On the Regularity Implied by the Assumptions of Geometry

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Does curvature control all derivatives of a connection Γ?

Example:

$\Gamma^k_{ij} = g^{kl}(\partial_i g_{jl} + \partial_j g_{il} - \partial_l g_{ij})$,

for g_{ij} a Lorentzian metric

Does curvature control all derivatives of a connection Γ?

$$\Gamma \in W^{1,p}$$

$$\Big\downarrow \frac{\partial}{\partial y}$$

$$\text{Riem}(\Gamma) \in L^p$$

- $\Gamma \in L^p$ means $\int |\Gamma|^p \, dx < \infty$ component-wise
- $\Gamma \in W^{1,p}$ means $\Gamma \in L^p$ & $\partial\Gamma \in L^p$

Does curvature control all derivatives of a connection Γ?

$$\Gamma \in W^{1,p} \xrightarrow{\frac{\partial}{\partial y}} \mathrm{Riem}(\Gamma) \in L^p$$

"Optimal Regularity"

$$\Gamma \in L^{2p} \xrightarrow{\frac{\partial}{\partial x}} \mathrm{Riem}(\Gamma) \in L^p$$

"Non-optimal Regularity"

$\mathrm{Riem}(\Gamma) \sim \mathrm{Curl}(\Gamma)$
makes this possible

Does curvature control all derivatives of a connection Γ?

$$y \to x, \qquad \frac{\partial x}{\partial y} \in W^{1,p}$$

"Optimal Regularity"	"Non-optimal Regularity"
$\Gamma \in W^{1,p}$	$\Gamma \in L^{2p}$
$\downarrow \frac{\partial}{\partial y}$	$\downarrow \frac{\partial}{\partial x}$
$\mathrm{Riem}(\Gamma) \in L^p$	$\mathrm{Riem}(\Gamma) \in L^p$

$$\mathrm{Riem}(\Gamma) \to \frac{\partial x}{\partial y} \cdot \mathrm{Riem}(\Gamma)$$

$$\Gamma \to \Gamma + \partial\left(\frac{\partial x}{\partial y}\right)$$

Does curvature control all derivatives of a connection Γ?

$$\Gamma \in W^{1,p} \;\Big\downarrow\; \tfrac{\partial}{\partial y} \;\; \mathrm{Riem}(\Gamma) \in L^p$$

$$\Gamma \in L^{2p} \;\Big\downarrow\; \tfrac{\partial}{\partial x} \;\; \mathrm{Riem}(\Gamma) \in L^p$$

$y \leftarrow x\,?$

<u>Question:</u> $\exists\, x \to y\,?$

<u>Examples:</u>
GR shock waves;
Schwarzschild radius

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$$\Gamma \in L^{2p} \quad \Big\downarrow \tfrac{\partial}{\partial x} \quad \mathrm{Riem}(\Gamma) \in L^p$$

<u>Thm 1:</u> $\exists\, x \to y\,!$

Thm 1: (R. & Temple, 2019/2021.)

Let $\Gamma \in L^{2p}$ with $\mathrm{Riem}(\Gamma) \in L^p$ in x-coordinates, ($n/2 < p < \infty$).

Then, locally, there exists a coord. transformation $x \to y$ with Jacobian $J \in W^{1,2p}$, such that $\Gamma \in W^{1,p}$ (optimal regularity) in y-coordinates.

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Proof is based on the "*Regularity Transformation (RT-)equations*":

$$\begin{cases} \Delta\tilde{\Gamma} = \delta d\Gamma - \delta\big(dJ^{-1} \wedge dJ\big) + d(J^{-1}A), \\ \Delta J = \delta(J\Gamma) - \langle dJ; \tilde{\Gamma}\rangle - A, \\ d\vec{A} = \overrightarrow{\mathrm{div}}\big(dJ \wedge \Gamma\big) + \overrightarrow{\mathrm{div}}\big(J\, d\Gamma\big) - d\big(\overrightarrow{\langle dJ; \tilde{\Gamma}\rangle}\big), \\ \delta\vec{A} = v, \end{cases}$$

a non-linear system of partial differential equations, which determines the Jacobian J of a regularising coordinate transformation.

The RT-equations are elliptic regardless of metric signature!

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Applications to General Relativity (GR):

- Extension of Kazdan-DeTurck's optimal regularity theorem [1982] from (positive) Riemannian to non-Riemannian metrics and General Relativity.

  Note: $\Gamma \in W^{1,p} \;\Leftrightarrow\; g \in W^{2,p}$.

