Consistent massive graviton on general backgrounds

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Plan

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Linearised field equations around a background solution

Application to the constraint analysis

Results in different cases

Motivations

- Have a better understanding of massive spin-2 fields.
- Explain the accelerated expansion of the Universe by a modification of GR at long distance.

Motivations and history of massive gravity

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Massive gravity : a brief historical review

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$$S_{h,m} = -\frac{1}{2}\bar{M}_h^2 \int d^4x \; h_{\mu\nu}\left[\mathcal{E}^{\mu\nu\rho\sigma} + \frac{\bar{m}^2}{2}\left(\eta^{\rho\mu}\eta^{\sigma\nu} - \eta^{\mu\nu}\eta^{\rho\sigma}\right)\right]h_{\rho\sigma}$$

$$\mathcal{E}_{\mu\nu}{}^{\rho\sigma}h_{\rho\sigma} \equiv -\tfrac{1}{2}\left[\delta^\rho_\mu\delta^\sigma_\nu\Box + \eta^{\rho\sigma}\partial_\mu\partial_\nu - \delta^\rho_\mu\partial^\sigma\partial_\nu - \delta^\rho_\nu\partial^\sigma\partial_\mu - \eta_{\mu\nu}\eta^{\rho\sigma}\Box + \eta_{\mu\nu}\partial^\rho\partial^\sigma\right]h_{\rho\sigma}$$

$$\boxed{\delta\bar{E}_{\mu\nu} \equiv \mathcal{E}_{\mu\nu}{}^{\rho\sigma}h_{\rho\sigma} + \frac{\bar{m}^2}{2}\Big(h_{\mu\nu} - h\,\eta_{\mu\nu}\Big) = 0}$$

- Field eqs. for a massive graviton that has 5 degrees of freedom.

  - $\partial^\nu\delta\bar{E}_{\mu\nu} \implies$ 4 vector constraints : $\partial^\mu h_{\mu\nu} - \partial_\nu h = 0$.
  - Taking another divergence : $2\partial^\mu\partial^\nu\delta\bar{E}_{\mu\nu} + \bar{m}^2\eta^{\mu\nu}\delta\bar{E}_{\mu\nu} = -\frac{3}{2}\bar{m}^4 h$.
  - Scalar constraint $\boxed{h = 0.}$

- **It is the only linear massive gravity theory free of ghost.**
- **But it needs to be generalized to a non-linear theory.**

Motivations and history of massive gravity

Massive gravity : a brief historical review

- Fierz-Pauli linear massive gravity theory (1939),
- van Dam, Veltman and Zakharov (vDVZ) discontinuity (1970): FP does not recover GR in the massless limit,
- Vainshtein mechanism (1972): have to take into account the non-linearities,
- Boulware Deser (BD) ghost (1972): a ghost-like 6th dof reappears in any non-linear massive gravity theory,
- de Rham, Gabadadze and Tolley (dRGT) theory (2011): non-linear theory free of the BD ghost.

The dRGT massive gravity theory [de Rham, Gabadadze, Tolley, 2010]

$$\boxed{S = M_g^2 \int \mathrm{d}^4x \sqrt{|g|} \Big[R(g) - 2m^2 V(S;\beta_n) \Big],}$$

$$V(S;\beta_n) = \sum_{n=0}^{3} \beta_n e_n(S)\,,$$

- Square-root matrix $S^\mu{}_\nu = \left[\sqrt{g^{-1}f}\right]^\mu{}_\nu$,
- $e_n(S)$ elementary symmetric polynomials:

$$e_0(S) = 1\,, \quad e_1(S) = \mathrm{Tr}[S]\,, \quad e_2(S) = \frac{1}{2}\Big(\mathrm{Tr}[S]^2 - \mathrm{Tr}[S^2]\Big)\,,$$
$$e_3(S) = \frac{1}{6}\Big(\mathrm{Tr}[S]^3 - 3\mathrm{Tr}[S]\mathrm{Tr}[S^2] + 2\mathrm{Tr}[S^3]\Big)$$

- **No BD ghost.**

Bimetric theory (1) [Hassan, Rosen 2012]

$$S = M_g^2 \int \mathrm{d}^4x \Big[\sqrt{|g|} R(g) + \alpha^2 \sqrt{|f|} R(f) - 2m^2 \sqrt{|g|} V(S;\beta_n) \Big],$$

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$$e_3(S) = \frac{1}{6}\Big(\mathrm{Tr}[S]^3 - 3\mathrm{Tr}[S]\mathrm{Tr}[S^2] + 2\mathrm{Tr}[S^3]\Big)\,, \quad e_4(S) = \det(S)\,.$$

