Zerilli equations within a modified theory of General Relativity

Peter O. Hess
Instituto de Ciencias Nucleares
Universidad Nacional Autónoma de México
and
Enrique López Moreno
Facultad de Ciencias
UNAM

Extension of the GR theory:

$$g_{00} = \left(1 - \frac{2m_0}{r}\right) \rightarrow \left(1 - \frac{2m(r)}{r}\right)$$

$$m(r) = m_0\left[1 - \frac{b}{4}\left(\frac{m_0}{r}\right)^2\right]$$

$$\text{example}: \; b = \frac{64}{27}$$

Origin of the addition term: Vacuum flutuations

Pseudo-Complex General Relativity: Hess and Greiner, IJPME 18(2009),51; Hess, Schäfer, Greiner, Springer book 2015.

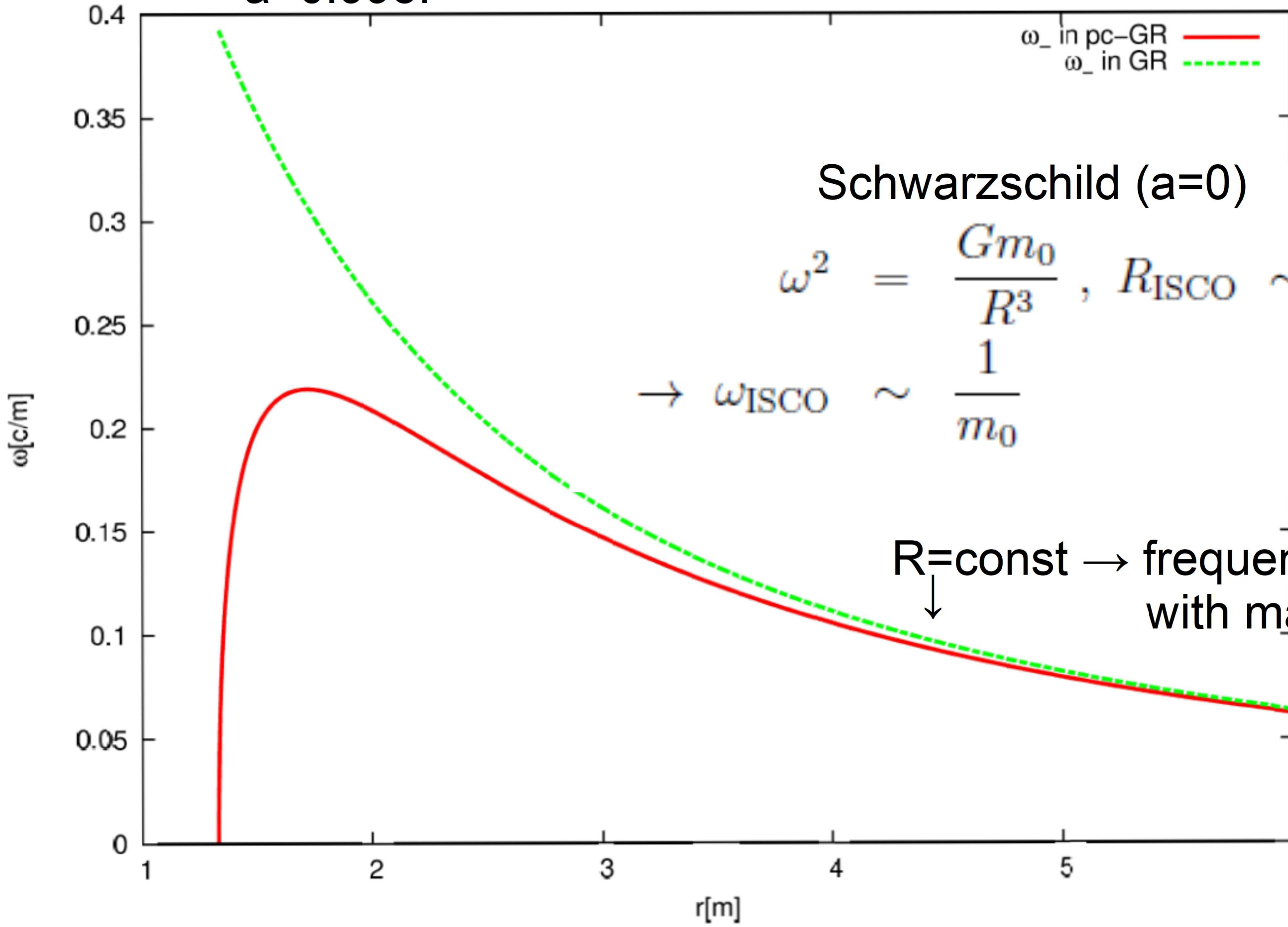
Gravitational wave event

$$M_c = \widetilde{M}_c F_\omega(r) = \frac{c^3}{G}\left[\frac{5}{96\pi^{\frac{8}{3}}}\frac{df_{gw}}{dt}f_{gw}^{-\frac{11}{3}}\right]^{\frac{3}{5}}$$

$x = \frac{r}{R_S}$	$F_\omega^{-1}(r)$	Final mass	$\widetilde{d}_L$	z (approx.)
0.6673	527	15,812	140,884	≈12.8
0.6675	401	12,023	107,000	≈10
0.6678	295	8,846	78,765	≈7.7
0.6682	218	6,544	58,304	≈6
0.6690	144	4,308	38,428	≈4
0.7000	10.8	323	3,002	≈0.5

Hess, MNRAS 462 (2016), 3026.

a=0.995:



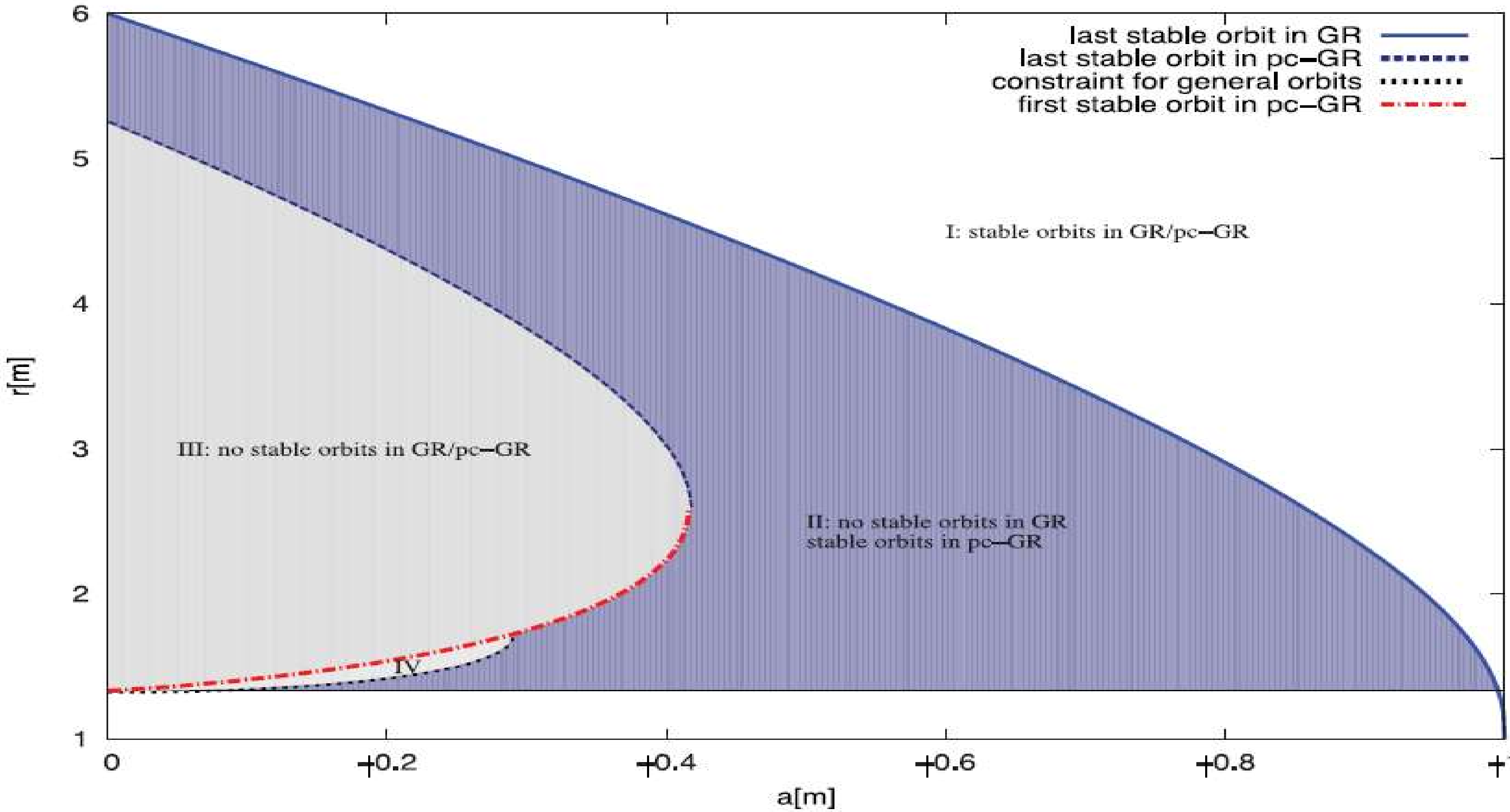
Schwarzschild (a=0)

