# Wormholes from Siegel modular forms in string theory: A black hole counting story

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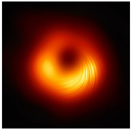
arXiv:2007.10302, 2112.10023, 2211.06873

TÉCNICO LISBOA

Black holes: gravitational solutions to Einstein's field equations.



Behave as thermodynamic systems w/ entropy:
(Bekenstein+Hawking in the 70's)
$A_H$: area of event horizon

$$S_{BH} = \frac{A_H}{4} + c_1 \log A_H + \frac{c_2}{A_H} + \cdots + \alpha_n e^{-\beta_n A_H} + \ldots$$

Boltzmann: black hole microstates $S_{BH} = \log d_{\text{micro}}$, $d_{\text{micro}} \in \mathbb{N}$
Central question in quantum gravity: microstates ? $d_{micro}$?
Goal:

$$Z(\phi) = \sum_{n \in \mathbb{Z}} d(n) e^{n\phi} \,, \quad d(n) = \int_C d\phi \quad Z(\phi) e^{-n\phi} \tag{1}$$

$$d \approx e^{\frac{A_H}{4}} \tag{2}$$

Invariant under symmetries.

# BPS black holes in string theory

- Heterotic string theory compactified on a six-torus:

  $\frac{1}{4}$ BPS black holes.

- BPS: supersymmetric. Asymptotically flat black holes.
- Single-centre black holes. Near horizon geometry is $AdS_2 \times S^2$,

$$ds_4^2 = v_* \left( -r^2 - 1)dt^2 + \frac{dr^2}{(r^2 - 1)} + d\Omega_2^2 \right)$$

- Dyonic black holes:

  electric-magnetic charges $(q_I, p^I)$ (several Maxwell fields $F^I$),

  charge bilinears $m = p \cdot p, \quad n = q \cdot q, \quad \ell = q \cdot p$

- Supported by complex scalar fields $Y^I$.

  Attractor mechanism: near horizon geometry supported by constant scalar fields $Y^I(q, p)$. $S_{BH}(q, p)$.

$$d(m, n, \ell), \quad m, n, \ell \in \mathbb{Z} \quad , \quad \Delta \equiv 4mn - \ell^2 > 0 \tag{3}$$

$$\log\ d(m, n, \ell) = \pi \sqrt{\Delta} + \cdots = \frac{A_H}{4} + \ldots \tag{4}$$

Number theoretic objects: Modular forms and Siegel modul forms.

$$\text{Complex upper half plane } \mathbb{D} = \{\tau \in \mathbb{C}, Im(\tau) > 0\} \tag{5}$$

$$f_k(\frac{a\tau + b}{c\tau + d}) \quad = \quad (c\tau + d)^k f_k(\tau), \ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in Sl(2, \mathbb{Z}) \tag{6}$$

# Siegel modular form of degree 2

Siegel's upper half plane $\mathcal{H}_2$:

$$\mathcal{H}_2 = \{\Omega \in \mathrm{Mat}(2\times 2, \mathbb{C}) : \Omega^T = \Omega\ ,\ \mathrm{Im}\Omega > 0\}$$

$$\Omega = \begin{pmatrix} \rho & v \\ v & \sigma \end{pmatrix} \quad , \quad \rho_2 > 0,\ \sigma_2 > 0,\ \det(\mathrm{Im}\Omega) > 0$$

Siegel modular group $Sp(4,\mathbb{Z})$ acts on $\mathcal{H}_2$ as

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathrm{Sp}(4,\mathbb{Z}) \quad , \quad A^T D - C^T D = I_2$$

$$\Omega \mapsto \Omega' = (A\Omega + B)(C\Omega + D)^{-1} \quad , \quad (\rho, \sigma, v) \mapsto (\rho', \sigma', v')$$

A Siegel modular form $\Phi_k$ of weight $k \in \mathbb{N}$ is a holomorphic function $\Phi_k : \mathcal{H}_2 \to \mathbb{C}$ s.t.

$$\Phi_k((A\Omega + B)(C\Omega + D)^{-1}) = \det(C\Omega + D)^k\, \Phi_k(\Omega)$$

# Number theory: meromorphic Siegel modular form

Heterotic string theory on $T^6$: $\frac{1}{4}$ BPS states.

Microstate degeneracies $d(m, n, \ell)$ given in terms of the Fourier coefficients of $1/\Phi_{10}$. $\Phi_{10}$ Igusa cusp form of weight 10.

Dijkgraaf, Verlinde, Verlinde, arXiv: 9607026

$$d(m, n, \ell) = \int_C d\sigma\, dv\, d\rho \, \frac{e^{-2\pi i(m\rho + n\sigma + \ell v)}}{\Phi_{10}(\rho, \sigma, v)}$$

Three contour integrations. Since $1/\Phi_{10}$ is meromorphic Siegel modular form, $d(m, n, \ell)$ depends on the choice of the integration contour $C$. $\Delta = 4mn - \ell^2$.

