# General relativistic solutions in the Minimal Theory of Bigravity

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# Introduction

- Predictions from general relativity (GR) are consistent with experimental and observational data
- Nevertheless, modified gravity theories have been extensively studied for various purposes, especially, for explaining the dark energy and dark matter problems
- Massive gravity and bigravity theories correspond to the minimal modification of GR
- dRGT massive gravity was the first formulation of nonlinear massive gravity free from the Boulware-Deser ghosts

de Rham, Gabadadze, and Tolley (2010)

- dRGT massive gravity was then extended to HR bigravity by promoting the fiducial metric to another dynamical field

Hassan and Rosen (2011)

# HR bigravity

Hassan and Rosen (2011)

$$S = \frac{1}{2\kappa^2}\int d^4x\left[\sqrt{-g}R[g]+\tilde{\alpha}^2\sqrt{-f}R[f]-m^2\sqrt{-g}\sum_{i=0}^{4}c_{4-i}e_i(\mathcal{K})+\mathcal{L}_m\right]$$

- $g_{\mu\nu}$ and $f_{\mu\nu}$; physical and fiducial metrics
- $\mathcal{K}^{\mu}{}_{\alpha}\mathcal{K}^{\alpha}{}_{\nu}=g^{\mu\alpha}f_{\alpha\nu}$, and

$$e_0(\mathcal{K})=1,\quad e_1=[\mathcal{K}],\quad e_2=\frac{1}{2}\left([\mathcal{K}]^2-\left[\mathcal{K}^2\right]\right),$$
$$e_3(\mathcal{K})=\frac{1}{6}\left([\mathcal{K}^3]-3[\mathcal{K}^2][\mathcal{K}]+2[\mathcal{K}]^3\right),$$
$$e_4(\mathcal{K})=\frac{1}{24}\left([\mathcal{K}]^4-6[\mathcal{K}]^2[\mathcal{K}^2]+3[\mathcal{K}^2]^2+8[\mathcal{K}][\mathcal{K}^3]-6[\mathcal{K}]^4\right)$$

- $c_i$ $(i=0,\cdots,4)$; dimensionless coupling constants
- $2+5$ DOFs;
  scalar and vector DOFs causing instabilities in cosmolgy
  $\Longrightarrow$ Minimal Theory of Bigravity with $2+2$ DOFs

# Minimal Theory of Bigravity (MTBG)

De Felice, Larrouturou, Mukohyama, and Oliosi (2020)

- ADM decomposition

$$g_{\mu\nu}dx^\mu dx^\nu = -N^2dt^2 + \gamma_{ij}\left(dx^i + N^i dt\right)\left(dx^j + N^j dt\right)$$
$$f_{\mu\nu}dx^\mu dx^\nu = -M^2dt^2 + \phi_{ij}\left(dx^i + M^i dt\right)\left(dx^j + M^j dt\right)$$

- $\mathcal{K}^i{}_k\mathcal{K}^k{}_j = \gamma^{ik}\phi_{kj}$ and $\tilde{\mathcal{K}}_j{}^k\tilde{\mathcal{K}}_k{}^i = \gamma_{jk}\phi^{ki}$
- interaction terms constructed by

$$\mathcal{H}_0 := \sum_{n=0}^{3} c_{4-n}e_n(\mathcal{K}), \quad \text{and} \quad \tilde{\mathcal{H}}_0 := \sum_{n=0}^{3} c_n e_n(\tilde{\mathcal{K}})$$

$$e_0(X) = 1,\ e_1(X) = [X],\ e_2(X) = \frac{1}{2}\left([X]^2 - \left[X^2\right]\right),\ e_3(X) = \det(X)$$

$c_i$ $(i = 0, \cdots, 4)$; dimensionless coupling constants

- extrinsic curvatures

$$K_{ij} := \frac{1}{2N}\left(\partial_t\gamma_{ij} - D_iN_j - D_jN_i\right), \quad \Phi_{ij} := \frac{1}{2M}\left(\partial_t\phi_{ij} - \tilde{D}_iM_j - \tilde{D}_jM_i\right)$$

