Gravity of static thin discs around black holes

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CHARLES UNIVERSITY
Faculty of mathematics
and physics

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What is it about?

An explicit **exact** superposition of the **Schwarzschild black hole** encircled by a **disc**.

- exact (vacuum) solution to Einstein equations

Assumptions:

- axially symmetric thin disc $\Longrightarrow$ distributional source $\propto \delta(z)$
- zero overall net rotation $\Longrightarrow$ static metric

Based on:

[1]Petr Kotlařík and David Kofroň. "Black Hole Encircled by a Thin Disk: Fully Relativistic Solution". *ApJ* 941.1 (2022), p. 25

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Weyl metric

Gravitational field of **static** and **axially symmetric** vacuum spacetimes is described by

$$ds^2 = -e^{2\nu}\,dt^2 + \rho^2 e^{-2\nu}\,d\phi^2 + e^{2\lambda-2\nu}(d\rho^2 + dz^2)\ , \qquad (1)$$

where t, ρ, ϕ, z are the Weyl cylindrical coordinates and $\nu(\rho, z), \lambda(\rho, z)$.

Vacuum Einstein equations then

$$\Delta\nu = 0\ , \qquad \lambda_{,\rho} = \rho(\nu_{,\rho}^2 - \nu_{,z}^2)\ , \qquad \lambda_{,z} = 2\rho\nu_{,\rho}\nu_{,z}\ , \qquad (2)$$

where Δ is 3D Laplace operator in cylindrical coordinates (ρ, z).

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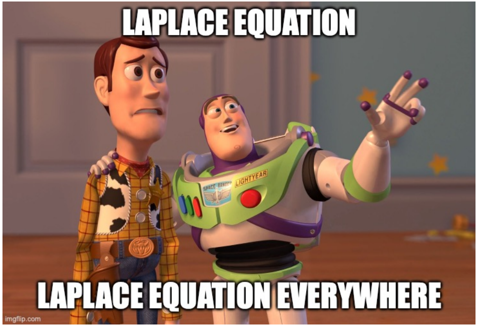

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1. linearity
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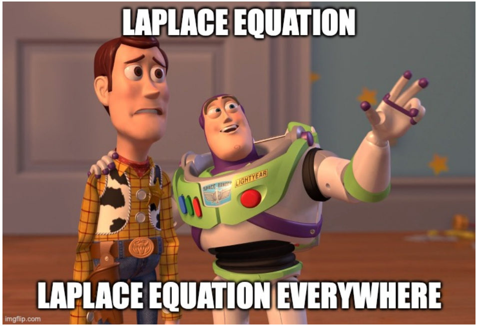

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Two families of discs (empty in their center)

1. by an explicit integration of the axially symmetric Green function
2. from the well-known Kuzmin-Toomre family using linearity (discs obtained by Vogt and Letelier)

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Potential in terms of the Bessel functions

The axisymmetric Green function of the Laplace equation can be expressed as

$$\mathcal{G}(\rho,\rho',z,z') = -2\pi\rho' \int_0^\infty J_0(s\rho')J_0(s\rho)e^{-s|z-z'|}\,\mathrm{d}s\ , \tag{3}$$

where J_0 is the zero-order Bessel function of the first kind and s an auxilliary real variable.

Potential due to a thin source of surface density $w(\rho)$ placed at $z'=0$

$$\nu(\rho,z) = -2\pi \int_0^\infty \int_0^\infty \rho' w(\rho')J_0(s\rho')J_0(s\rho)e^{-s|z|}\,\mathrm{d}s\,\mathrm{d}\rho'\ . \tag{4}$$

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Strategy

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Our strategy now:

1. perform the integration for elementary density terms
2. construct reasonable physical discs by simple superposition of elementary terms

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Elementary density terms

Consider the density as

$$w(\rho) = \rho^{2l} , \qquad \rho \in (0, b) , \qquad l > 0 \text{ integer.} \tag{5}$$