$1/\Phi_{10}$ captures degeneracies of single-centre ($\Delta > 0$) as well as of two-centre black holes ($\Delta < 0$). Ashoke Sen, arXiv:0705.3874

Need to select a contour $C$ that only captures single-centre degeneracies $d(m, n, \ell)$, with $\Delta > 0$.

# Rademacher expansion: a classic example

Dedekind's eta function $\eta : \mathbb{D} \to \mathbb{C}$ : $\quad \eta^{24}(q) = q \prod_{m=1}^{\infty} (1 - q^m)^{24}$

Meromorphic modular form of weight $k = -12$: $\quad q = e^{2\pi i \tau}, \mathrm{Im}\tau > 0$

$$\frac{1}{\eta^{24}(\tau)} = \sum_{n=-1}^{\infty} d(n)\, e^{2\pi i n \tau} \quad , \quad d(n) = \int_z^{z+1} \frac{e^{-2\pi i n \tau}}{\eta^{24}(\tau)}\, d\tau \quad , \quad n > 0$$

Polar coefficient: $d(-1) = 1$. For $n > 0$, modular properties:

$$\eta^{-24}(\tau) = (c\tau + d)^{-12}\, \eta^{-24}\left(\frac{a\tau + b}{c\tau + d}\right)$$

$$d(n) = \int_z^{z+1} \frac{e^{-2\pi i n \tau}}{\eta^{24}(\tau)}\, d\tau = d(-1) \frac{2\pi}{n^{13/2}} \sum_{c>0} \frac{K(n, -1, c)}{c}\, I_{13}\left(\frac{4\pi\sqrt{n}}{c}\right)$$

Rademacher expansion: polar coefficients, classical Kloosterman sums $K$, Bessel functions $I_{13}$

Uses: modular symmetry $SL(2, \mathbb{Z})$, Ford circles.

# Exact expression for $d(m,n,\ell)$ with $\Delta = 4mn - \ell^2 > 0$

**Theorem:** $d(m,n,\ell) \in \mathbb{N}$, $\tilde{\Delta} = 4m\tilde{n} - \tilde{\ell}^2 < 0$

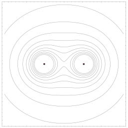
$$d(m,n,\ell) = (-1)^{\ell+1} \sum_{\gamma=1}^{+\infty} \sum_{\tilde{\ell} \in \mathbb{Z}/2m\mathbb{Z}} \left( 2\pi \sum_{\substack{\tilde{n} \geq -1, \\ \tilde{\Delta} < 0}} c_m^F(\tilde{n},\tilde{\ell}) \frac{\mathrm{Kl}(\frac{\Delta}{4m}, \frac{\tilde{\Delta}}{4m}, \gamma, \psi)_{\ell\tilde{\ell}}}{\gamma} \left(\frac{|\tilde{\Delta}|}{\Delta}\right)^{23/4} I_{23/2}\left(\frac{\pi}{\gamma m}\sqrt{\Delta|\tilde{\Delta}|}\right) \right.$$

$$- \delta_{\tilde{\ell},0} \sqrt{2m}\, d(m) \frac{\mathrm{Kl}(\frac{\Delta}{4m}, -1; \gamma, \psi)_{\ell 0}}{\sqrt{\gamma}} \left(\frac{4m}{\Delta}\right)^6 I_{12}\left(\frac{2\pi}{\gamma}\sqrt{\frac{\Delta}{m}}\right)$$

$$+ \frac{1}{2\pi} d(m) \sum_{\substack{g \in \mathbb{Z}/2m\gamma\mathbb{Z} \\ g = \tilde{\ell} \bmod 2m}} \frac{\mathrm{Kl}(\frac{\Delta}{4m}, -1 - \frac{g^2}{4m}; \gamma, \psi)_{\ell\tilde{\ell}}}{\gamma^2}$$

$$\left. \left(\frac{4m}{\Delta}\right)^{25/4} \int_{-1/\sqrt{m}}^{1/\sqrt{m}} dx' f_{\gamma,g,m}(x') (1 - mx'^2)^{25/4} I_{25/2}\left(\frac{2\pi}{\gamma\sqrt{m}}\sqrt{\Delta(1 - mx'^2)}\right) \right),$$

with



$$c_m^F(\tilde{n},\tilde{\ell}) = \sum_{\substack{a>0, c<0 \\ b \in \mathbb{Z}/a\mathbb{Z},\ ad-bc=1 \\ 0 \leq \frac{b}{a} + \frac{\tilde{\ell}}{2m} < -\frac{1}{ac}}} \Big((ad+bc)\tilde{\ell} + 2ac\tilde{n} + 2bdm\Big)\, d(c^2\tilde{n} + d^2 m + cd\tilde{\ell})\, d(a^2\tilde{n} + b^2 m + ab\tilde{\ell})$$

$$\frac{1}{\eta^{24}(\tau)} = \sum_{n=-1}^{\infty} d(n)\, e^{2\pi i \tau n},$$ two $SL(2,\mathbb{Z})$