## Hamiltonian formulation

$$
\begin{aligned}
H_{\rm MTBG} &= H_{\rm pre} - \int d^3x \Big[\lambda \left(C_0 - \tilde{C}_0\right) + \lambda^i \left(C_i - \beta \tilde{C}_i\right) \\
&\quad + \tilde{\lambda} \left( \sqrt{\gamma}\gamma^{ij} D_{ij} \left(\frac{C_0}{\sqrt{\gamma}}\right) + \sqrt{\phi}\phi^{ij}\tilde{D}_{ij}\left(\frac{\tilde{C}_0}{\sqrt{\phi}}\right)\right)\Big] , \\
H_{\rm pre} &= -\int d^3x \left(N\mathcal{R}_0 + N^i \mathcal{R}_i + M\tilde{\mathcal{R}}_0 + M^i \tilde{\mathcal{R}}_i\right)
\end{aligned}
$$

- 2nd-class Hamiltonian and momentum constraints

$$
\mathcal{R}_0 = \mathcal{R}_0^{\rm GR}[\gamma,\pi] - \frac{m^2}{2\kappa^2}\sqrt{\gamma}\mathcal{H}_0 \approx 0, \quad \tilde{\mathcal{R}}_0 = \tilde{\mathcal{R}}_0^{\rm GR}[\phi,\pi_\phi] - \frac{m^2}{2\kappa^2}\sqrt{\phi}\tilde{\mathcal{H}}_0 \approx 0,
$$

$$
\mathcal{R}_i = 2\sqrt{\gamma}\gamma_{ij}D_k\left(\frac{\pi^{jk}}{\sqrt{\gamma}}\right) \approx 0, \quad \tilde{\mathcal{R}}_i = 2\sqrt{\phi}\phi_{ij}\tilde{D}_k\left(\frac{\pi_\phi^{jk}}{\sqrt{\phi}}\right) \approx 0
$$

  except that $\mathcal{R}_i + \tilde{\mathcal{R}}_i \approx 0 \Longrightarrow$ a joint 3D diffeomorphism
- Lagrange multipliers $\lambda$, $\bar{\lambda}$, $\lambda^i$;
  associated with 5 additional 2nd-class constraints
- phase space dimensions; $24 - 2\times 3 - 1 \times 10 = 2 \times (2+2)$

# Lagrangian formulation

$$
\begin{aligned}
S_{\rm MTBG} &= \frac{1}{2\kappa^2}\int dtd^3x\,(\mathcal{L}_{\rm pre}+\mathcal{L}_{\rm con}+\mathcal{L}_m)\\
\mathcal{L}_{\rm pre} &:= \sqrt{-g}R[g]+\tilde{\alpha}^2\sqrt{-f}R[f]-m^2\left(N\sqrt{\gamma}\mathcal{H}_0+M\sqrt{\phi}\tilde{\mathcal{H}}_0\right)\\
\mathcal{L}_{\rm con} &:= \sqrt{\gamma}\alpha_{1\gamma}\left(\lambda+\Delta_\gamma\bar{\lambda}\right)+\sqrt{\phi}\alpha_{1\phi}\left(\lambda-\Delta_\phi\bar{\lambda}\right)\\
&+ \sqrt{\gamma}\alpha_{2\gamma}\left(\lambda+\Delta_\gamma\bar{\lambda}\right)^2+\sqrt{\phi}\alpha_{2\phi}\left(\lambda-\Delta_\phi\bar{\lambda}\right)^2\\
&- m^2\left[\sqrt{\gamma}U^i{}_k(\gamma,\phi)D_i\lambda^k-\beta\sqrt{\phi}\tilde{U}_k{}^i(\gamma,\phi)\tilde{D}_i\lambda^k\right]\\
\text{and}\quad \alpha_{1\gamma} &:= -m^2U^p{}_q(\gamma,\phi)K^q{}_p,\quad \alpha_{1\phi}:=m^2\tilde{U}^p{}_q(\gamma,\phi)\Phi^q{}_p,\\
\alpha_{2\gamma} &:= -\frac{m^4}{4N}\left(U^p{}_q(\gamma,\phi)-\frac{1}{2}U^k{}_k(\gamma,\phi)\delta^p{}_q\right)U^q{}_p(\gamma,\phi),\\
\alpha_{2\phi} &:= -\frac{m^4}{4M\tilde{\alpha}^2}\left(\tilde{U}_q{}^p(\gamma,\phi)-\frac{1}{2}\tilde{U}_k{}^k(\gamma,\phi)\delta_q{}^p\right)\tilde{U}_p{}^q(\gamma,\phi)
\end{aligned}
$$