Then, the radial integration gives

$$\nu^{(2l)}(\rho, z) = \pi b^{2l+2} l! \sum_{j=1}^{l+1} \frac{(-2)^j}{b^j (l+1-j)!} \mathcal{I}_{(-j,j,0)} , \tag{6}$$

where we have denoted

$$\mathcal{I}_{(\alpha,\beta,\gamma)} = \int_0^\infty s^\alpha J_\beta(sb) J_\gamma(s\rho) e^{-s|z|} \, \mathrm{d}s . \tag{7}$$

The Laplace transform of Bessel-function products

Our hero

$$\mathcal{I}_{(\alpha,\beta,\gamma)} = \int_0^\infty s^\alpha J_\beta(sb) J_\gamma(s\rho) e^{-s|z|}\,\mathrm{d}s$$

Conway[1] studied this kind of integral thoroughly – the lowest orders can be computed directly

$$\mathcal{I}_{(0,0,0)} = \frac{1}{\pi\sqrt{b\rho}} k K(k)\,, \qquad \text{where } k := \frac{2\sqrt{b\rho}}{\sqrt{(\rho+b)^2+z^2}}$$

$$\mathcal{I}_{(0,1,1)} = \frac{1}{\pi\sqrt{b\rho}} \left[\frac{2-k^2}{k} K(k) - \frac{2}{k} E(k)\right]\,,$$

$$\mathcal{I}_{(0,1,0)} = \frac{1}{b}\left\{\frac{|z|k}{2\pi\sqrt{b\rho}}\left[\frac{\rho-b}{\rho+b}\Pi\left(\frac{4b\rho}{(\rho+b)^2}, k\right) - K(k)\right] + H(b-\rho)\right\}\,,$$

[1] J. T. Conway. "Analytical solutions for the Newtonian gravitational field induced by matter within axisymmetric boundaries". *MNRAS* 316.3 (2000), pp. 540–554.

The general form of elementary density-potential pair

The gravitational potential[2] due to the elementary density terms $w(\rho) = \rho^{2l}$ is

$$\nu^{(2l)}(\rho, z) = 2\pi \mathcal{P}_H^{(2l)} |z| H(b-\rho) - 4\mathcal{P}_E^{(2l)} \frac{\sqrt{b\rho}}{k} E(k) - \mathcal{P}_K^{(2l)} \frac{k}{\sqrt{b\rho}} K(k) - \mathcal{P}_\Pi^{(2l)} \frac{b-\rho}{b+\rho} \frac{z^2 k}{\sqrt{b\rho}} \Pi\left(\frac{4b\rho}{(b+\rho)^2}, k\right)$$

where $\mathcal{P}^{(2l)}$ are polynomials, e.g.

$2l = 0$ (const. density)	$\mathcal{P}_{H,\Pi}^{(0)}$	1
	$\mathcal{P}_E^{(0)}$	1
	$\mathcal{P}_K^{(0)}$	$b^2 - \rho^2$
$2l = 2$	$\mathcal{P}_{H,\Pi}^{(2)}$	$\frac{1}{3}(3\rho^2 - 2z^2)$
	$\mathcal{P}_E^{(2)}$	$\frac{1}{9}(b^2 + 4\rho^2 - 11z^2)$
	$\mathcal{P}_K^{(2)}$	$\frac{1}{9}(5b^4 - b^2\rho^2 + 4b^2z^2 - 4\rho^4 + 5z^4 + 16\rho^2z^2)$

[2] P. K., D. Kofroň, and O. Semerák. "Static Thin Disks with Power-law Density Profiles". *ApJ* 931 (2022), p. 161.