$$\omega^2 = \frac{Gm_0}{R^3}\ , \ R_{\mathrm{ISCO}} \sim m_0$$

$$\rightarrow \omega_{\mathrm{ISCO}} \sim \frac{1}{m_0}$$

R=const → frequency increases with mass!

Nielsen and Birnholz, Astronomische Nachrichten 339(4) (2018), 298; arXiv; gr-qc:1708.0333



last stable orbit in GR
last stable orbit in pc−GR
constraint for general orbits
first stable orbit in pc−GR
I: stable orbits in GR/pc−GR
II: no stable orbits in GR
stable orbits in pc−GR
III: no stable orbits in GR/pc−GR
IV
r[m]
a[m]
1
2
3
4
5
6
0
+0.2
+0.4
+0.6
+0.8
+1

Stability of the Schwarzschild solution

$$ds^2 = e^{2\nu}dt^2 - e^{2\psi}\left(d\phi - \omega dt - q_r dr - q_\theta d\theta\right)^2 - e^{2\mu_r}dr^2 - e^{2\mu_\theta}d\theta^2$$

$$\text{axial modes}(-):$$
$$\omega \neq 0\ ,\ q_r \neq 0\ ,\ q_\theta \neq 0$$
$$\text{polar modes}(+):$$
$$\delta\nu \neq 0\ ,\ \delta\mu_r \neq 0\ ,\ \delta\mu_\theta \neq 0\ ,\ \delta\psi \neq 0$$

Chandrasekhar’s book; The mathematical theory of black holes

(a) Regge-Wheeler

Axial modes, negative parity

$$\left(\frac{d^2}{dr_*^2}+\omega^2\right)Z^{(-)} = V^{(-)}Z^{(-)}$$

$$V^{(-)} = \frac{\Delta}{r^5}\left\{\mu^2 r + 2r + 2mr - 6m + 2m'r\right\}$$

m=m(r)!