# Exact Rademacher type expansion for $1/\Phi_{10}$

Area law:

$\gamma = 1,\ \tilde{n} = -1,\ \tilde{\ell} = m : \qquad d(m,n,\ell) \approx e^{\pi\sqrt{\Delta}} = e^{\frac{1}{4}A_H}$

Rademacher type expansion that we obtained arises as:

- A sum over residues of the quadratic poles of $1/\Phi_{10}$
- Expansion uses two $SL(2,\mathbb{Z})$ subgroups
- Expansion encoded in degeneracies of the perturbative $\frac{1}{2}$ BPS states! $c^F_m(\tilde{n},\tilde{\ell}) = \sum L\, d(M)\, d(N)$ bound state degeneracy
- Exponentially suppressed corrections: $e^{\pi\sqrt{\Delta|\tilde{\Delta}|}/\gamma m}$

Integrate $1/\Phi_{10}$ over $\rho$ , change of variables $(\sigma, v) \to (\tau_1, \tau_2)$, connect to macroscopics QEF:

$$d(m,n,\ell)_{\Delta>0} = \sum_{SL^2(2,\mathbb{Z}),\,\Sigma} \frac{e^{i\pi\varphi}}{\gamma} \frac{1}{(ac)^{13}} \int_{\Gamma_2} \frac{d\tau_2}{\tau_2^2} \left( \int_{\Gamma_1} d\tau_1 \, f(\tau_1,\tau_2) \right)$$

$$f(\tau_1,\tau_2) = \left[ 26 + \frac{2\pi}{n_2} \frac{\left(m(\tau_1^2+\tau_2^2)+n-\ell\tau_1\right)}{\tau_2} \right] \frac{e^{\frac{\pi}{n_2}\frac{m(\tau_1^2+\tau_2^2)+n-\ell\tau_1}{\tau_2}}}{\eta^{24}(\rho'_*)\,\eta^{24}(\sigma'_*)\,\tau_2^{12}}$$

$$\begin{aligned} \rho'_* &= -\frac{a}{c}\frac{\tau_1+i\tau_2}{\gamma} - \frac{b}{c}\frac{\alpha}{\gamma} - \frac{a}{c}\Sigma \\ \varphi &= \frac{2}{n_2}\left(-\tfrac{1}{2}j\ell - m_1 n + n_1 m\right) \\ \Gamma_1 : \quad \tau_1 &= \frac{\ell}{2m} + i\tau_2\,(-1+2y) \quad , \quad 0<y<1 \\ \Gamma_2 : \quad \tau_2 &= \frac{\sqrt{\Delta}}{2m} + i\,t \quad , \quad -\infty < t < \infty \end{aligned}$$

- Measure:

$$\left[26+\frac{2\pi}{n_2}\frac{\left(m(\tau_1^2+\tau_2^2)+n-\ell\tau_1\right)}{\tau_2}\right]\Big|_* = 26+2\,\frac{A_H}{4\,n_2}$$

absence of $\log A_H$ term in $N=4$ black hole entropy,

$S_{BH} = \frac{A_H}{4} + 0\ \log A_H + \ldots$ Banerjee, Jatkar, Sen, arXiv: 0810.3472

- Terms $\frac{1}{\eta^{24}(\rho'_*)\,\eta^{24}(\sigma'_*)\,\tau_2^{12}}$:

underlying bound state structure of Rademacher picture not manifest here $(c_m^F = \sum L\, d(M)\, d(N))$

Consider $n_2 = 1$: $\frac{1}{\eta^{24}(\tau)\,\eta^{24}(-\bar\tau)}\frac{1}{(\tau-\bar\tau)^{12}}$ how to account for this?

2D JT gravity point of view: $S_{JT} = -\frac{1}{8\pi G_2}(\frac{1}{2}\int_M d^2x\sqrt{g}R + \int_{\partial M} dt\sqrt{h}K) - \frac{1}{2}\int_M d^2x\sqrt{g}\Phi(R+\frac{2}{v_*}) - \int_{\partial M} dt\sqrt{h}(K-\frac{1}{\sqrt{v_*}}))$

# Macroscopic interpretation

Global Euclidean $AdS_2$: supported by constant dilaton field $\Phi_0 = v_*$

$$ds^2 = \frac{v_*}{\sin^2\sigma}\left(dT^2 + d\sigma^2\right) \quad , \quad -\pi < \sigma < 0 \quad , \quad T \cong T + 2\pi\tau_{2\,*} \; ,$$

Proposal:

Add 24 chiral + 24 antichiral periodic scalar fields (critical closed bosonic string), time-independent classical configuration: $T^{\rm cl}_{\mu\nu} = 0$.