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$2l = 4$	$\mathcal{P}_{H,\Pi}^{(4)}$	$\frac{1}{15}(15\rho^4 + 8z^4 - 40\rho^2 z^2)$
	$\mathcal{P}_E^{(4)}$	$\frac{1}{225}(9b^4 + 16b^2\rho^2 - 47b^2z^2 + 64\rho^4 + 274z^4 - 607\rho^2z^2)$
	$\mathcal{P}_K^{(4)}$	$\frac{1}{225}(81b^6 - b^4\rho^2 + 8b^4z^2 - 16b^2\rho^4 - 107b^2z^4 + 192b^2\rho^2z^2 - 64\rho^6 - 154z^6 - 267\rho^2z^4 + 768\rho^4z^2)$
$2l = 6$	$\mathcal{P}_{H,\Pi}^{(6)}$	$\frac{1}{35}(35\rho^6 - 16z^6 + 168\rho^2z^4 - 210\rho^4z^2)$
	$\mathcal{P}_E^{(6)}$	$\frac{1}{1225}(25b^6 + 36b^4\rho^2 - 107b^4z^2 + 64b^2\rho^4 + 306b^2z^4 - 631b^2\rho^2z^2 + 256\rho^6 - 1452z^6 + 8132\rho^2z^4 - 5175\rho^4z^2)$
	$\mathcal{P}_K^{(6)}$	$\frac{1}{1225}(325b^8 - b^6\rho^2 + 12b^6z^2 - 4b^4\rho^4 - 59b^4z^4 + 80b^4\rho^2z^2 - 64b^2\rho^6 + 586b^2z^6 - 2819b^2\rho^2z^4 + 1536b^2\rho^4z^2 - 256\rho^8 + 892z^8 - 800\rho^2z^6 - 10307\rho^4z^4 + 6144\rho^6z^2)$

Negative powers?

To obtain the potential due to the density terms of **negative** powers, we can perform an inversion with respect to the outer rim $\rho = b$, (i.e. Kelvin transformation)

$$\rho \to \frac{b^2\rho}{\rho^2 + z^2}\,, \qquad z \to \frac{b^2 z}{\rho^2 + z^2}\,, \tag{8}$$

and the corresponding potential transform

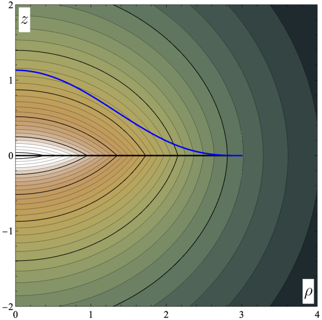
$$\nu^{(2l)} \longrightarrow \frac{b}{\sqrt{\rho^2 + z^2}}\nu^{(2l)}\left(\frac{b^2\rho}{\rho^2 + z^2}, \frac{b^2 z}{\rho^2 + z^2}\right) \quad \Longrightarrow \quad w(\rho) \equiv \rho^{2l} \longrightarrow \frac{b^3}{\rho^3} w(b^2/\rho) = \frac{b^{3+4l}}{\rho^{3+2l}}\,.$$

So, the density-potential pair

$$w(\rho) = \rho^{-3-2l}\,, \quad \nu_{\mathrm{i}}^{(-3-2l)} = \frac{b^{-2-4l}}{\sqrt{\rho^2 + z^2}}\nu^{(2l)}\left(\frac{b^2\rho}{\rho^2 + z^2}, \frac{b^2 z}{\rho^2 + z^2}\right)\,. \tag{9}$$

Physical discs

$$w^{(m,2l)} = W\left(1 - \frac{\rho^{2l}}{b^{2l}}\right)^m$$



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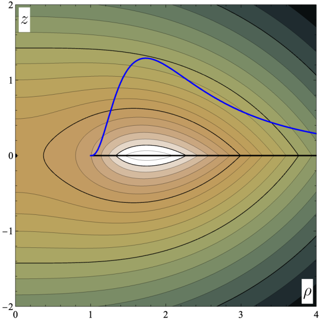
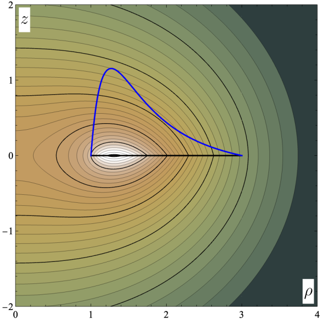


$$w = -W_0 + \frac{W_{-3}}{\rho^3} - \frac{W_{-5}}{\rho^5}$$



Summary

The potential of all these discs have been found in **closed-forms**.