- No BD ghost.
- **Interchange symmetry:**

$$\alpha^{-1} g_{\mu\nu} \leftrightarrow \alpha f_{\mu\nu}\,, \quad \alpha^{4-n}\beta_n \leftrightarrow \alpha^n \beta_{4-n}\,.$$

Bimetric theory (2)

$$S = M_g^2 \int \mathrm{d}^4x \Big[\sqrt{|g|} R(g) + \alpha^2 \sqrt{|f|} R(f) - 2m^2 \sqrt{|g|} V(S; \beta_n) \Big],$$

$V(S;\beta_n) = \sum_{n=0}^{4} \beta_n e_n(S)$ and $S^\mu{}_\nu = [\sqrt{g^{-1}f}]^\mu{}_\nu$.

Field equations

$$\begin{cases} E_{\mu\nu} \equiv \quad \mathcal{G}_{\mu\nu} + m^2\, V_{\mu\nu} = 0\,, \\ \tilde{E}_{\mu\nu} \equiv \quad \tilde{\mathcal{G}}_{\mu\nu} + \dfrac{m^2}{\alpha^2}\, \tilde{V}_{\mu\nu} = 0\,. \end{cases}$$

$$\begin{aligned} V_{\mu\nu} &= g_{\mu\rho} \sum_{n=0}^{3} (-1)^n \beta_n \sum_{k=0}^{n} (-1)^k e_k(S) [S^{n-k}]^\rho_\nu\,, \\ \tilde{V}_{\mu\nu} &= f_{\mu\rho} \sum_{n=0}^{3} (-1)^n \beta_{4-n} \sum_{k=0}^{n} (-1)^k e_k(S^{-1}) [S^{k-n}]^\rho_\nu\,. \end{aligned}$$

Linearised field equations around a background solution

Metrics expansion: $g_{\mu\nu} \to g_{\mu\nu} + \delta g_{\mu\nu}$, $f_{\mu\nu} \to f_{\mu\nu} + \delta f_{\mu\nu}$

$$\begin{cases} \delta E_{\mu\nu} \equiv & \delta\mathcal{G}_{\mu\nu} + m^2 \delta V_{\mu\nu} = 0\,, \\ \delta\tilde{E}_{\mu\nu} \equiv & \delta\tilde{\mathcal{G}}_{\mu\nu} + \dfrac{m^2}{\alpha^2}\delta\tilde{V}_{\mu\nu} = 0 \end{cases}$$

$$\delta\mathcal{G}_{\mu\nu} = -\tfrac{1}{2}\Big[\delta^\rho_\mu\delta^\sigma_\nu\nabla^2 + g^{\rho\sigma}\nabla_\mu\nabla_\nu - \delta^\rho_\mu\nabla^\sigma\nabla_\nu - \delta^\rho_\nu\nabla^\sigma\nabla_\mu - g_{\mu\nu}g^{\rho\sigma}\nabla^2 + g_{\mu\nu}\nabla^\rho\nabla^\sigma + g_{\mu\nu}R^{\rho\sigma} - \delta^\rho_\mu\delta^\sigma_\nu R\Big]\delta g_{\rho\sigma}\,,$$

$$\delta V_{\mu\nu} = g^{\rho\sigma}V_{\sigma\nu}\delta g_{\mu\rho} - g_{\mu\rho}\sum_{n=1}^{3}(-1)^n\beta_n\sum_{k=1}^{n}(-1)^k\Bigg\{\frac{1}{2}\left[S^{n-k}\right]^\rho{}_\nu\sum_{m=1}^{k}(-1)^m e_{k-m}(S)\left[S^{m-2}\delta S^2\right]^\sigma{}_\sigma + e_{k-1}(S)\sum_{m=0}^{n-k}\left[S^m\delta S S^{n-k-m}\right]^\rho{}_\nu\Bigg\}.$$

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$$\delta V_{\mu\nu} = g^{\rho\sigma}V_{\sigma\nu}\delta g_{\mu\rho} - g_{\mu\rho}\sum_{n=1}^{3}(-1)^n\beta_n\sum_{k=1}^{n}(-1)^k\Big\{\frac{1}{2}\left[S^{n-k}\right]^\rho{}_\nu\sum_{m=1}^{k}(-1)^m e_{k-m}(S)\left[S^{m-2}\delta S^2\right]^\sigma{}_\sigma + e_{k-1}(S)\sum_{m=0}^{n-k}\left[S^m\delta S S^{n-k-m}\right]^\rho{}_\nu\Big\}\,.$$

Method 1: Variation of the matrix S

To linearised the field equations we first need to obtain the perturbed matrix S.