1-loop partition function of periodic scalars:

$$Z^{1-{\rm loop}} = \frac{1}{\eta^{24}(\tau)\,\eta^{24}(-\bar\tau)}\,\frac{1}{(\tau-\bar\tau)^{12}}$$

$< T^{\rm quan}_{\mu\nu} > \neq 0$, backreacts on the dilaton

$$\Phi_0 + \Phi = \Phi_0 - 24\,\mathcal{E}\left[2\pi\tau_{2\,*}\right]\left(1 - \frac{\sigma + \frac{\pi}{2}}{\tan\sigma}\right) \quad , \quad -\pi < \sigma < 0$$

The resulting solution (trumpet + dilaton) is interpreted as an 2D Euclidean wormhole solution.

Garcia-Garcia, Godet, arXiv:2010.11633

# Conclusions

Summarizing:

- Rademacher picture: Finite seed data to generate black hole entropy.
- Macroscopic interpretation: 2D picture.
  - $\frac{1}{4}A_H$: JT gravity.
  - $\eta^{24}$ contributions: Euclidean wormhole on the double trumpet.
- Further Checks: HOLOGRAPHY! See Gabriel Cardoso's talk

Thanks!

Reproduce $d(m,n,\ell)$ by a suitable quantum gravity path integral, quantum entropy function (QEF). Ashoke Sen, arXiv:0805.0095, 0809.3304

- Functional integral over all fields in string theory in an Euclidean background $B$ that asymptotes to a specific Euclidean $AdS_2 \times S^2$ solution fixed by the attractor mechanism. $W = \sum_B W_B$
- Background $B$: supported by Abelian gauge potentials $A^I$, constant complex scalar fields $Y^I$
- Using supersymmetric localization: QEF is a finite-dimensional integral over $\{\phi^I\}$, where $Y^I = \frac{1}{2}(\phi^I + ip^I)$. Dabholkar, Gomes, Murthy, 2010
- QEF$\rightarrow$ Rademacher picture. $c^F_m(\tilde{n},\tilde{\ell})$ in measure.

Macroscopic interpretation:

- Saddle point analysis: $\tau_* = (\tau_1 + i\tau_2)_* = \frac{\ell}{2m} + i\frac{\sqrt{\Delta}}{2m}$

$$\frac{\pi}{n_2}\frac{\left(m(\tau_1^2+\tau_2^2)+n-\ell\tau_1\right)}{\tau_2}\Big|_* + \pi i\varphi = \frac{1}{n_2}\left[\frac{A_H}{4} + 2\pi i\left(-\tfrac{1}{2}j\ell - m_1 n + n_1 m\right)\right]$$

# Macroscopic interpretation

- Semi-classical interpretation in terms of sums over space-time backgrounds: $\mathbb{Z}_{n_2}$ orbifolds of Euclidean $AdS_2 \times S^2$

$$ds^2 = v_* \left( (r^2 - 1) d\theta^2 + \frac{dr^2}{r^2 - 1} + d\psi^2 + \sin^2 \psi \, d\phi^2 \right)$$
$$0 \leq \theta < \frac{2\pi}{n_2} \quad , \quad 0 \leq \phi < \frac{2\pi}{n_2},$$

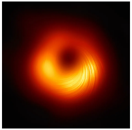
supported by gauge potentials $A^I_\theta$ that acquire a constant real part when orbifolding,

$$A^I_\theta = -ie^I_* \, (r - 1) \, d\theta + \mathrm{Re} A^I_\theta$$

$$\frac{1}{n_2} \left[ \frac{A_H}{4} + \pi i \left( q \cdot \mathrm{Re} A_\theta - p \cdot \mathrm{Re} \tilde{A}_\theta \right) \right]$$

$(\mathrm{Re} A_\theta, \mathrm{Re} \tilde{A}_\theta)$ expressed in terms of $(q_I, p^I; m_1, n_1, j)$ , symplectic vector under S-duality

# Three approaches to BPS black hole entropy



1. Number theory:
   $d(m, n, \ell)$: meromorphic Siegel modular form.
   Exact expression as a Rademacher type expansion. C, Nampuri, Rosselló, arXiv: 2112.10023

2. Quantum entropy function: Ashoke Sen, arXiv:0805.0095
   $d(m, n, \ell)$ from a quantum gravity path integral: sum over space-time geometries that asymptote to a product geometry $AdS_2 \times S^2$.

3. Conformal quantum mechanics:
   $AdS_2/CFT_1$ correspondence: Maldacena, arXiv:9711200
   $d(m, n, \ell)$ from a conformal quantum mechanics model (DFF model). de Alfaro, Fubini, Furlan, 1976