The discs empty in their center are suitable for a superposition with the Schwarzschild black hole

$$\nu_{\mathrm{Schw}} = \frac{1}{2} \ln \frac{d_1 + d_2 - 2M}{d_1 + d_2 + 2M} , \qquad d_{1,2} \equiv \sqrt{\rho^2 + (|z| \mp M)^2} \tag{10}$$

done by a simple sum

$$\nu = \nu_{\mathrm{Schw}} + \nu_{\mathrm{disc}} . \tag{11}$$

The second metric function λ has to be (when needed) integrated numerically

$$\lambda = \int_{\mathrm{axis}}^{\rho,z} \rho(\nu_{,\rho}^2 - \nu_{,z}^2)\,\mathrm{d}\rho + 2\rho\nu_{,\rho}\nu_{,z}\,\mathrm{d}z . \tag{12}$$

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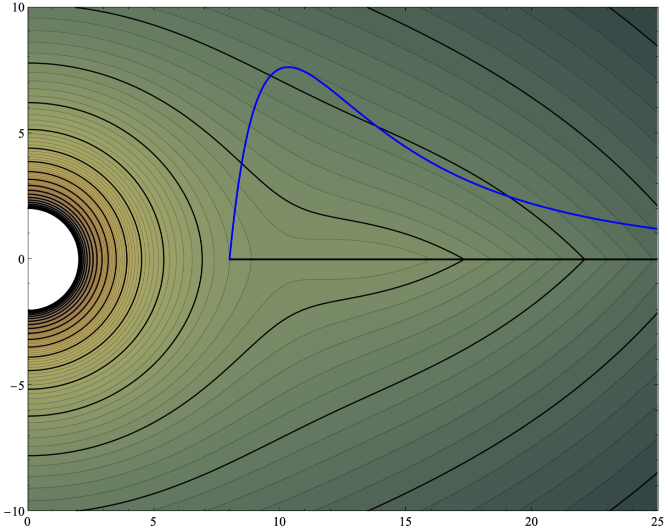

Figure 1: Contourplot of the total potential in Schwarzschild coordinates.

BH+disc where λ is known?

$$\lambda_{,\rho} = \rho(\nu_{,\rho}^2 - \nu_{,z}^2) \tag{13}$$

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Whereas ν adds linearly, the non-linearity of Einstein equations clearly manifests in λ.

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Kuzmin-Toomre family of thin discs

Kuzmin and Toomre[3,4] obtained a family of thin discs which Newtonian densities are

$$w_n = \frac{(2n+1)b^{2n+1}}{2\pi} \frac{\mathcal{M}}{(\rho^2+b^2)^{n+3/2}} \,, \tag{15}$$

where $\mathcal{M}$ is the discs total mass and b is a constant, described by the gravitational potential

$$\nu_n = -\frac{\mathcal{M}}{(2n-1)!!} \sum_{k=0}^{n} \frac{(2n-k)!}{2^{n-k}(n-k)!} \frac{b^k}{r_b^{k+1}} P_k\left(|\cos\theta_b|\right) \,, \tag{16}$$

where P_l are the Legendre polynomials, and

$$r_b^2 := \rho^2 + (|z|+b)^2 \,, \qquad |\cos\theta_b| := \frac{|z|+b}{r_b} \,. \tag{17}$$

[3] G. G. Kuzmin. "A stationary Galaxy model admitting triaxial velocity distribution". *Astr. Zh.* 33 (1956), pp. 27–45.