- $\lambda$, $\bar{\lambda}$, and $\lambda^i$ as well as the metric variables contribute to dynamics as the auxiliary variables
- the same FLRW cosmology with HR bigravity

# Schwarzschild-de Sitter solutions

- Spatially-flat (Gullstrand-Painlevé) coordinates

$$\begin{aligned} g_{\mu\nu}dx^\mu dx^\nu &= -dt^2 + (dr + N^r(r)dt)^2 + r^2\theta_{ab}d\theta^a d\theta^b \\ f_{\mu\nu}dx^\mu dx^\nu &= C_0^2\Big[ - dt^2 + (dr + N_f^r(r)dt)^2 + r^2\theta_{ab}d\theta^a d\theta^b\Big] \\ \lambda &= \lambda(r), \qquad \bar{\lambda} = \bar{\lambda}(r), \qquad \lambda^r = \lambda^r(r), \qquad \lambda^a = 0 \end{aligned}$$

Equations for $\lambda$ and $\bar{\lambda}$

$$\left(C_0^2 c_1 + 2C_0 c_2 + c_3\right) \mathcal{F}\left(\lambda(r), \bar{\lambda}'(r), \bar{\lambda}''(r)\right) = 0,$$

$$\left(C_0^2 c_1 + 2C_0 c_2 + c_3\right) \mathcal{G}\left(\lambda(r), \lambda'(r), \lambda''(r), \bar{\lambda}''(r), \bar{\lambda}^{(3)}(r), \bar{\lambda}^{(4)}(r)\right) = 0$$

- Self-accelerating branch $C_0^2 c_1 + 2C_0 c_2 + c_3 = 0$

$$N^r(r) = \sqrt{\frac{2M}{r} + \frac{\Lambda_{(g)}}{3}r^2}, \quad N^r_{(f)}(r) = \sqrt{\frac{2M_f}{r} + \frac{C_0^2\Lambda_{(f)}}{3}r^2},$$

$$\lambda(r) = \lambda^r(r) = 0, \qquad \bar{\lambda}(r) = d_0 + \frac{d_1}{r}$$

$$\Lambda_{(g)} = \frac{m^2\left(-2C_0^3 c_1 - 3C_0^2 c_2 + c_4\right)}{2}, \quad \Lambda_{(f)} = \frac{m^2\left(c_0 + 2C_0^{-1} c_1 + C_0^{-2} c_2\right)}{2\tilde{\alpha}^2}$$

- GR stellar solutions with matter in the spatially-flat coordinates are solutions in the self-accelerating branch, satisfying the regularity boundary conditions at the center
- Normal branch; $\mathcal{F} = \mathcal{G} = 0$
  no solution in the spatially-flat coordinates
- Instead, in the normal branch, Schwarzschild-de Sitter metrics in the coordinates with $K = -3b_0 = \mathrm{constant}$ with mass $M$ and effective cosmological constant $\Lambda_{\mathrm{eff}}$ can be a solution

$$
\begin{aligned}
g_{\mu\nu}\, dx^\mu\, dx^\nu &= -F(r)\, dt^2 + \left[\frac{1}{\sqrt{F(r)}} dr + \left(b_0 r - \frac{\kappa_0}{r^2}\right) dt\right]^2 + r^2 \theta_{ab} d\theta^a d\theta^b \\
f_{\mu\nu}\, dx^\mu\, dx^\nu &= C_1^2 g_{\mu\nu} dx^\mu dx^\nu \\
F(r) &= 1 - \frac{2M}{r} - \frac{1}{3}\Lambda_0\, r^2 + \frac{\kappa_0^2}{r^4}
\end{aligned}
$$

where

$$
\Lambda_0 = \Lambda_{\mathrm{eff}} - 3b_0^2, \qquad \Lambda_{\mathrm{eff}} := \frac{1}{2} m^2 \left(c_1 C_1^3 + 3c_2 C_1^2 + 3c_3 C_1 + c_4\right),
$$

$$
c_0 C_1^3 + c_1 C_1^2 \left(3 - \tilde{\alpha}^2 C_1^2\right) + 3c_2 C_1 \left(1 - \tilde{\alpha}^2 C_1^2\right) + c_3 \left(1 - 3\tilde{\alpha}^2 C_1^2\right) - \tilde{\alpha}^2 c_4 C_1 = 0
$$

