Black Hole Solutions

in Non-Minimally Coupled Weyl Connection Gravity

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The Weyl connection introduces a vector field that provides non-metricity properties:

$$D_\lambda g_{\mu\nu} = A_\lambda g_{\mu\nu},$$

where A_λ is the Weyl vector field and $D_\lambda g_{\mu\nu} = \nabla_\lambda g_{\mu\nu} - \bar{\bar{\Gamma}}^\rho_{\mu\lambda} g_{\rho\nu} - \bar{\bar{\Gamma}}^\rho_{\nu\lambda} g_{\rho\mu}$.

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The **generalized Ricci tensor** is given by:

$$\bar{R}_{\mu\nu} = R_{\mu\nu} + \underbrace{\frac{1}{2}A_\mu A_\nu + \frac{1}{2}g_{\mu\nu}\left(\nabla_\lambda - A_\lambda\right)A^\lambda + \tilde{F}_{\mu\nu} + \frac{1}{2}\left(\nabla_\mu A_\nu + \nabla_\nu A_\mu\right)}_{\bar{\bar{R}}_{\mu\nu}},$$

where $\tilde{F}_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ is the strength tensor of the Weyl field.

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The **scalar curvature** is given by:

$$\bar{R} = R + \underbrace{3\nabla_\lambda A^\lambda - \frac{3}{2}A_\lambda A^\lambda}_{\bar{\bar{R}}}.$$

Non-minimal matter–curvature coupling model, with Weyl connection, considering action functional:

$$S = \int \left(\kappa f_1(\bar{R}) + f_2(\bar{R})\mathcal{L}\right) \sqrt{-g} d^4x,$$

where $f_1(\bar{R})$ and $f_2(\bar{R})$ are generic functions of the scalar curvature.

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Varying the action with respect to the vector field, we obtain the **constraint-like equations**:

$$\nabla_\lambda \Theta(\bar{R}) = -A_\lambda \Theta(\bar{R}),$$

where $\Theta(\bar{R}) = F_1(\bar{R}) + F_2(\bar{R})\mathcal{L}$ and $F_i(\bar{R}) = \frac{df_i(\bar{R})}{d\bar{R}}$, $i \in \{1, 2\}$.

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Varying the action with respect to the metric, we obtain the **field equations**:

$$\left(R_{\mu\nu} + \bar{\bar{R}}_{(\mu\nu)}\right) \Theta(\bar{R}) - \frac{1}{2} g_{\mu\nu} f_1(\bar{R}) = \frac{f_2(\bar{R})}{2} T_{\mu\nu}.$$

Let us consider the electromagnetic Lagrangian density:

$$\mathcal{L}^{(EM)} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu},$$

where $F_{\mu\nu} = \partial_\mu \Phi_\nu - \partial_\nu \Phi_\mu$ is the Faraday tensor and Φ_μ is the electromagnetic four-potential.

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The variation with respect to the four-potential leads to the inhomogeneous modified **Maxwell equations**:

$$\nabla_\mu (f_2(\bar{R})F^{\mu\nu}) = 0.$$

Static line element in spherical coordinates:

$$ds^2 = -\alpha(r)dt^2 + \beta(r)dr^2 + r^2(d\theta^2 + \sin^2(\theta)d\phi^2),$$

where $\alpha(r)$ and $\beta(r)$ are arbitrary functions of the distance, r.

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A general Weyl vector takes the form:

$$A_\mu = (A_0(r), A_1(r), A_2(r), A_3(r)),$$

where $A_i(r)$, with $i \in \{0, 1, 2, 3\}$, are arbitrary functions of the distance.

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$$A_\mu = (0, A(r), 0, 0),$$

such as

$$A(r) \geq 0, \; \forall r.$$

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such as

$$A_0(r) \neq 0$$

and

$$A_0'(r) + (A_1(r) - \alpha'(r))A_0(r) = 0.$$

In the vacuum case, the field Equations take the form of a pure $f(R)$ gravity with the Weyl connection:

$$\left(R_{\mu\nu} + \bar{\bar{R}}_{(\mu\nu)}\right) F_1(\bar{R}) - \frac{1}{2} g_{\mu\nu} f_1(\bar{R}) = 0.$$

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Taking the trace of these equations:

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The scalar curvature needs to be zero:

$$\bar{R} = 0.$$

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$$\alpha(r) = 1 - \frac{2M}{r} + \frac{2(\omega + 3M)}{\omega}\frac{r}{\omega} + \frac{\omega + 4M}{\omega}\left(\frac{r}{\omega}\right)^2,$$

$$\beta(r) = \frac{1 + 6\left(\frac{M}{\omega}\right)}{1 - \frac{2M}{r} + \frac{2(\omega+3M)}{\omega}\frac{r}{\omega} + \frac{\omega+4M}{\omega}\left(\frac{r}{\omega}\right)^2},$$

$$A(r) = \frac{2}{r + \omega},$$

where $M > 0$ is the black hole mass and $\omega > 0$ that we can call Weyl's constant.