[4] A. Toomre. "On the Distribution of Matter Within Highly Flattened Galaxies.". *ApJ* 138 (1963), p. 385.

Vogt and Letelier[5] took a particular superposition of the Kuzmin-Toomre discs

$$\nu^{(m,n)} = W^{(m,n)} \sum_{k=0}^{n} (-1)^k \binom{n}{k} \frac{\nu_{m+k}}{2m+2k+1} , \tag{18}$$

where $W^{(m,n)}$ is a constant. The disc's Newtonian density

$$w^{(m,n)} = W^{(m,n)} \frac{\mathcal{M} b^{2m+1}}{2\pi} \frac{\rho^{2n}}{(\rho^2 + b^2)^{m+n+3/2}} . \tag{19}$$

The special case $m = 0$ comes also from the inversion around $\rho = b$ (Kelvin transformation) – inverted Kuzmin-Toomre discs.

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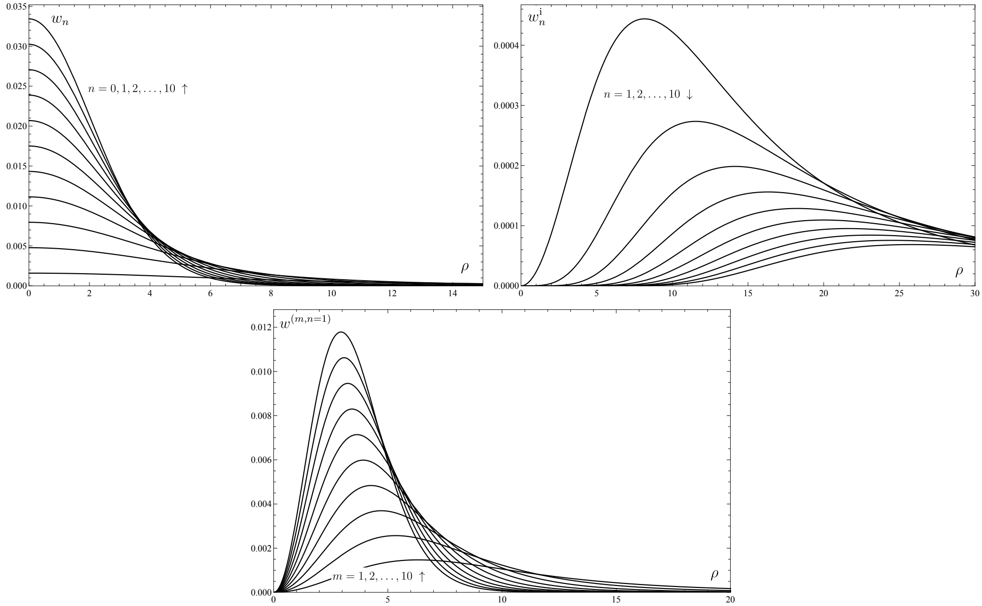
w_n
$n = 0, 1, 2, \dots, 10 \uparrow$
ρ
w_n^{i}
$n = 1, 2, \dots, 10 \downarrow$
ρ
$w^{(m,n=1)}$
$m = 1, 2, \dots, 10 \uparrow$
ρ

Ressumation of the potential

Fixing the constant $W^{(m,n)}$, so the total mass of the disc is $\mathcal{M}$, the Vogt-Letelier potential can be recast into a form

$$\nu^{(m,n)} = -(2m+1)\mathcal{M}\binom{m+n+1/2}{n}\sum_{j=0}^{m+n}\mathcal{Q}_j^{(m,n)}\frac{b^j}{r_b^{j+1}}P_j(\cos\theta_b)\,, \qquad (20)$$

where

$$\mathcal{Q}_j^{(m,n)} = \begin{cases} \frac{2^{j-m}(2m-j-1)!!}{(2m+1)(m-j)!}\,{}_3F_2\left(j-n, \frac{j+m+1}{2}, \frac{j+m+2}{2}; j+1, \frac{2j+2m+3}{2}; 1\right) & \text{if } j \leq m \\ \frac{(-1)^{j-m}j!}{(2j+1)!!}\binom{n}{j-m}\,{}_3F_2\left(\frac{j+1}{2}, \frac{j+2}{2}, j-m-n; \frac{2j+3}{2}, j-m+1; 1\right) & \text{if } j > m\,, \end{cases}$$

and ${}_3F_2$ are the generalized hypergeometric functions.

