = 0$

$$R_{\alpha\beta} - \frac{1}{2} g_{\alpha\beta} R = -\frac{1}{2} T_{\alpha\beta}$$ - Einstein's equations

$$T_{\alpha\beta} = \begin{pmatrix} E & 0 \\ 0 & -Ph_{\mu\nu} \end{pmatrix}$$ - homogeneous and isotropic energy-momentum tensor

$$h_{\mu\nu}(t,x) := \frac{a^2 \eta_{\mu\nu}}{\left(1+bx^2\right)^2}$$

$$R_{0\mu} = -\frac{4\dot{b}x_\mu}{\left(1+bx^2\right)^2} = 0 \Longrightarrow b = \text{const}$$

Definition. Diffeomorphism $i: \; x \mapsto x'$ is called *isometry* if it preserves metric

$$g(x) = i^* g(x')$$

where i^* is the pullback of $\boldsymbol{i}$.

In coordinates: $$g_{\alpha\beta}(x) = \frac{\partial x'^{\gamma}}{\partial x^{\alpha}} \frac{\partial x'^{\delta}}{\partial x^{\beta}} g_{\gamma\delta}(x')$$

Infinitesimal isometry is generated by Killing vector field $K = K^{\alpha}\partial_{\alpha}$

$$x^{\alpha} \mapsto x'^{\alpha} = x^{\alpha} + \varepsilon K^{\alpha}, \qquad \varepsilon \ll 1$$

$$L_K g = 0 \quad \Leftrightarrow \quad \nabla_{\alpha} K_{\beta} + \nabla_{\beta} K_{\alpha} = 0$$ where $K_{\alpha} := g_{\alpha\beta} K^{\beta}$

$$\nabla_{\alpha} K_{\beta} := \partial_{\alpha} K_{\beta} - \Gamma_{\alpha\beta}{}^{\gamma} K_{\gamma}$$

Definition. A manifold $\mathbb{M}$ is called *homogeneous at a point* $p \in \mathbb{M}$ if there are infinitesimal isometries which map this point into any other point from some neighborhood $\mathbb{U}_p$. A manifold $\mathbb{M}$ is called *homogeneous* if it is homogeneous at each point $p \in \mathbb{M}$.

Definition. A manifold $\mathbb{M}$ is called *isotropic at a point* $p \in \mathbb{M}$ if there are such infinitesimal isometries with the Killing 1-forms $K = dx^{\alpha} K_{\alpha} \in \Lambda_1(p)$, preserving this point, that the external derivative $dK \in \Lambda_2(p)$ take any value in the space of two forms $\Lambda_2(p)$. A manifold $\mathbb{M}$ is called *isotropic* if it is isotropic at each point $p \in \mathbb{M}$.

Theorem. Any isotropic (pseudo-)Riemannian manifold $(\mathbb{M}, g)$ is homogeneous.

Theorem. Maximal dimension of the Lie algebra of infinitesimal isometries is equal to $n(n+1)/2$. If dimension of Lie algebra is maximal, then this manifold is isotropic and homogeneous, and it is a space of constant curvature.

See, for example, S. Weinberg. *Gravitation and cosmology.* 1972

$$R_{\alpha\beta\gamma\delta} = \frac{R}{n(n-1)}\left(g_{\alpha\gamma}g_{\beta\delta} - g_{\alpha\delta}g_{\beta\gamma}\right), \qquad R = \text{const}$$

- constant curvature space

Manifolds with maximally symmetric submanifolds

$\mathbb{M} = \mathbb{R} \times \mathbb{S}$ - space-time

$t \in \mathbb{R}$ - time

$x \in \mathbb{S}$ - constant curvature space slices

$n = 4$ $\quad K_i = K_i^{\mu}(x)\partial_{\mu}, \quad i = 1, \ldots, 6$ - Killing vectors on space slices $\mathbb{S}$

$$g_{\alpha\beta} = \begin{pmatrix} g_{00}(t,x) & g_{0\nu}(t,x) \\ g_{\mu 0}(t,x) & h_{\mu\nu}(t,x) \end{pmatrix}$$

$h_{\mu\nu}(t,x)$ - constant curvature metric on $\mathbb{S}$ depending on t as a parameter

Infinitesimal isometry: $t \mapsto t' = t$

$$x^{\mu} \mapsto x'^{\mu} = x^{\mu} + \varepsilon K^{\mu}, \quad \varepsilon \ll 1 \qquad (*)$$

$\mathbb{T}(\mathbb{S}) \ni \left\{ K^{\mu}(x) \right\} \mapsto \left\{ 0, K^{\mu}(t,x) \right\} \in \mathbb{T}(\mathbb{M})$ - extension from $\mathbb{S}$ to $\mathbb{M}$

Theorem. Let metric on $\mathbb{M} = \mathbb{R} \times \mathbb{S}$ be invariant with respect to transformations $(*)$, then there is a coordinate system in which the metric is block diagonal

$$ds^2 = dt^2 + h_{\mu\nu} dx^\mu dx^\nu, \qquad (**)$$

where $h_{\mu\nu}(t, x)$ is the metric of constant curvature on $\mathbb{S}$ for all $t \in \mathbb{R}$ depending on time t as a parameter. In this coordinate system the Killing vector fields do not depend on time $K = K(x)$.