Sylvester equation: $AX - XB = C$

$$S^{\mu}{}_{\nu}\,(\delta S)^{\nu}{}_{\sigma} + (\delta S)^{\mu}{}_{\nu}\,S^{\nu}{}_{\sigma} = \delta[S^2]^{\mu}{}_{\sigma}\,.$$

- **Unique explicit solution for δS iff S and $-$S do not have common eigenvalues** $\iff$ $\mathbb{X} \equiv e_3\mathbb{1} + e_1 S^2$ is invertible.

$$\delta S = \frac{1}{2}\mathbb{X}^{-1}\sum_{k=1}^{4}\sum_{m=0}^{k-1}(-1)^m e_{4-k}(S) S^{k-m-2}\delta S^2 S^m\,,$$

- δS contains more than 30 terms in massive gravity (60 in bimetric gravity)!

Method 2: Redefined fluctuation variables

Redefinition of the perturbation variable

$$
\begin{aligned}
\delta g_{\mu\nu} &= \left(\delta^{\beta}_{\mu} S^{\lambda}_{\nu} + \delta^{\beta}_{\nu} S^{\lambda}_{\mu}\right) \delta g'_{\beta\lambda} \\
\delta f_{\mu\nu} &= \left(\delta^{\beta}_{\mu} [S^{-1}]^{\lambda}_{\nu} + \delta^{\beta}_{\nu} [S^{-1}]^{\lambda}_{\mu}\right) \delta f'_{\beta\lambda}
\end{aligned}
$$

- We can express all other variation of variables as a function of $\delta g'_{\beta\lambda}$ and $\delta f'_{\beta\lambda}$.

Method 2: Redefined fluctuation variables

Redefinition of the perturbation variable

$$\begin{aligned}\delta g_{\mu\nu} &= \left(\delta^\beta_\mu S^\lambda_\nu + \delta^\beta_\nu S^\lambda_\mu\right)\delta g'_{\beta\lambda}\\ \delta f_{\mu\nu} &= \left(\delta^\beta_\mu [S^{-1}]^\lambda_\nu + \delta^\beta_\nu [S^{-1}]^\lambda_\mu\right)\delta f'_{\beta\lambda}\end{aligned}$$

- We can express all other variation of variables as a function of $\delta g'_{\beta\lambda}$ and $\delta f'_{\beta\lambda}$.

Invertibility of the definitions

$$g^{-1}\delta g = S\, g^{-1}\delta g' + g^{-1}\delta g'\, S$$

- Sylvester equation: **unique solution for $g^{-1}\delta g'$ iff S and $-$S do not have common eigenvalues.**
- $\delta S = -g^{-1}\delta g' S^2 + S^{-1} g^{-1}\delta f' S^{-1}$: only two terms in δS.

Search for a scalar constraint

Counting the degrees of freedom

- 4 vector constraints: $\nabla^\nu \delta E_{\mu\nu} = 0$
- Scalar constraint: unlike in the F-P theory, it cannot be obtained from a linear combination of $g^{\mu\nu}\delta E_{\mu\nu}$ and $\nabla^\mu\nabla^\nu\delta E_{\mu\nu}$.

Search for a scalar constraint

Counting the degrees of freedom

- 4 vector constraints: $\nabla^\nu \delta E_{\mu\nu} = 0$
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Generalised traces and divergences of the field equations

1. We define all possible ways of tracing $\delta E_{\mu\nu}$ with $S^\mu{}_\nu$:

$$\Phi_i^{(g,f)} \equiv [S^i]^{\mu\nu}\,\delta E^{(g,f)}_{\mu\nu}\,, \qquad 0 \leq i \leq 3$$
$$\Psi_i^{(g,f)} \equiv [S^i]^{\mu\nu}\nabla_\nu\nabla^\lambda\,\delta E^{(g,f)}_{\lambda\mu} \qquad 0 \leq i \leq 3\,.$$

2. Find a linear combination of these 16 scalars for which the 2nd derivative terms vanish:

$$\sum_{i=0}^{3}\left(u_i\,\Phi_i^{(g)} + v_i\,\Psi_i^{(g)}\right) + \sum_{i=0}^{3}\left(U_i\,\Phi_i^{(f)} + V_i\,\Psi_i^{(f)}\right) \sim 0,$$

A particular case in massive gravity: the beta 1 model

We assume $\beta_2 = \beta_3 = 0$ and $f_{\mu\nu}$ arbitrary but non-dynamical.

Field equations

$$\mathcal{G}_{\mu\nu} + m^2\Big[\beta_0\, g_{\mu\nu} + \beta_1\, g_{\mu\rho}\big(e_1(S)\delta^\rho_\nu - S^\rho{}_\nu\big)\Big] = 0\,,$$

It can be solved for $S^\mu{}_\nu$:

$$S^\rho{}_\nu = \frac{1}{\beta_1 m^2}\left[R^\rho{}_\nu - \frac{1}{6}\delta^\rho_\nu R - \frac{m^2\beta_0}{3}\,\delta^\rho_\nu\right].$$

- **It is only possible in the β_1 model.**
- It can be used to eliminate any occurrences of S (or f) in the linearised field equations.