The second metric function λ

The potential is separated in r_b and θ_b. We can rewrite the equations for λ in terms of (r_b, θ_b) and integrate them over the r_b coordinate[6]

$$\lambda^{(m,n)} = -(2m+1)^2 \binom{m+n+1/2}{n}^2 \mathcal{M}^2 \sin^2\theta_b \sum_{k,l=0}^{m+n} \mathcal{B}_{k,l}^{(m,n)} \frac{b^{k+l}}{r_b^{k+l+2}} \mathcal{P}_{k,l}(\theta_b) \quad (21)$$

where

$$\mathcal{B}_{k.l}^{(m,n)} = \frac{\mathcal{Q}_l^{(m,n)} \mathcal{Q}_k^{(m,n)}}{k+l+2} \quad (22)$$

$$\mathcal{P}_{k,l} \equiv (k+1)(l+1)P_k P_l + 2(k+1)|\cos\theta_b| P_k P_l' - \sin^2\theta_b P_k' P_l' \quad (23)$$

$$P_k' \equiv \frac{\mathrm{d}}{\mathrm{d}|\cos\theta_b|} P_k(|\cos\theta_b|) \quad (24)$$

[6] J. Bičák, D. Lynden-Bell, and C. Pichon. "Relativistic Discs and Flat Galaxy Models". *MNRAS* 265 (1993), p. 126.

The Laplace equation is linear $\Longrightarrow$ the total gravitational potential is a simple sum

$$\nu = \nu_{\mathrm{Schw}} + \nu_{\mathrm{disc}} \,. \tag{25}$$

... easy!

The total gravitational potential

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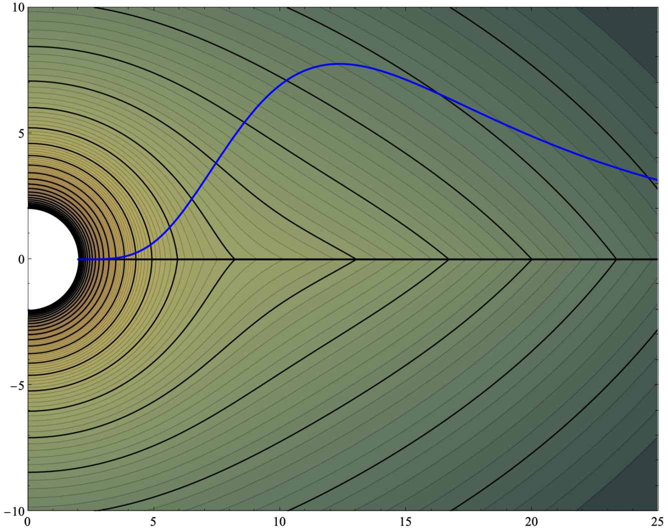

Figure 2: Contourplot of the total potential in Schwarzschild coordinates.

The second metric function λ

The metric function λ does not superpose that simply. In fact, we denoted

$$\lambda = \lambda_{\mathrm{Schw}} + \lambda_{\mathrm{disc}} + \lambda_{\mathrm{int}} \ , \tag{26}$$

where $\lambda_{\mathrm{Schw}}, \lambda_{\mathrm{disc}}$ are contributions from the black hole and the disc, and

$$\lambda_{\mathrm{int},\rho} = 2\rho(\nu_{\mathrm{Schw},\rho}\nu_{\mathrm{disc},\rho} - \nu_{\mathrm{Schw},z}\nu_{\mathrm{disc},z}) \ , \tag{27}$$

$$\lambda_{\mathrm{int},z} = 2\rho(\nu_{\mathrm{Schw},\rho}\nu_{\mathrm{disc},z} + \nu_{\mathrm{Schw},z}\nu_{\mathrm{disc},\rho}) \ . \tag{28}$$