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Close enough to the BH:
$\alpha(r) \approx 1 - \frac{2M}{r}$;

Far enough to the BH:
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Ricci Scalar:

$$R = -\frac{12(r+\omega)((4M+\omega)r - M\omega)}{\omega^2(6M+\omega)r^2};$$

Kretschmann Invariant:

$$K = \frac{48M^2}{r^6}\left(\frac{r+\omega}{6M+\omega}.\right)^2$$

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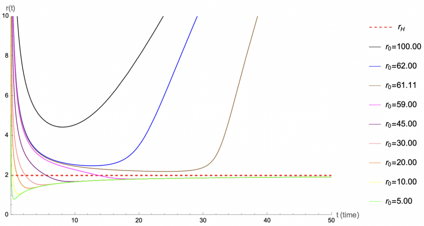

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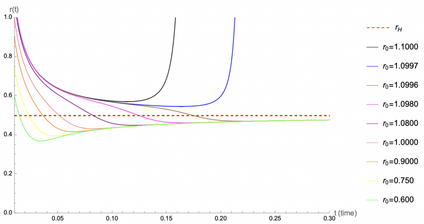

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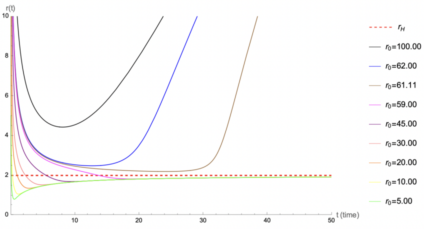

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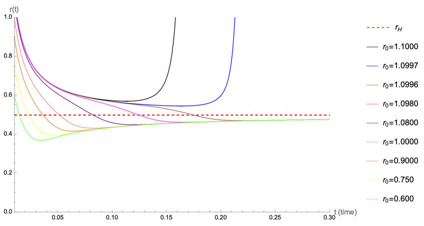

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Some thermodynamics quantities for a black hole, from the **quantum tunneling method**:

$$T^{(BH)} = T_0 \left(1 + 6\frac{M}{\omega}\right)^{-3/2};$$

$$S^{(BH)} = \frac{8\pi}{9} \left(1 + 6\frac{M}{\omega}\right)^{-3/2} (3M+\omega)(6M+\omega);$$

$$C_V^{(BH)} = C_0 \left(1 + 6\frac{M}{\omega}\right)^{-3/2} \frac{6M+\omega}{\omega - 3M};$$

$$F^{(BH)} = -\left(M + \omega + \frac{\omega^2}{9M}\right).$$

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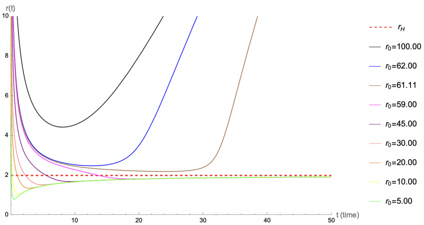

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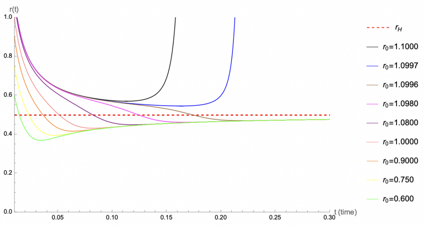

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Some thermodynamics quantities for a black hole, from the so-called **“area approach”**:

$$T^{(A)} = T_0\frac{4M+\omega}{\omega};$$
$$S^{(A)} = 4\pi\left(\frac{M\omega}{4M+\omega}\right)^2;$$
$$C_V^{(A)} = C_0\left(\frac{\omega}{4M+\omega}\right)^2;$$
$$F^{(A)} = \frac{M(8M+\omega)}{2(4M+\omega)}.$$

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$$A_0(r) = \frac{1}{\omega}\left(1 - \frac{2M}{r}\right),$$

$$A_1(r) = \frac{2r}{r^2 - 4\omega^2},$$

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Event horizons:

$$r_H^{(M)} = 2M \text{ and } r_H^{(\omega)} = 2\omega$$

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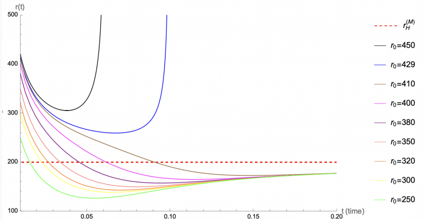
Ricci Scalar: $R = \frac{3(r-M)}{\omega^2 r}$;

Kretschmann Invariant:

$$K = \frac{48M}{r^6}\left(\frac{4M\omega^2(47-2Mr+r^2)-r^2(44M-2r(1+M^2)+Mr^2)}{192\omega^2}\right)$$