Proof.

D. Hilbert. *Mathematische Annalen,* 15(1924)1
Eisenhart, *Continuous Groups of Transformations,* (1933)

Fix one space slice $t = \text{const}$ with coordinates x^μ, $\mu = 1, 2, 3.$

Take orthogonal (timelike) vector field n on this slice.

Construct geodesics going through each point of the space slice along n.

Let s be the canonical parameter along geodesics.

Choose the coordinate system (s, x^μ) on $\mathbb{M}$.

Then $g_{00} = 1$ by construction.

Geodesic equations ⟹ $\partial_0 g_{0\mu} = 0$ with initial condition $g_{0\mu}(0, x) = 0$

⟹ unique solution $g_{0\mu}(s, x) = 0$ ⟹ Metric has form $(**)$

For metric $(**)$ the (00) and $(\mu\nu)$ components of the Killing equations are identically satisfied.

The (0μ) component ⟹ $\partial_s K^\mu = 0$ ⟹ $K^\mu = K^\mu(x)$

Theorem. Under the conditions of the previous theorem metric on space slices has the form

$$h_{\mu\nu}(t,x) = a^2(t)\hat{g}_{\mu\nu}(x)$$

where $a(t) > 0$ is the scale factor and $\hat{g}_{\mu\nu}(x)$ is a constant curvature metric on space slices $\mathbb{S}$, depending only on x.

Proof.

Killing equations:

$$\nabla_\mu K_\nu + \nabla_\nu K_\mu = 0 \quad \Leftrightarrow \quad h_{\mu\rho}\partial_\nu K^\rho + h_{\nu\rho}\partial_\mu K^\rho + K^\rho \partial_\rho h_{\mu\nu} = 0$$

Previous theorem: $K = K(x)$

$$\dot{h}_{\mu\rho}\partial_\nu K^\rho + \dot{h}_{\nu\rho}\partial_\mu K^\rho + K^\rho \partial_\rho \dot{h}_{\mu\nu} = 0 \quad \Leftrightarrow \quad L_K \dot{h}_{\mu\nu} = 0$$

$\dot{h}_{\mu\nu} = f(t)h_{\mu\nu}$ - the most general second rank tensor

$$t \mapsto t' \qquad dt' = f(t)dt$$

$$\frac{dh_{\mu\nu}}{dt'} = h_{\mu\nu} \quad \Leftrightarrow \quad h_{\mu\nu}(t',x) = \mathrm{C}e^{t'}\hat{g}_{\mu\nu}(x), \quad C = \mathrm{const}$$

Two functions ?

$$g_{\alpha\beta} = \begin{pmatrix} 1 & 0 \\ 0 & \dfrac{a^2\eta_{\mu\nu}}{\left(1+bx^2\right)^2} \end{pmatrix}$$

$a(t) > 0$

$b(t)$

- two arbitrary functions on time. We cannot remove one of these functions by coordinate transformation without appearing off diagonal terms.

$\nabla_\mu K_\nu + \nabla_\nu K_\mu = 0$ - Killing equation is satisfied

Killing vector in stereographic coordinates does depend on time ⇒

The metric $g_{\alpha\beta}$ is not homogeneous and isotropic in four dimensions.

$R^{(4)} = R^{(4)}(t, x^2)$ is not homogeneous.

For homogeneous and isotropic matter $b = \text{const}$ due to Einstein's equations

Conclusion

Theorem. Let space-time be the topological product $\mathbb{M} = \mathbb{R} \times \mathbb{S}$. Let it be homogeneous and isotropic. Then, up to a coordinate transformation, the most general metric is

$$ds^2 = dt^2 + a^2 \hat{g}_{\mu\nu} dx^\mu dx^\nu ,$$

where $\hat{g}_{\mu\nu}(x)$ is the metric of constant curvature on $\mathbb{S}$ and $a(t) > 0$ is a scale factor. In this coordinate system the Killing vector fields do not depend on time

$$K = K(x)$$

Definition. A space-time $\mathbb{M} = \mathbb{R} \times \mathbb{S}$ is called homogeneous and isotropic if

1) every section of constant time is a three dimensional space of constant curvature,
2) extrinsic curvature of a submanifold $\mathbb{S} \to \mathbb{M}$ is homogeneous and isotropic.

Extrinsic curvature $$K_{\mu\nu} = -\frac{1}{2}\frac{d}{dt} h_{\mu\nu}(t, x)$$