# GR collapsing solution in the spatially-flat coordinates

$$\begin{aligned} g^{(\pm)}_{\mu\nu}dx^\mu dx^\nu &= -dt^2 + \left(dr + N^r_{(\pm)}(t,r)dt\right)^2 + r^2\theta_{ab}d\theta^a d\theta^b, \\ f^{(\pm)}_{\mu\nu}dx^\mu dx^\nu &= C_0^2\left[-b^2 dt^2 + \left(dr + N^r_{(f)}(t,r)dt\right)^2 + r^2\theta_{ab}d\theta^a d\theta^b\right], \\ \lambda(t,r) &= \lambda^r(t,r) = 0, \qquad \bar{\lambda}(t,r) = \text{constant} \end{aligned}$$

- $b$; relative difference in the proper times
- exterior Schwarzschild regions

$$N^{r(+)} = \sqrt{\frac{r_{(g)}}{r}}, \qquad N^{r(+)}_{(f)} = \sqrt{\frac{r_{(f)}}{r}}$$

- interior spatially-flat FLRW regions with pressure-less dust

$$N^{r(-)} = N^{r(-)}_{(f)} = -r\frac{a_{,t}}{a(t)}$$

$$T^{\mu\nu}_{(g)} = \frac{\rho_{(g,0)}}{a(t)^3}v^\mu_{(g)}v^\nu_{(g)}, \quad T^{\mu\nu}_{(f)} = \frac{\rho_{(f,0)}}{a(t)^3}v^\mu_{(f)}v^\nu_{(f)} \Longrightarrow a(t) = c_{(g)}(-t)^{\frac{2}{3}}$$

- GR collapsing solution in the self-accelerating branch with $C_0^2 c_1 + 2C_0 c_2 + c_3 = 0$ and

$$\Lambda = \Lambda_f = 0 \quad \Longrightarrow \quad c_4 - 2C_0^3 c_1 - 3C_0^2 c_2 = 0, \quad C_0^2 c_0 + 2C_0 c_1 + c_2 = 0$$

- Matter interfaces follow the trajectories

$$(t, r) = \left(T_{(g)}(\tau_{(g)}), R_{(g)}(\tau_{(g)})\right), \qquad \left(T_{(f)}(\tau_{(f)}), R_{(f)}(\tau_{(f)})\right)$$

$$(-T_{(g)}) = \frac{2R_{(g)}^{3/2}}{3r_{(g)}^{1/2}}, \qquad (-T_{(f)}) = \frac{2R_{(f)}^{3/2}}{3r_{(f)}^{1/2}}$$

- $b$ is fixed by the ratio of the two energy densities

$$b^2 = \frac{\tilde{\alpha}^2}{C_0^2} \frac{\rho_{(g,0)}}{\rho_{(f,0)}}$$

No fine-tuning between the physical and fiducial sectors

# GR collapsing solution in the spatially-closed coordinates

Oppenheimer and Snyder (1939), Kanai, Siino, and Hosoya (2011)

$$
\begin{aligned}
g^{(+)}_{\mu\nu} dx^\mu dx^\nu &= -dt^2 + \frac{1}{1-q_{0,(g)}}\left(dr + \sqrt{\frac{r_{(g)}}{r} - q_{0,(g)}}\,dt\right)^2 + r^2\theta_{ab}d\theta^a d\theta^b, \\
f^{(+)}_{\mu\nu} dx^\mu dx^\nu &= C_0^2\left[-b^2 dt^2 + \frac{1}{1-q_{0,(f)}}\left(dr + \sqrt{\frac{r_{(f)}}{r} - b^2 q_{0,(f)}}\,dt\right)^2 + r^2\theta_{ab}d\theta^a d\theta^b\right] \\
g^{(-)}_{\mu\nu} dx^\mu dx^\nu &= -dt^2 + \frac{1}{1-\frac{r^2}{a(t)^2}}\left(dr - r\frac{\dot{a}(t)}{a(t)}dt\right)^2 + r^2\theta_{ab}d\theta^a d\theta^b, \\
f^{(-)}_{\mu\nu} dx^\mu dx^\nu &= C_0^2\left[-b^2 dt^2 + \frac{1}{1-\frac{r^2}{a(t)^2}}\left(dr - r\frac{\dot{a}(t)}{a(t)}dt\right)^2 + r^2\theta_{ab}d\theta^a d\theta^b\right]
\end{aligned}
$$

