8th International Work shop on Astronomy and Relativistic Astrophysics 2018 Peru

Equations of boson-fermion star and the basic equation discussions under Newtonian approximation

Zhi-Fu Gao   **Ci-Xin Chen***

** Xinjiang Astronomical Observatory, Chinese Academy of Sciences, Urumqi, Xinjiang, China. email: zhifugao@xao.ac.cn

*Department of Astronomy, University of Sciences and Technology of China, Chinese Academy of Sciences, Hefei, Anhui, China

Outline

- Introduction
- Fluid Boson Stars
- Equilibrium Equation of Boson-Fermi System in Newtonian Approximation
- Viral Equations for a Boson-Fermi system
- Summarry

The Standard model of particle physics

Sheldon Lee Glashow
Prize share: 1/3

Abdus Salam
Prize share: 1/3

Steven Weinberg
Prize share: 1/3

- The Standard model of particle physics takes quark model as structure carrier, and is built and developed gradually on the basis of the unified theory of electro-weak and quantum chromodynamics. Glasow et al. are the founders 。

- The Standard Model is a theory concerning the electromagnetic, weak and strong nuclear interactions. It divides particles into the two categories of fermions(including quarks and leptons) and bosons (including gluons, photons, W and Z bosons and Higgs boson) according to their and can explain the properties and interactions of elementary particles 。

- All the particles but the Higgs boson have been experimentally supported and validated. The Higgs boson has no spin or electric charge, but has mass。

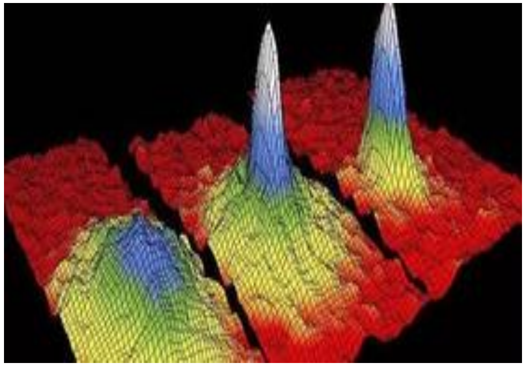
- The differences in their spins make fermions and bosons have completely different properties. A fermion has a semi-integer spin and obeys the Pauli exclusion principle. The boson has integer spin and does not follow the Pauli exclusion principle。

Bose-Einstein Condensate

- **In 1924, a young Indian physicist S. Bose proposed a new idea of distinguishable identical particles. Einstein extended Bose's statistical method on photons to atoms, This is what we call Bose-Einstein condensate (BEC).**

- **In 1938,London proposed that the superfluid of liquid helium (He4) was essentially a quantum statistical phenomenon, Until the end of last century, this research made a breakthrough.**

- **Recently, the study on BEC has developed rapidly and a series of new phenomena were observed, such as the coherence in BEC, Josephson effect, spiral, ultra-cold Fermi atomic gas. In neutron stars, superconductivity and superfluid belong to BEC phenomenon.**。



3 Nobel price honors

Stevn Chu; Claude Cohen-Tannoudji; William Daniel Phillips

The concept of Bose stars

- The concept of Bose stars was first proposed by Ruffini and Bonazzola in 1969.   It is generally believed to be a dense star formed by the collapse of a boson cloud with a spin of 0 under the action of self-gravity.

- Bose stars contain at least one scalar field are considered as a macroscopic BEC phenomenon under the action of gravity. If a scalar field exists in nature, it is possible to form a gravitational binding through the Jeans instability.

- There is no degenerate pressure inside the planet, but it will not collapse indefinitely. The way that Boson stars prevent infinite gravitational collapse is by the Heisenberg uncertainty principle。

Fluid Boson stars

2.1 Scale fields describing Boson stars

The Lagrangian density of a complex scalar field coupled to its own gravitational field reads

$$L = \frac{R}{16\pi G} - \partial_\mu \Phi^* \partial^\mu \Phi - m^2 \Phi^* \Phi, \tag{1}$$

where $R := g^{\mu\nu} R_{\mu\nu}$ is the Ricci curvature scale ($g^{\mu\nu}$ is the metric tensor and $\mu, \nu = (0,1,2,3)$, $R_{\mu\nu}$ is the Ricci tensor), G is the Newton gravitational constant in natural units, and the asterisk in the equation above denotes complex conjugation, ∂_μ denotes the covariant derivative and m is the mass of a boson. Using the variational principle, we obtain a coupled Einstein-Klein-Gordon equation

$$G_{\mu\nu} := R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = -8\pi G T_{\mu\nu}, \tag{2a}$$

$$\frac{1}{\sqrt{-g}} \partial_\mu (g^{\mu\nu} \sqrt{-g} \partial_\nu \Phi) - m^2 \Phi = 0, \tag{2b}$$

where $T_{\mu\nu}$ is the energy-momentum tensor

$$T_{\mu\nu} = \partial_\mu \Phi^* \partial_\nu \Phi + \partial_\mu \Phi \partial_\nu \Phi^* - g_{\mu\nu} (\partial_\alpha \Phi^* \partial^\alpha \Phi + m^2 \Phi^* \Phi). \tag{3}$$