Notice that (27), (28) are linear in ν_{disc}, hence λ_{int} satisfies the same recurrence relations as ν_{disc}. Namely

$$\lambda_{\mathrm{int}}^{(0,n+1)} = \lambda_{\mathrm{int}}^{(0,n)} + \frac{b}{2(n+1)}\frac{\partial}{\partial b}\lambda_{\mathrm{int}}^{(0,n)} \ , \qquad \lambda_{\mathrm{int}}^{(0,0)} = -\frac{\mathcal{M}}{r_b}\left(\frac{d_1}{b+M} - \frac{d_2}{b-M}\right) - \frac{2\mathcal{M}M}{b^2-M^2} \ ,$$

$$\frac{(2m+1)(2n+3)}{2m+2n+3}\lambda_{\mathrm{int}}^{(m+1,n)} = \lambda_{\mathrm{int}}^{(m,n)} + \frac{4m(n+1)}{2m+2n+3}\lambda_{\mathrm{int}}^{(m,n+1)} - b\frac{\partial}{\partial b}\lambda_{\mathrm{int}}^{(m,n)} \ .$$

The second metric function λ

The metric function λ does not superpose that simply. In fact, we denoted

$$\lambda = \lambda_{\mathrm{Schw}} + \lambda_{\mathrm{disc}} + \lambda_{\mathrm{int}} \ , \tag{26}$$

where $\lambda_{\mathrm{Schw}}, \lambda_{\mathrm{disc}}$ are contributions from the black hole and the disc, and

$$\lambda_{\mathrm{int},\rho} = 2\rho(\nu_{\mathrm{Schw},\rho}\nu_{\mathrm{disc},\rho} - \nu_{\mathrm{Schw},z}\nu_{\mathrm{disc},z}) \ , \tag{27}$$

$$\lambda_{\mathrm{int},z} = 2\rho(\nu_{\mathrm{Schw},\rho}\nu_{\mathrm{disc},z} + \nu_{\mathrm{Schw},z}\nu_{\mathrm{disc},\rho}) \ . \tag{28}$$

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Power-law discs vs regular discs around the black hole

power-law discs	**regular discs**
only potential known explicitly	full metric
inner edge	without edge
higher derivatives of curvature might diverge at the inner rim	field is regular everywhere outside of horizon

Two simple physical interpretations of the disc are possible

- a single component ideal fluid with a certain surface density ($e^{\nu-\lambda}\sigma$) and azimuthal pressure ($e^{\nu-\lambda}P$) which keeps the orbits at their radius
- two identical counter-orbiting dust components with proper surface densities ($\sigma_+ = \sigma_- \equiv \sigma/2$) following circular geodesics

Both characteristics σ and P are encoded in the jump of the normal derivative of the gravitational potential

$$\sigma + P = \frac{\nu_{,z}(z=0^+)}{2\pi} = w(\rho) , \qquad P = \frac{\nu_{,z}(z=0^+)}{2\pi}\rho\nu_{,\rho} = w(\rho)\rho\nu_{,\rho} . \qquad (29)$$

Physical interpretation

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Physical interpretation

We checked the basic physical properties. For wide range of $M, \mathcal{M}, b$, the discs have

- reasonable relativistic density and pressure profiles
- densities and pressures are positive
- all energy conditions are satisfied

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Thank you for your attention.

Question time!

Brief summary

- explicit potentials of static power-law discs around Schwarzschild BH
- explicit full metric of static regular discs around Schwarzschild BH (constructed from Kuzmin-Toomre solution)
- closed-form formulas

Thank you for your attention.

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