- interior spatially-closed FLRW with pressure-less dust

$$t(\eta) = \frac{\kappa^2}{6}\rho_{0,(g)}\left(\eta + \sin\eta\right), \quad a(\eta) = \frac{\kappa^2}{6}\rho_{0,(g)}\left(1+\cos\eta\right), \quad dt = a(\eta)d\eta$$

- fine-tuning between the physical and fiducial sectors

$$b = 1, \qquad \rho_{(f,0)} = \frac{\tilde{\alpha}^2}{C_0^2}\rho_{(g,0)} \quad \Longrightarrow \quad r_{(f)} = r_{(g)}$$

# Odd-parity perturbations in self-accelerating branch

$$
\begin{aligned}
g_{\mu\nu} dx^\mu dx^\nu &= -dt^2 + (dr + N^r(r)dt)^2 + r^2\theta_{ab} d\theta^a d\theta^b \\
&+ 2\sum_{\ell,m} \left(r^2 H_t(t,r)dt + rH_r(t,r)(dr + N^r(r)dt)\right) E_a{}^b \partial_b Y_{\ell m}(\theta^c) d\theta^a \\
&+ r^2 \sum_{\ell,m} H_3(t,r) E_{(a}{}^c \tilde{\nabla}_{b)} \tilde{\nabla}_c Y_{\ell m}(\theta^c) d\theta^a d\theta^b, \\
f_{\mu\nu} dx^\mu dx^\nu &= C_0^2 \Big[ - b^2 dt^2 + \left(dr + N^r_{(f)}(r)dt\right)^2 + r^2 \theta_{ab} d\theta^a d\theta^b \\
&+ 2\sum_{\ell,m} \left(r^2 K_t(t,r)dt + rK_r(t,r)(dr + N^r_{(f)}(r)dt)\right) E_a{}^b \partial_b Y_{\ell m}(\theta^c) d\theta^a \\
&+ r^2 \sum_{\ell,m} K_3(t,r) E_{(a}{}^c \tilde{\nabla}_{b)} \tilde{\nabla}_c Y_{\ell m}(\theta^c) d\theta^a d\theta^b \Big], \\
\lambda = 0, & \qquad \bar{\lambda} = 0, \quad \lambda^r = 0, \quad \lambda^a = \sum_{\ell,m} \Lambda(t,r) E^{ab} \partial_b Y_{\ell m}
\end{aligned}
$$

Schwarzschild-de Sitter background with $C_0^2 c_1 + 2C_0 c_2 + c_3 = 0$

$$
N^r(r) = \sqrt{\frac{\Lambda_{(g)} r^2}{3} + \frac{r_{(g)}}{r}}, \quad N^r_{(f)}(r) = \sqrt{\frac{C_0^2 b^2 \Lambda_{(f)} r^2}{3} + \frac{r_{(f)}}{r}},
$$

$$
\Lambda_{(g)} = \frac{m^2 \left(c_4 - 2C_0^3 c_1 - 3C_0^2 c_2\right)}{2}, \qquad \Lambda_{(f)} = \frac{\left(C_0^2 c_0 + 2C_0 c_1 + c_2\right) m^2}{2C_0^2 \tilde{\alpha}^2}
$$

## $\ell \geq 2$ modes

- gauge-invariant variables

$$H_t = \bar{H}_t + \frac{1}{2}\left(\dot{H}_3 - N^r H_3'\right), \quad H_r = \bar{H}_r + \frac{1}{2} r H_3', \quad K_3 = \bar{K}_3 + H_3,$$

$$K_t = \bar{K}_t + \frac{1}{2}\left(\dot{K}_3 - N^r_{(f)} K_3'\right), \quad K_r = \bar{K}_r + \frac{1}{2} r K_3', \quad \Lambda$$

- dynamical variables; $\chi_h$ and $\chi_k$

$$\bar{H}_t = \bar{H}_t(\chi_h), \quad \bar{H}_r = \bar{H}_r\left(\chi_h, \chi_k, \bar{K}_3, \Lambda\right),$$
$$\bar{K}_t = \bar{K}_t(\chi_k), \quad \bar{K}_r = \bar{K}_r\left(\chi_h, \chi_k, \bar{K}_3, \Lambda\right)$$