(a) Zerilli

Polar modes, positive parity

$$\left(\frac{d^2}{dr_*^2} + \omega^2\right) Z^{(+)} = V^{(+)} Z^{(+)}$$

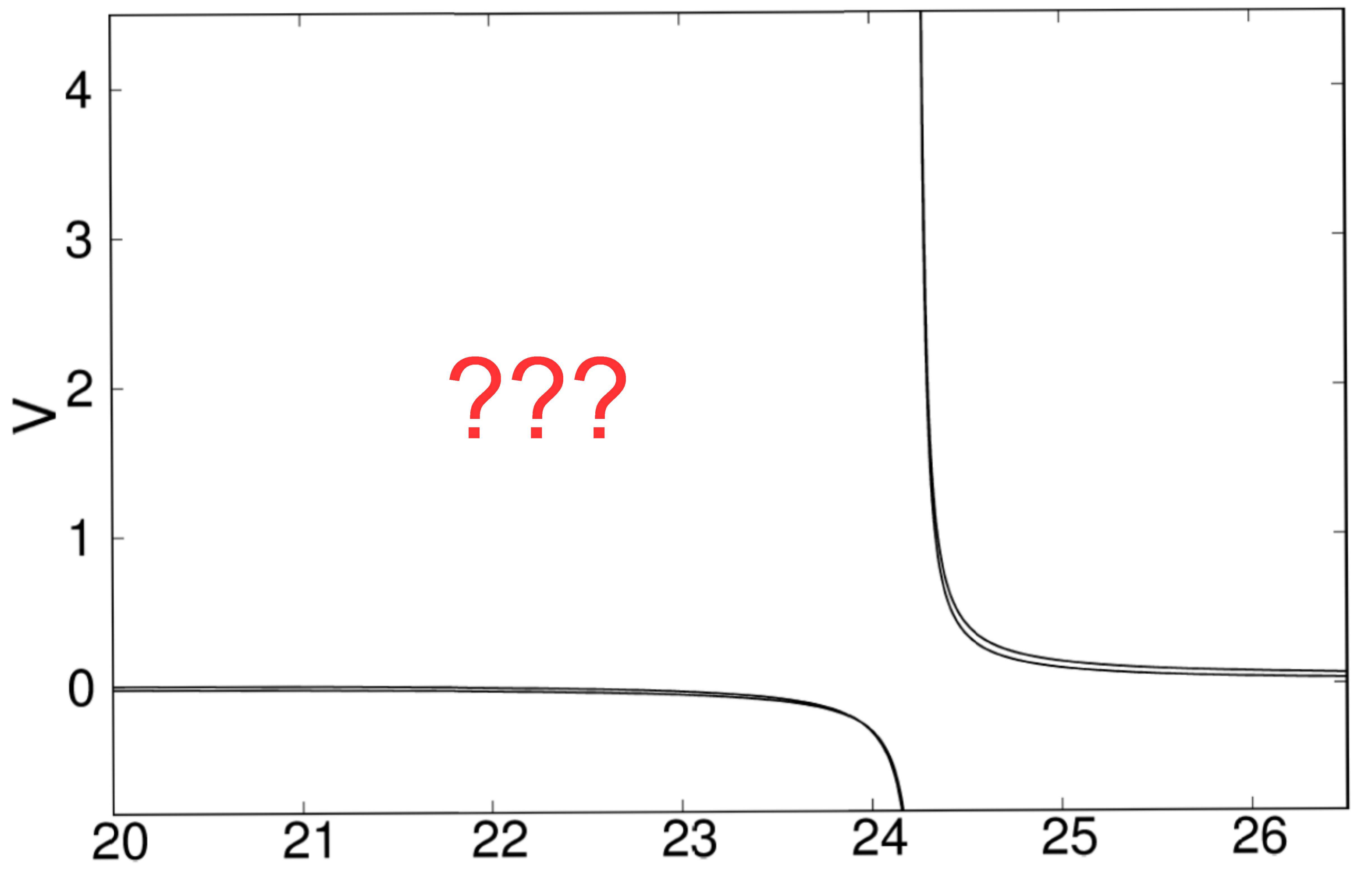
$$V^{(+)} \approx \frac{a}{(r - r_0)}$$

$$V^{(+)} = \frac{F(m, m', m'', n)}{\{r^3[n(-2m + r) - 2m'(3m + nr)]\}}$$

$$
\begin{aligned}
&F(m, m', m'', n) = \\
&-(n * (-2m + r)^2 (3m + nr)(-(\tfrac{1}{r}) + \tfrac{(3mn(-m+(2+n)r))}{(3m+nr)^3} \\
&+ \tfrac{(m'(3m^2+3mnr-nr^2))}{((-2*xm+r)*(3*xm+xn*r)**2)} - \tfrac{((-5*xm+2*r)*(-3*xm**2-3*xm*xn*r+xn*r**2))}{(r*(-2*xm+r)*(3*xm+xn*r)**2)} \\
&+ \tfrac{(n(-6mm'r-2m'nr^2))}{(3m+nr)^3} + \tfrac{((-5m+2r+m'r)(-3mm'r-m'nr^2))}{(r(-2m+r)(3m+nr)^3)} + \tfrac{(3mm'+3m'2r+3mm''r+2m'nr+m''}{nr^2)}(3m + nr)^2 \\
&+(1 + 2m' - \tfrac{2m}{r} - \tfrac{(2(-2m+r))}{r} + \tfrac{(3m*(-2m+r))}{(r(3m+nr))})((1 + n)/(n(-2m + r)) \\
&- \tfrac{1}{n}(\tfrac{(-2*m')}{(-2m+r)^2} + \tfrac{m^2}{(r(-2m+r)^2)} + \tfrac{(m'2r)}{(-2m+r)^2} + \tfrac{1}{(-2m+r)} + \tfrac{m'}{(-2m+r)} - \tfrac{m}{(r(-2m+r))}))))
\end{aligned}
$$

In pcGR no relation between the axial and polar solutions!
This expression reduced to the standaqrd one, when the Mass function is constant.