Here Eq.(2a) describes the properties of space-time in the stellar immediate vicinity, while Eq. (2b) gives an equation of state of the scalar field. The physical state of a boson star is usually described by the Einstein-Klein-Godon equation of a scalar field coupling to the gravitational field

$$ds^2 = -B(r)dt^2 + A(r)dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2), \tag{4}$$

2.2 The ground state solution to Einstein-Klein-Gordon equation in a scalar field

Employing a covariant divergence formula, then Eq.(2b) is rewritten as

$$\frac{1}{\sqrt{-g}}\partial_l(g^{lk}\sqrt{-g}\partial_k\Phi)+\frac{1}{B(r)}\partial_0^2\Phi-m^2\Phi=0. \tag{7}$$

when we investigate a solution to the Klein–Gordon equation, the scalar field $\Phi(r,t)$ is usually treated as a function of separable variables. Considering a spherically symmetric boson star, it has

$$\Phi(r,t)=R(r)Y_l^m(\theta,\phi)e^{-i\omega t}, \tag{8}$$

where $Y_l^m(\theta,\phi)$ is a spherical harmonics function. By introducing a method of second quantization of a scalar field, the general solution to Eq. (7) is given as

$$\Phi(r,t)=\sum a_{nlm}R_{nl}(r)Y_m^l e^{-i\omega t}+\sum b_{nlm}R_{nl}(r)Y_m^{l*}e^{i\omega t}. \tag{9}$$

where $a_{nlm}=b_{nlm}^{\dagger}$, i.e., a_{nlm} and b_{nlm} are a pair of Hermitian conjugate operators. Since all the particles are in the ground state, i.e., $n=1$ and $l=0$, and the scale field is limited to a real scale field, i.e., $\Phi=\Phi^*$, then there is a relation of $a_{nlm}=b_{nlm}$, which is obtained from Eq. (9). Inserting Eq. (8) into Eq.(7) yields a radial component $R(r)$ of Φ,

$$\frac{dR_{nl}^2}{dr^2}+(\frac{2}{r}+\frac{dB/dr}{2B}-\frac{dA/dr}{2A})\frac{dR_{nl}}{dr}+A[\frac{w^2}{B}+m^2-\frac{l(l+1)}{r^2}]R_{nl}=0\,. \tag{10}$$

Considering the ground state of scale field, the expression above is simplified as

$$\frac{dR_{10}^2}{dr^2}+\left(\frac{2}{r}+\frac{dB/dr}{2B}-\frac{dA/dr}{2A}\right)\frac{dR_{10}}{dr}+A\left(\frac{w^2}{B}+m^2\right)R_{10}=0. \quad (11)$$

Treating the energy-momentum-tensor, $T_{\mu\nu}$, as an operator, one can obtain an average value of $T_{\mu\nu}$ in the ground state $|G>=|N,0,0,0,\ldots>$ (Camenzind 2007; Mukhnaov & Winitzki,2007),

$$<G|T_0^0G|>=\frac{-1}{2m}N\left[\frac{E_{10}^2}{B}-m^2\right)R_{10}^2+\frac{(dR_{10}/dr)^2}{A}], \quad (12a)$$

$$<G|T_1^1G|>=\frac{1}{2m}N\left[\frac{E_{10}^2}{B}+m^2\right)R_{10}^2+\frac{(dR_{10}/dr)^2}{A}], \quad (12b)$$

$$<G|T_2^2G|>=\frac{1}{2m}N\left[\frac{E_{10}^2}{B}+m^2\right)R_{10}^2+\frac{(dR_{10}/dr)^2}{A}], \quad (12c)$$

$$<G|T_3^3G|>=\frac{1}{2m}N\left[\frac{E_{10}^2}{B}+m^2\right)R_{10}^2+\frac{(dR_{10}/dr)^2}{A}], \quad (12d)$$

where $E_{10}=w$ denotes the eigenenergy of the ground state. In the same way, if the conserved current, J^{μ}, in Eq.(5) is treated as an operator, we can obtain an average value of $J_{\mu\nu}$ in the ground state,

$$<G|J^0|G>=E_{01}N\frac{R_{10}^2}{mB(r)}. \quad (13)$$

Inserting Eq.(11) and Eq.(12a) into the Einstein field equation of Eq.(2a), we obtain two independent equations

$$\frac{dA/dr}{A^2r} + \frac{1-1/A}{r^2} = \frac{4\pi GN}{m}[(\frac{E_{10}^2}{B} - m^2)R_{10}^2 + \frac{(dR_{10}/dr)^2}{A}] \ , \qquad (14)$$

and

$$\frac{-dB/dr}{ABr} + \frac{1-1/A}{r^2} = \frac{4\pi GN}{m}[(\frac{E_{10}^2}{B} + m^2)R_{10}^2 + \frac{(dR_{10}/dr)^2}{A}] \ . \qquad (15)$$