- shadowy modes; $\bar{K}_3$ and $\Lambda$

$$r^2 \bar{K}_3'' + \left\{4r - \frac{2r^3\left(\tilde{\alpha}^2 C_0^2 b + 1\right) m^2 (b-1)(C_0 c_1 + c_2)}{\left(C_0^2 \tilde{\alpha}^2 b + 1\right) r^2 (C_0 c_1 + c_2)(b-1) m^2 + 2b(\ell^2 + \ell - 2)\tilde{\alpha}^2}\right\} \bar{K}_3'$$
$$- \left\{(\ell+2)(\ell-1) + \frac{m^2 (b-1) r^2 \left(\tilde{\alpha}^2 b C_0^2 + 1\right)(C_0 c_1 + c_2)}{2\tilde{\alpha}^2 b}\right\} \bar{K}_3$$
$$= \mathcal{F}\left[\dot{\chi}_h', \chi_h'', \dot{\chi}_h, \chi_h', \chi_h; \dot{\chi}_k', \chi_k'', \dot{\chi}_k, \chi_k', \chi_k; \Lambda'', \Lambda', \Lambda\right],$$

$$r^2 \Lambda'' + 4r\Lambda' - (\ell+2)(\ell-1)\Lambda = 0$$

- $b \neq 1$; propagate with different speeds $c_h = 1$ and $c_k = b$

$$
\begin{aligned}
&\ddot{\chi}_h + M_1(r)\chi_h + M_2(r)\chi_k + L_1(r)\dot{\chi}_h' + L_2(r)\chi_h'' + L_3(r)\dot{\chi}_h + L_4(r)\chi_h' \\
&+P_1(r)\dot{\chi}_k' + P_2(r)\chi_k'' + P_3(r)\dot{\chi}_k + P_4(r)\chi_k' \\
&= Q_1(r)\dot{\bar{K}}_3' + Q_2(r)\bar{K}_3'' + Q_3(r)\bar{K}_3' + S_1(r)\dot{\Lambda}' + S_2(r)\Lambda'' + S_3(r)\Lambda', \\
&\ddot{\chi}_k + \tilde{M}_1(r)\chi_k + \tilde{M}_2(r)\chi_h + \tilde{L}_1(r)\dot{\chi}_k' + \tilde{L}_2(r)\chi_k'' + \tilde{L}_3(r)\dot{\chi}_k + \tilde{L}_4(r)\chi_k' \\
&+\tilde{P}_1(r)\dot{\chi}_h' + \tilde{P}_2(r)\chi_h'' + \tilde{P}_3(r)\dot{\chi}_h + \tilde{P}_4(r)\chi_h' \\
&= \tilde{Q}_1(r)\dot{\bar{K}}_3' + \tilde{Q}_2(r)\bar{K}_3'' + \tilde{Q}_3(r)\bar{K}_3' + \tilde{S}_1(r)\dot{\Lambda}' + \tilde{S}_2(r)\Lambda'' + \tilde{S}_3(r)\Lambda'
\end{aligned}
$$

- $b = 1$; decoupled and propagate with the same speeds

## $\ell = 1$ mode

for a slowly rotating Kerr-de Sitter solution in the physical sector

$$
H_t = -\frac{1}{1-(N^r)^2}\left(\omega_0 + \frac{2J_{(g)0}}{r^3}\right), \qquad H_r = \frac{rN^r}{1-(N^r)^2}\left(\omega_0 + \frac{2J_{(g)0}}{r^3}\right)
$$

the metric in the fiducial sector differs from it

$$
\begin{aligned}
K_t &= H_t + \frac{2}{r^3}\left(\frac{1-N^r N^r_{(f)}}{1-(N^r)^2}J_{(g)0} - J_{(f)0}\right) + \left(\frac{1-N^r N^r_{(f)}}{1-(N^r)^2}\omega_0 - k_3(t)\right) \\
K_r &= H_r,
\end{aligned}
$$

$J_{(g)0}$, $J_{(f)0}$; angular momenta

# Summary

- the Minimal Theory of Bigravity with $2+2$ DOFs
- static spherically symmetric GR solutions in the spatially-flat coordinates in the self-accelerating branch
- static spherically symmetric GR solutions in the coordinates with $K =$ constant in the normal branch
- GR collapsing solutions depending on the choice of the coordinates
- odd-parity perturbations governed by the two dynamical modes and sourced by the two elliptic (shadowy) modes

Future subjects

1. even-parity perturbations
2. dynamics in the normal branch
3. $\cdots$

## Muito Obrigado !