Schwarzschild-Like BH: $A_\mu = (A_0(r), A_1(r), 0, 0)$



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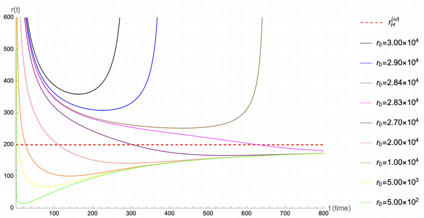

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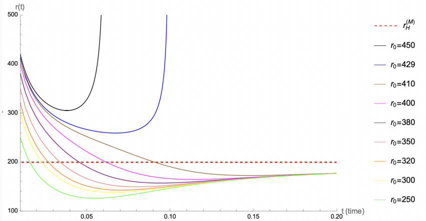

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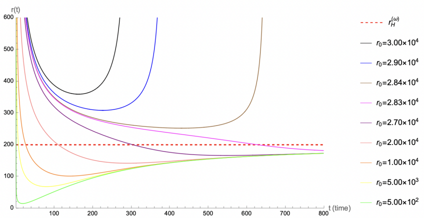

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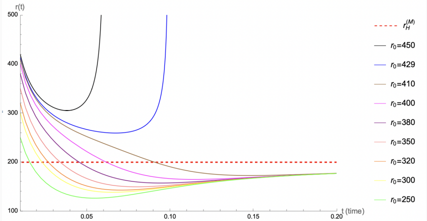
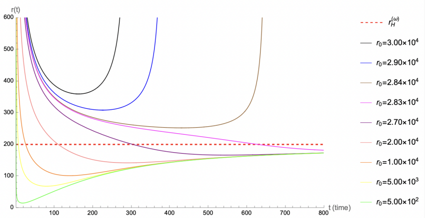


Geodesic representation of the Schwarzschild-like black hole, considering $\omega \gg M$, for the parameters $\omega = 10^4$ and $M = 1$:



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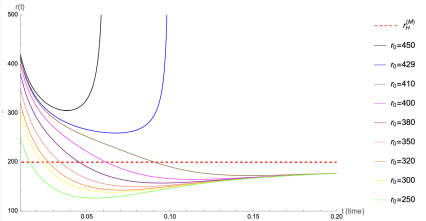

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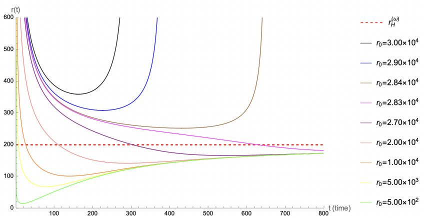

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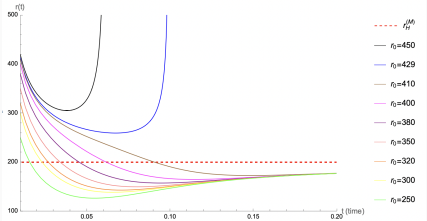

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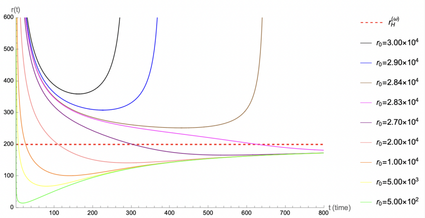

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Schwarzschild-Like BH: Cosmological Constant Background

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The scalar curvature needs to be zero:

$$\bar{R} = 0.$$

The vacuum solutions are also solutions when we consider a cosmological constant background: it is a mathematical reparametrization like $f(\bar{R}) = f_1(\bar{R}) - 2\Lambda f_2(\bar{R})$.

There are no solutions in vacuum, so let's consider $\mathcal{L} = \mathcal{L}^{(EM)} + \mathcal{L}^{(\Lambda)}$.

Reissner–Nordstrøm-Like BH: Cosmological Constant Background

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$$\alpha(r) = 1 - \frac{2M}{r} + \frac{\tilde{Q}^2}{r^2} + \frac{2\left(\omega^2 + 3M\omega + 2\tilde{Q}^2\right)}{\omega^2}\frac{r}{\omega} + \frac{4M+\omega}{\omega}\left(\frac{r}{\omega}\right)^2,$$

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$$A(r) = \frac{2}{r+\omega},$$

where $M > 0$ is the mass, $\omega > 0$ is the Weyl constant and $\tilde{Q}$ is the dressed/corrected charge.

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Scalar Curvature: $\bar{R} = \frac{36\tilde{Q}^2}{\left(\omega^2+6M\omega+6\tilde{Q}^2\right)(r+\omega)^2}$;

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- Reissner–Nordstrøm-like solution can be found with an extra linear term in r and corrected/dressed charge and cosmological constant in the solution for the metric functions.

Thank You for Your Attention!

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