Combining the Bianchi identity with Eqs. (12b,c) and (13-15), one obtain a normalization relation,

$$\int \sqrt{-g} < G|J^0|G > d^3x = N \ . \qquad (16)$$

Inserting Eq. (13) into Eq. (16), we have

$$\int \frac{4\pi E_{10}}{m} R_{10}^2 \sqrt{\frac{A}{B}} r^2 dr = 1 \ . \qquad (17)$$

Using a numerical simulation method, we get a relation of R_{10} and r,

$$\frac{d^2R_{10}}{dr^2} + (\frac{2}{r} + \frac{\frac{dB}{dr}}{2B} - \frac{\frac{dA}{dr}}{2A})\frac{dR_{10}}{dr} + A[\frac{w^2}{B} + m^2]R_{10} = 0 \ , \qquad (18)$$

Equations of (11),(14),(15),(17) and (18) collectively describe the ground-state properties of a complex scalar field.

The properties of a fluid Boson star

A fluid boson star discussed here refers to a scalar field system mixed with fermions, Fermions inside a boson-fermion star are always treated as an ideal fluid, whose energy-momentum tensor is written as

$$T_{\mu\nu} = (P+\rho)u_\mu u_\nu + g_{\mu\nu}P, \tag{19}$$

where P is the fluid pressure, ρ is the fluid density and $u^\mu = dx^\mu/d\tau$ is a four-dimensional speed. For a static fluid $u^i = (u^0, 0, 0, 0)$, since $ds^2 = g_{00}(dx^0)^2$. and $u^0 = dx^0/d\tau = -dx^0/ds$, we have

$$u_0 = g_{0i}u^i = g_{00}u^0 = \sqrt{-g_{00}}\,, \quad u_1 = u_2 = u_3 = 0. \tag{20}$$

Utilizing the metric of Eq.(4), we express Eq.(19) as

$$T_{\mu\nu} = \begin{pmatrix} \rho B(r) & 0 & 0 & 0 \\ 0 & PA(r) & 0 & 0 \\ 0 & 0 & r^2 P & 0 \\ 0 & 0 & 0 & r^2 \sin^2\theta P \end{pmatrix} \tag{21}$$

Inserting Eq.(21) into the Einstein field equation, we get an equilibrium equation for the Fermi fluid,

$$\frac{dP}{dt} = -\frac{P+\rho}{2}\frac{dB(r)}{dr}\frac{1}{B}\,. \tag{22}$$

Boson Fermi stars

Assuming that the system is composed of a cold boson-fermion fluid, the total energy-momentum tensor could be expressed as the sum of two terms,

$$T_{\mu\nu} = T_{\mu\nu}^{(B)} + T_{\mu\nu}^{(F)}, \tag{28}$$

where $T_{\mu\nu}^{(B)}$ denotes the energy-momentum-tensor of a real scale field and $T_{\mu\nu}^{(F)}$ denotes the energy-momentum tensor of a strongly degenerate Fermi fluid. Inserting Eq. (3) and Eq. (21) into Eq. (28), we have

$$\begin{aligned} T_{\mu\nu}^{(F)} &= (P+\rho)u_\mu u_\nu + g_{\mu\nu}P , \\ T_{\mu\nu}^{(B)} &= \partial_\mu \Phi^* \partial_\nu \Phi + \partial_\mu \Phi \partial_\nu \Phi^* - g_{\mu\nu}(\partial_\alpha \Phi^* \partial^\alpha \Phi + m^2 \Phi^* \Phi) \ . \end{aligned} \tag{29}$$

Combining Eq.(12) with Eqs.(21) and (29), for a spherically symmetric static boson-fermion star in the ground state, its energy-momentum tensor is given by

$$< T_0^0 >= -\frac{1}{2m} N[(\frac{E_{10}^2}{B} - m^2)R_{10}^2 + \frac{(\frac{dR_{10}}{dr})^2}{A}] + B_\rho \ , \tag{30a}$$

$$< T_1^1 >= \frac{1}{2m} N[(\frac{E_{10}^2}{B} + m^2)R_{01}^2 + \frac{(\frac{dR_{10}}{dr})^2}{A}] + AP \ , \tag{30b}$$

$$< T_2^2 >= \frac{1}{2m} N[(\frac{E_{10}^2}{B} + m^2)R_{01}^2 - \frac{(\frac{dR_{10}}{dr})^2}{A}] + r^2 P \ , \tag{30c}$$

$$< T_3^3 >= \frac{1}{2m} N[(\frac{E_{10}^2}{B} + m^2)R_{10}^2 - \frac{(\frac{dR_{10}}{dr})^2}{A}] + r^2 \sin^2\theta P \ . \tag{30d}$$

Inserting Eq.(30) into the Einstein field equation of Eq.(2a) and making use of variable substitution:

$x := mr$, $\sigma(x) := \sqrt{8\pi G}R(r)$, $w := \frac{\omega}{m}$, $\overline{\rho}(t) := \frac{4\pi G}{m^2}\rho(t)$ and $\overline{P}(t) := \frac{4\pi G}{m^2}P(t)$, we obtain

$$A' = xA^2[2\overline{\rho} + (\frac{w^2}{B} + 1)\sigma^2 + \frac{\sigma'^2}{A}] - \frac{A}{x}(A-1) \quad , \tag{31}$$

and

$$B' = xAB[2\overline{P} + (\frac{w^2}{B} - 1)\sigma^2 + \frac{\sigma'^2}{A}] + \frac{B}{x}(A-1) \quad , \tag{32}$$

where the apostrophe signifies the derivative of x. Comparing Eqs.(31,32) with Eqs.(14,15), it is easy to see that the Einstein field equation becomes different when a scalar field is coupled with an ideal Fermi fluid Inserting all of the variable substitutions above into Eq.(7) yields

$$\sigma'' = -[\frac{2}{x} + \frac{1}{2}(\frac{B'}{B} - \frac{A'}{A})]\sigma' - A(\frac{w^2}{B} - 1)\sigma \quad , \tag{33}$$

where $'' := d^2/d_x^2$.

Substituting Eq.,(35) into Eq.(34), we have

$$t' = -2\frac{B'}{B}\frac{\sinh t - 2\sinh(\frac{t}{2})}{\cosh t - 4\cosh(\frac{t}{2}) + 3} \quad . \tag{36}$$

For the fermions, if we apply the above variable substitutions to Eq.(26) of the fluid equilibrium, then have

$$t' = -\frac{dt}{d\overline{P}}\frac{B'}{2B}\left(\overline{P}+\overline{\rho}\right) \ , \tag{34}$$

where

$$\overline{K} = \overline{K}(m, m_n), \ \ \overline{\rho} = \overline{K}(\sinh t - t), \text{ and } \overline{P} = \frac{\overline{K}}{3}\left(\sinh t - 8\sinh\frac{t}{2} - 3t\right) \ . \tag{35}$$

Substituting Eq.,(35) into Eq.(34), we have

$$t' = -2\frac{B'}{B}\frac{\sinh t - 2\sinh(\frac{t}{2})}{\cosh t - 4\cosh(\frac{t}{2}) + 3} \ . \tag{36}$$

Equilibrium Equation of Boson-Fermi Star in Newtonian Approximation

The metric of V in the Newtonian approximation is given by

$$g_{\mu\nu} = \begin{bmatrix} -1-2V & 0 & 0 & 0 \\ 0 & 1-2V & 0 & 0 \\ 0 & 0 & 1-2V & 0 \\ 0 & 0 & 0 & 1-2V \end{bmatrix}$$

3.2The flow conservation equation of a Boson system

A scalar field $\Phi(r,t)$ is treated as by the separation of variables, $\Phi(r,t) = \phi(r)e^{-iwt}$, then one gets $\Phi_{,0} \propto w\Phi$. It is reasonably assumed that

$$\frac{w}{m} \ll 1 + O(\varepsilon^{1/2}) \ , \ \text{and} \ |T^{00}| \gg |T^{0i}| \gg |T^{ij}| \qquad (53)$$

for a weak field approximation (Silverira & Sousa 1995). Inserting the metric of gravitation potential V into $T^{\mu\nu}$, the energy-momentum tensor in the ground state

$$T^{00} = (1-4V)\Phi^*_{,0}\Phi_{,0} + \sum_i \Phi^*_{;i}\Phi_{,i} + m^2(1-2V)\Phi^*\Phi \,. \qquad (54)$$

Using the assumptions in Eq. (53) Eq. (54) is written as

$$T^{00} = \frac{1}{2}(m^2 + w^2)\Phi^*\Phi \approx m^2\Phi^*\Phi \,. \qquad (55)$$

where V is considered as a small quantity. Compared with $\nabla^2 V = 4\pi G\rho$, the density in a Boson system is then defined as $\rho^{(B)} := m^2\Phi^*\Phi$. The covariant divergence of the energy-momentum tensor in Eq. (3) is given by

$$T^{i\nu}_{;\nu} = \frac{1}{\sqrt{-g}}\partial_\nu[\sqrt{-g}(\partial^\mu\Phi\partial^\nu\Phi^* + \partial^\mu\Phi^*\partial^\nu\Phi)]$$
$$+\Gamma^\mu_{\nu\alpha}(\partial^\alpha\Phi\partial^\nu\Phi^* + \partial^\alpha\Phi^*\partial^\nu\Phi) - g^{\mu\nu}\partial_\nu(\partial^\alpha\Phi\partial_\alpha\Phi^* + m^2\Phi\Phi^*)\,. \quad (56)$$

Substituting the approximation relation $\Phi_{,i}/\Phi_{,0} \sim O(\varepsilon^{1/2})$ (Silveira & de Sousa 1995) into Eq. (56), we have

$$T^{0\nu}_{;\nu} = O(\varepsilon^{1/2}) + 2m^3\Phi^*\Phi + 4m^3\Phi^*\Phi \sim 6m^3\Phi^*\Phi\,, \quad (57a)$$

$$T^{i\nu}_{;\nu} \approx -m^2\partial_i(\Phi^*\Phi) + m^2\Phi^*\Phi\partial_i V + O(\varepsilon^2) - \partial_i(m^2\Phi^*\Phi) + O(\varepsilon^2)$$
$$\approx -2m^2\partial_i(\Phi^*\Phi) + m^2\Phi^*\Phi\partial_i V + O(\varepsilon^2)\,. \quad (57b)$$