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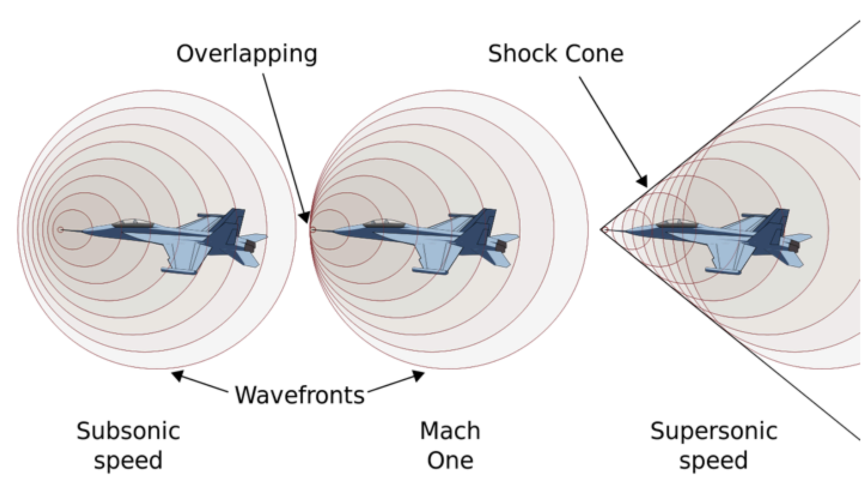
- Shock wave solutions of the Einstein-Euler equations are non-singular!
  (Their Lipschitz continuous metrics can be mapped to optimal regularity.)
  $\Rightarrow$ Geodesic curves, locally inertial coordinates & Newtonian limit exist!



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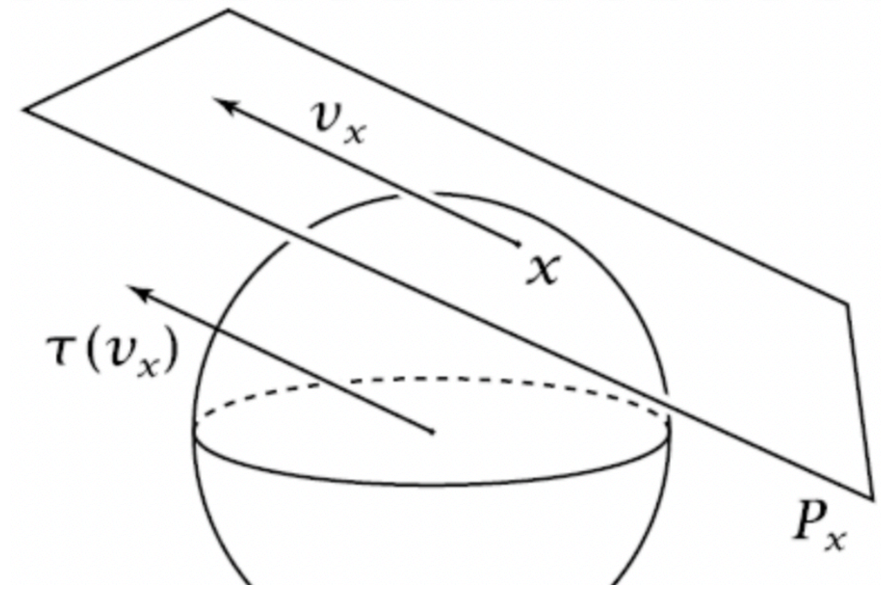
- Shock wave solutions of the Einstein-Euler equations are non-singular! (Their Lipschitz continuous metrics can be mapped to optimal regularity.) $\Rightarrow$ Geodesic curves, locally inertial coordinates & Newtonian limit exist!
- Extension Uhlenbeck compactness from Riemannian [K. Uhlenbeck, 1982] to non-Riemannian geometry and General Relativity:

Thm 2: Strong L^p- and weak $W^{1,p}$-convergence of affine connections with connections and curvature uniformly bounded in L^∞.

Thm 1: Optimal Regularity.

Thm 2: Uhlenbeck compactness.



Thm's 1 & 2 address only connections on tangent bundles.

Thm 3: [R. & Temple, 2021]
Theorems 1 & 2 extend to vector bundles,
with compact and non-compact gauge groups,
over non-Riemannian base manifolds.
(Yang-Mills gauge theories of Particle Physics.)

Thank you very much for your attention!

1. M.R. & B. Temple, "On the Regularity Implied by the Assumptions of Geometry", (2019/2021), 100 pages. [arXiv:1912.12997]
2. M.R. & B.Temple, "On the Regularity Implied by the Assumptions of Geometry II - Connections on Vector Bundles", (2021), 40 pages. [arXiv:2105.10765]
3. M. R. & B. Temple, "Optimal regularity and Uhlenbeck compactness for General Relativity and Yang-Mills Theory", (2022), 18 pages. [arXiv:2202.09535]