A particular case: the beta 1 model

- In the $\boldsymbol{\beta_1}$ model, we can express the linearised field equations as a function of $\boldsymbol{g_{\mu\nu}}$ and its curvature.
- **We now take these equations as our starting point, no more assuming that $g_{\mu\nu}$ is a background solution.**

A particular case: the beta 1 model

- In the $\boldsymbol{\beta_1}$ model, we can express the linearised field equations as a function of $\boldsymbol{g_{\mu\nu}}$ and its curvature.
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The fifth scalar constraint

$$-\frac{m^2\,\beta_1\,e_4}{2}\,\Phi_0 - e_3\,\Psi_0 + e_2\,\Psi_1 - e_1\,\Psi_2 + \Psi_3 = 0\,.$$

- **Massive graviton (with 5 dof) propagating on a single arbitrary background.**

$$\overline{\Psi} = [S^{-1}]^{\mu\nu}\nabla_\nu\nabla^\lambda\,\delta E_{\lambda\mu} = \frac{1}{e_4}\left(e_3\,\Psi_0 - e_2\,\Psi_1 + e_1\,\Psi_2 - \Psi_3\right).$$

1. $\boldsymbol{\beta_3 = 0}$

$$\boxed{\frac{m^2\,\beta_1}{2}\,\Phi_0 + m^2\,\beta_2\,\Phi_1 + \overline{\Psi} = 0\,.}$$

Beyond the beta 1 model: general massive gravity

$$\overline{\Psi} = [S^{-1}]^{\mu\nu}\nabla_\nu\nabla^\lambda\,\delta E_{\lambda\mu} = \tfrac{1}{e_4}\left(e_3\,\Psi_0 - e_2\,\Psi_1 + e_1\,\Psi_2 - \Psi_3\right).$$

1. $\boldsymbol{\beta_3 = 0}$

$$\boxed{\frac{m^2\,\beta_1}{2}\,\Phi_0 + m^2\,\beta_2\,\Phi_1 + \overline{\Psi} = 0\,.}$$

2. $\boldsymbol{\beta_3 \neq 0}$

$$\frac{m^2\,\beta_1}{2}\,\Phi_0 + m^2\,\beta_2\,\Phi_1 - m^2\,\beta_3\left(\Phi_2 - e_1\Phi_1 + \tfrac{1}{2}e_2\,\Phi_0\right) + \overline{\Psi}$$
$$\sim m^2\beta_3\left(S^{\mu\lambda}[S^2]^{\nu\beta} - S^{\mu\nu}[S^2]^{\beta\lambda}\right)\nabla_\mu\nabla_\nu\delta g'_{\beta\lambda}\,.$$

- **It is not a covariant constraint but all the second time derivatives acting on the lapse and shifts vanish.**

Going back to bimetric theory

- Problematic terms: $[S^{-1}]^{\nu}{}_{\kappa}\nabla^{\kappa}\nabla^{\mu}\delta_f V_{\mu\nu} + [S]^{\nu}{}_{\kappa}\tilde{\nabla}^{\kappa}\tilde{\nabla}^{\mu}\delta_g \tilde{V}_{\mu\nu}$,
- Repeating the same analysis using the interchange symmetry, we could not find a covariant constraint.
- Performing a 3+1 decomposition $\Longrightarrow$ all the 2nd-time derivative acting on the lapse or shifts in the problemetic terms disappear.

$$\frac{m^2\,\beta_1}{2}\,\Phi_0^{(g)} + m^2\,\beta_2\,\Phi_1^{(g)} - m^2\,\beta_3\left(\Phi_2^{(g)} - e_1\Phi_1^{(g)} + \tfrac{1}{2}e_2\,\Phi_0^{(g)}\right)$$

$$+\overline{\Psi}^{(g)} + \text{ (interchange symmmetry } g \leftrightarrow f) = 0$$

Conclusion

- Linearised equations of bimetric theory in the general case,
- Condition for linearisation: S and $-S$ do not have common eigenvalues.
- Consistent theory for a massive graviton propagating in a single arbitrary background metric (β_1 model).
- Five covariant constraints in the metric formulation of massive gravity, when $\beta_3 = 0$.
- Non-covariant scalar constraint when $\beta_3 \neq 0$ and in bimetric theories.
- Interesting applications: cosmological pertubations, Higuchi bound, partial masslessness...