In the same way, we get the energy-momentum tensor $T^{\mu\nu}_{;\nu} = 0$ and the conservation equation

$$6m^3\Phi^*\Phi = 0\,, \quad (58a)$$

$$-2m^2\partial_i(\Phi^*\Phi) + m^2\Phi^*\Phi\partial_i V = 0\,, \quad (58b)$$

from the Bianchi identity. This equation describes the properties of flow equilibrium equation for a scalar field in Newtonian approximation.

3.3 The flow conservation equation of a Boson-Fermi system system in the Newtonian approximation

The total energy momentum tensor of a Bose-Fermi system can be simply expressed as the sum of two terms. The flow conservation of the system gives

$$T_{\mu\nu} = T_{\mu\nu}^{(B)} + T_{\mu\nu}^{(F)} = 0 \ . \tag{59}$$

Combining Eq.(59) with Eqs.(52) and (58), we have

$$-\frac{\partial P}{\partial t} + \frac{\partial}{\partial t}\frac{P+\rho}{1-|v|^2} + \nabla\cdot(\frac{P+\rho}{1-|v|^2}\vec{v}) + 6m^3\Phi^*\Phi = 0\ , \tag{60a}$$

$$\nabla P + \vec{v}\frac{\partial}{\partial t}\frac{P+\rho}{1-|v|^2} + (\vec{v}\cdot\nabla)(\frac{P+\rho}{1-|v|^2}\to v) + \frac{P+\rho}{1-|v|^2}(\nabla\cdot\vec{v})\vec{v} +$$
$$\frac{P+\rho}{1-|v|^2}(1+|v|^2)\nabla V - 4\frac{P+\rho}{1-|v|^2}(\nabla V\cdot\vec{v})\vec{v} - 2m^2\partial_i(\Phi^*\Phi) + m^2\Phi^*\Phi = 0\ . \tag{60b}$$

where $V = V_{(F)} + V_{(B)}$ denotes the total gravitational potential of the system. Inserting Eq. (60a) into Eq. (60b), we get

$$\nabla P + \frac{\partial P}{\partial t}\vec{v} - 6m^3\Phi^*\Phi\vec{v} + \frac{P+\rho}{1-|v|^2}\frac{\partial\vec{v}}{\partial t} + \frac{P+\rho}{1-|v|^2}(\vec{v}\cdot\nabla)\vec{v} +$$
$$\frac{P+\rho}{1-|v|^2}(1+|v|^2)\nabla V - 4\frac{P+\rho}{1-|v|^2}(\nabla V\cdot\vec{v})\vec{v} - 2m^2\nabla(\Phi^*\Phi)\partial_i V + m^2\Phi^*\Phi\nabla V = 0\ . \tag{61}$$

For the speed is non-relativistic, e.g., $v^2 \sim O(\varepsilon)$, all the quadratic terms of speed in Eq. (61) can be ignored and Eq. (61) is simplified as

$$\nabla P - \frac{\partial P}{\partial t}\vec{v} - 6m^3\Phi^*\Phi\vec{v} + (P+\rho)\nabla V - 2m^2\nabla(\Phi^*\Phi)\partial_i V + m^2\Phi^*\Phi\nabla V = 0\,. \tag{62}$$

In non-relativistic Newtonian approximation, $P \ll \rho$ and $\frac{d\vec{v}}{dt} = \frac{\partial \vec{v}}{\partial t} + (\vec{v}\cdot\nabla)\vec{v}$ (Pijush et al. 2011). Thus we obtain the fluid motion equation for a Boson-Fermi system,

$$\frac{d\vec{v}}{dt} = -\frac{1}{\rho}[\nabla P - 2m^2\nabla(\Phi^*\Phi) - 6m^3\Phi^*\Phi\vec{v}] - (1 + \frac{m^2\Phi^*\Phi}{\rho})\nabla V\ . \tag{63}$$

Comparing Eq. (63) with the motion equation of a Fermi system,

$$\frac{d\vec{v}}{dt} = -\frac{1}{\rho}\nabla P - \nabla V\ , \tag{64}$$

in the non-relativistic Newtonian approximation (Pijush et al. 2011), we find that the motion equation of a relativistic Fermi fluid has changed considerably after coupling the scalar field.

Viral Equations for a Boson-Fermi system

To investigate the equilibrium geometry of a Boson-Fermi system, it is necessary to construct viral equations with any order for the system. If we multiply both sides of Eq.(63) by ρ , then have

$$\rho \frac{du_i}{dt} = -[\frac{\partial P}{\partial x_i} - 2\frac{\partial (m^2\Phi^*\Phi)}{\partial x_i} - 6m^3\Phi^*\Phi u_i] - (\rho + m^2\Phi^*\Phi)\frac{\partial V}{\partial x_i} \ . \tag{65}$$

In order simplify Eq. (65), we define the energy density of Bosons $\rho_\phi := m^2\Phi^*\Phi$, denote ρ as ρ_f as, and then integrate both sides of the equation over the whole space. Thus we get

$$\int_\infty \rho_f \frac{du_i}{dt} d^3x = -\int_\infty (\frac{\partial P}{\partial x_i} - 2\frac{\partial \rho_\phi}{\partial x_i} - 6m\rho_\phi u_i) d^3x - \int_\infty (\rho_f + \rho_\phi)\frac{\partial V}{\partial x_i} d^3x \ . \tag{66}$$

For an isolated Boson-Fermi star, its total mass, M, is conserved,

$$\frac{d}{dt}M = \frac{d}{dt}\int_\infty (\rho_\phi + \rho_f) d^3x = 0\,. \tag{67}$$

Substituting Eq. (67) into Eq. (66) and considering boundary conditions, we get

$$\frac{d}{dt}\int_\infty \rho_f u_i d^3x = 6m\frac{dI_i(\phi)}{dt} - G\int\int_\infty \rho_f(x)\rho_f(x')\frac{x-x'}{|x-x'|^3} d^3x' d^3x - G\int\int_\infty \rho_\phi(x)\rho_\phi(x')\frac{x-x'}{|x-x'|^3} d^3x' d^3x, \tag{68}$$

where $I_i(\phi) = \int_V \rho_\phi(x) x_i d^3x$ is the first order torque of moment of inertia. The antisymmetry of the integrals of x and x' gives

$$\int_\infty \int_\infty \rho_f(x)\rho_f(x')\frac{x-x'}{|x-x'|^3}d^3xd^3x' = 0\,, \quad \int_\infty \int_\infty \rho_\phi(x)\rho_\phi(x')\frac{x-x'}{|x-x'|^3}d^3xd^3x' = 0\,. \tag{69}$$

Inserting Eq. (69) into Eq. (68), we have

$$\frac{d}{dt}\int_\infty \rho_f u_i d^3x = 6m\frac{dI_i(\phi)}{dt}\ , \tag{70}$$

where the term of $6m\frac{dI_i(\phi)}{dt}$ is caused by a scalar field. The integral range of Eq. (70) has changed from the full space to the boundary of Fermi fluid, because there is no scalar field term on its left side. Then, we get the first order viral equation of fluid motion,

$$\frac{d}{dt}\int_\infty \rho_f u_i d^3x = \frac{d}{dt}\int_V \rho_f u_i d^3x = 6m\frac{dI_i(\phi)}{dt}\ , \tag{71}$$

which works only on a Boson-Fermi system, while $\frac{d}{dt}\int_V \rho_f u_i dx = 0$ for a pure Fermi fluid system.

Then we get the second order viral equation of a Boson-Fermi system fluid,

$$\frac{d}{dt}\int_V \rho_f u_i x_j d^3x = 2E_{ij} + \delta_{ij}\prod - 2\delta_{ij}M_\phi + 6m\frac{dI_{ij}(\phi)}{dt} - P_{ij} \ . \tag{75}$$

Compared Eq.(72) with Eq.(75), it is found that there are more terms on the Fermi fluid in the latter. Using a similar approach, we can directly give more than three order viral equation of a Boson-Fermi system,

$$\frac{d}{dt}\int_V \rho_f u_i x_j x_k dx - 6m\int_\infty \rho_\phi u_i x_j x_k dx$$
$$= 2(E_{ij;k} + E_{ik;j}) + \delta_{ij}\prod_k + \delta_{ik}\prod_j - 2\delta_{ij}M_{\phi k} - 2\delta_{ik}M_{\phi j} + 6m\frac{dI_{(\phi)i}}{dt} - P_{ij;k} - P_{ik;j} \ . \tag{76}$$

In the previous works (e.g., Chandrasekhar 1969), by using viral equations, the equilibrium configuration of a fluid Fermi star has been investigated in detail are discussed in detail. It can be predicted that, after coupling a scalar field, the equilibrium configuration of a fluid Fermi star, especially the geometric shape of ellipsoid stars, will change substantially.

Summary and Expectation

There is accumulating evidence that scalar fields may exist in nature. The gravitational collapse of a boson cloud lead to the formation of a boson star. Here, we first examine the properties of a complex-scalar–field boson star, analyze the ground state solutions, and then analyzed the configuration of a star composed of bosons and fermions, and gave coupling equations. At last, we considered the hydrostatic equilibrium equation of the boson-fermion star, and gave the virial equation with different orders.

In our future work, we will explore the equilibrium configuration and the stability of a Boson-Fermi star and how a scalar field effect the pressure of the system by using higher order viral equations of fluid motions, and the related theoretical work is underway!

Thank you

for your attention!

Appendix

Massive mass of mini Bose star

Indeed, applying the uncertainty principle to a boson star by assuming it to be a macroscopic quantum state results in an excellent estimate for the maximum mass of a BS. One begins with the Heisenberg uncertainty principle of quantum mechanics

$$\Delta p \, \Delta x \geq \hbar \tag{3}$$

and assumes the BS is confined within some radius $\Delta x = R$ with a maximum momentum of $\Delta p = mc$ where m is the mass of the constituent particle

$$mcR \geq \hbar . \tag{4}$$

This inequality is consistent with the star being described by a Compton wavelength of $\lambda_C = h/(mc)$. We look for the maximum possible mass $M_{\max}$ for the boson star which will saturate the uncertainty bound and drive the radius of the star towards its Schwarzschild radius $R_S \equiv 2\,GM/c^2$. Substituting yields

$$\frac{2\,Gm\,M_{\max}}{c} = \hbar , \tag{5}$$

which gives an expression for the maximum mass

$$M_{\max} = \frac{1}{2}\frac{\hbar c}{Gm} . \tag{6}$$

Recognizing the Planck mass $M_{\text{Planck}} \equiv \sqrt{\hbar c/G}$, we obtain the estimate of $M_{\max} = 0.5\,M_{\text{Planck}}^2/m$. This simple estimate indicates that the maximum mass of the BS is inversely related to the mass of the constituent scalar field. We will see below in

The maximum mass of neutron star

The maximum mass of neutron stars can be estimated in a similar way. [2] The existence of these stars is the result of the balance between the attractive gravitational force and the pressure due to degenerate neutrons (fermions). Suppose there are N fermions confined in a region of size R. Then by Pauli's exclusion principle, each particle occupies a volume $1/n$, where $n \equiv N/R^3$ is the number density. Effectively, each particle has a size of $R/N^{1/3}$. Again, by the uncertainty principle we have $pR/N^{1/3} \sim \hbar$. Following the same argument as for the boson star case, we have $R \sim \hbar N^{1/3}/(mc)$, and hence $2GM_{\rm max}/c^2 \sim \hbar N^{1/3}/(mc) \sim \hbar {M_{\rm max}}^{1/3}/(m^{4/3}c)$. Thus we have $M_{\rm max} \sim 0.35{M_{\rm pl}}^3/m^2$. In contrast to the bosonic case, then, the maximum mass of fermionic stars scales as $M_{\rm max} \sim {M_{\rm pl}}^3/m^2$.

$$V\left(|\phi|^2\right) = m^2\,|\phi|^2 + \frac{\lambda}{2}\,|\phi|^4\,,$$

$$M_{\max} \approx 0.22\,\Lambda^{1/2} M_{\text{Planck}}/m = \left(0.1\ \text{GeV}^2\right) M_{\odot} \lambda^{1/2}/m^2$$

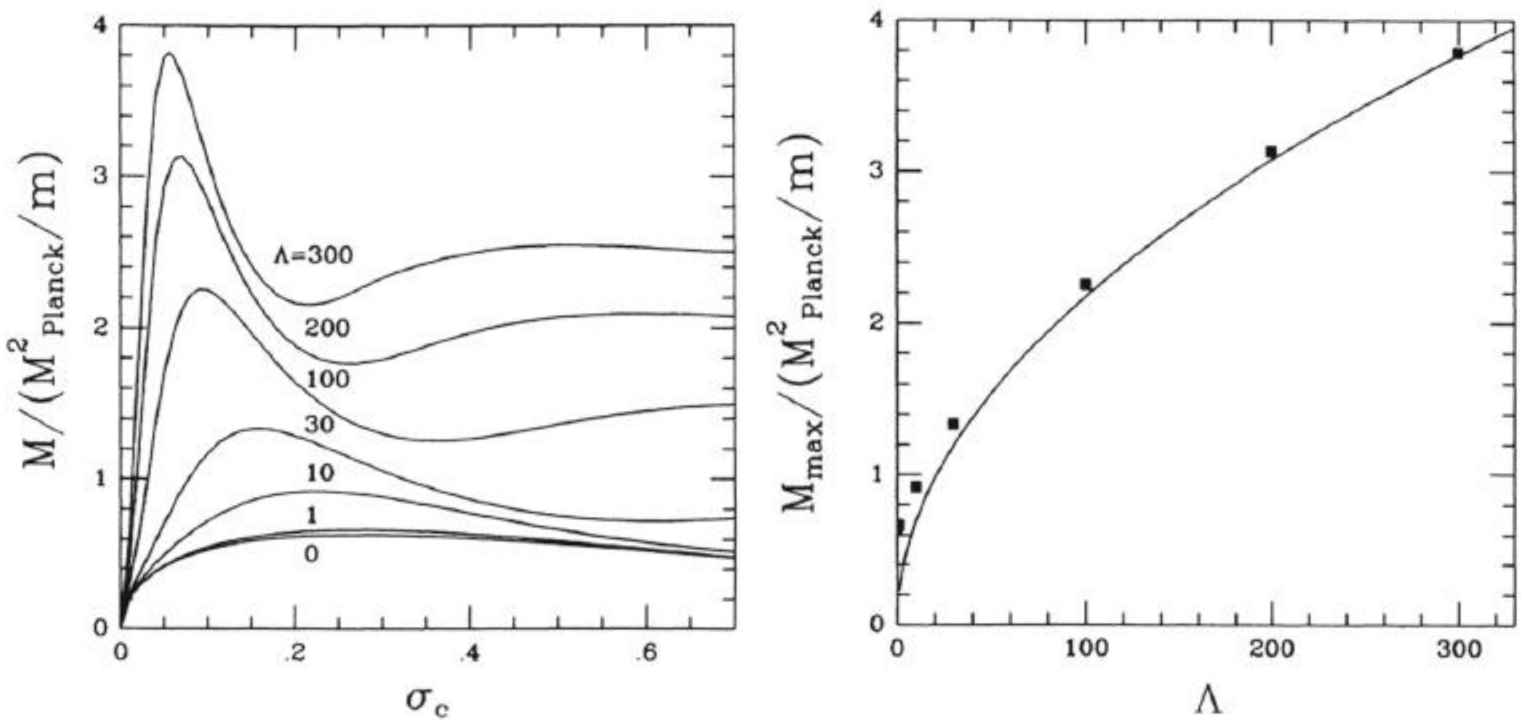


Fig. 3 Left: The mass of the boson star as a function of the central value of the scalar field in adimensional units $\sigma_c = \sqrt{4\pi\,G}\phi_c$. Right: Maximum mass as a function of Λ (squares) and the asymptotic $\Lambda \to \infty$ relation of Eq. (52) (solid curve). Reprinted with permission from Colpi et al. (1986); copyright by APS

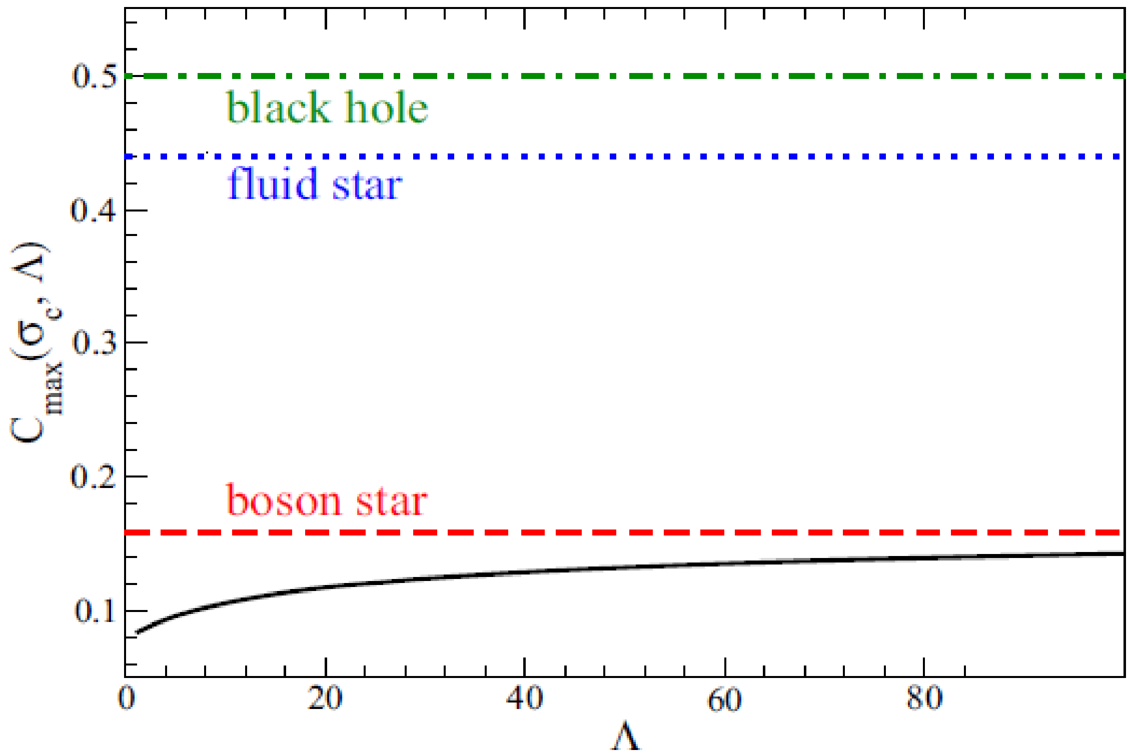


Fig. 4 The compactness of a stable boson star (black solid line) as a function of the adimensional self-interaction parameter $\Lambda \equiv \lambda / \left(4\pi\, Gm^2\right)$. The compactness is shown for the most massive stable star (the most compact BS is unstable). This compactness asymptotes for $\Lambda \to \infty$ to the value indicated by the red, dashed line. Also shown for comparison is the compactness of a Schwarzschild BH (green dot-dashed line), and the maximum compactness of a non-spinning neutron star (blue dotted line). Reprinted with permission from Amaro-Seoane et al. (2010); copyright by IOP