Zerilli equation, polar, positive parity modes. Potential:



Asymptotic behavior:

$$\psi(x) \longrightarrow \begin{cases} e^{-i\omega x}, & x \longrightarrow -\infty, \\ e^{i\omega x}, & x \longrightarrow \infty, \end{cases}$$

$$\xi = 1 - \frac{4}{3y} \text{, with } y = \frac{r}{m_0}$$

$$\text{Domain} : [0, 1]$$

(x=Tortoise coordinate)

$$\rightarrow \quad \Psi = (1-\xi)^{-2i\widetilde{\omega}} \xi^{\frac{4i\widetilde{\omega}}{3}} e^{\frac{2i\widetilde{\omega}}{(1-\xi)}} \Phi(\xi)$$

Units:

$$\begin{aligned} \widetilde{\omega} &= m_0 \omega \\ [\omega] &= \text{km}^{-1} \\ [m_0] &= \text{km} \end{aligned}$$

Order of relevent frequencies

$$
\begin{aligned}
\nu = 250 \text{ Hz} \;\rightarrow\; \omega &= (2\pi)250 \;\approx\; 1571\text{Hz} \\
1\text{Hz} &= \frac{10^{-5}}{3}\,\text{km}^{-1} \\
\omega &\approx 5.24\; 10^{-3}\;\text{km}^{-1} \\
\tilde{\omega} &= m_0\omega \\
m_0 = 60 * 1.5\text{ km} &= 90\text{km} \\
\tilde{\omega} &\approx 0.47
\end{aligned}
$$

I. e., when the mass m0 increases, also does the $\tilde{\omega}$ and it also decreases when the mass decreases. Similar for the bare Frequency omega.

The Asymptotic Iteration Method (AIM)

T.H. Cho et al., Advances in Mathematical Physics (2012), ID:281705

$$\chi'' = \lambda_0(x)\chi' + s_0(x)\chi, \quad \text{-->} \quad \chi^{(n+1)} = \lambda_{n-1}(x)\chi' + s_{n-1}(x)\chi$$

$$\lambda_n(x) = \lambda'_{n-1}(x) + s_{n-1}(x) + \lambda_0(x)\lambda_{n-1}(x), \qquad s_n(x) = s'_{n-1}(x) + s_0(x)\lambda_{n-1}(x).$$

Convergence condition: $\dfrac{s_n(x)}{\lambda_n(x)} = \dfrac{s_{n-1}(x)}{\lambda_{n-1}(x)} \equiv \beta(x)$

Quantization condition: $\delta_n = s_n\lambda_{n-1} - s_{n-1}\lambda_n = 0$
(It is a polynomial in the Frequencies→ look for zeros)

However, the dependence on x is of a disadvantage:

The improved AIM:

Expansion around the point x:

$$\lambda_n(\xi) = \sum_{i=0}^{\infty} c_n^i (x-\xi)^i,$$

$$s_n(\xi) = \sum_{i=0}^{\infty} d_n^i (x-\xi)^i,$$

Recursion relations for the expansion coefficients:

$$c_n^i = (i+1)c_{n-1}^{i+1} + d_{n-1}^i + \sum_{k=0}^{i} c_0^k c_{n-1}^{i-k},$$

$$d_n^i = (i+1)d_{n-1}^{i+1} + \sum_{k=0}^{i} d_0^k c_{n-1}^{i-k}.$$

New quantization condition: $d_n^0 c_{n-1}^0 - d_{n-1}^0 c_n^0 = 0,$ Polynomial in ω

Some remarks on how to get a “quick” convergence

- Choose a compact support.
- Get the asymptotics right.
- Expand around the maximum or minimum of the potential.

Axial modes: GR

$\tilde{\omega}$

Ordered according to raising imaginary part:

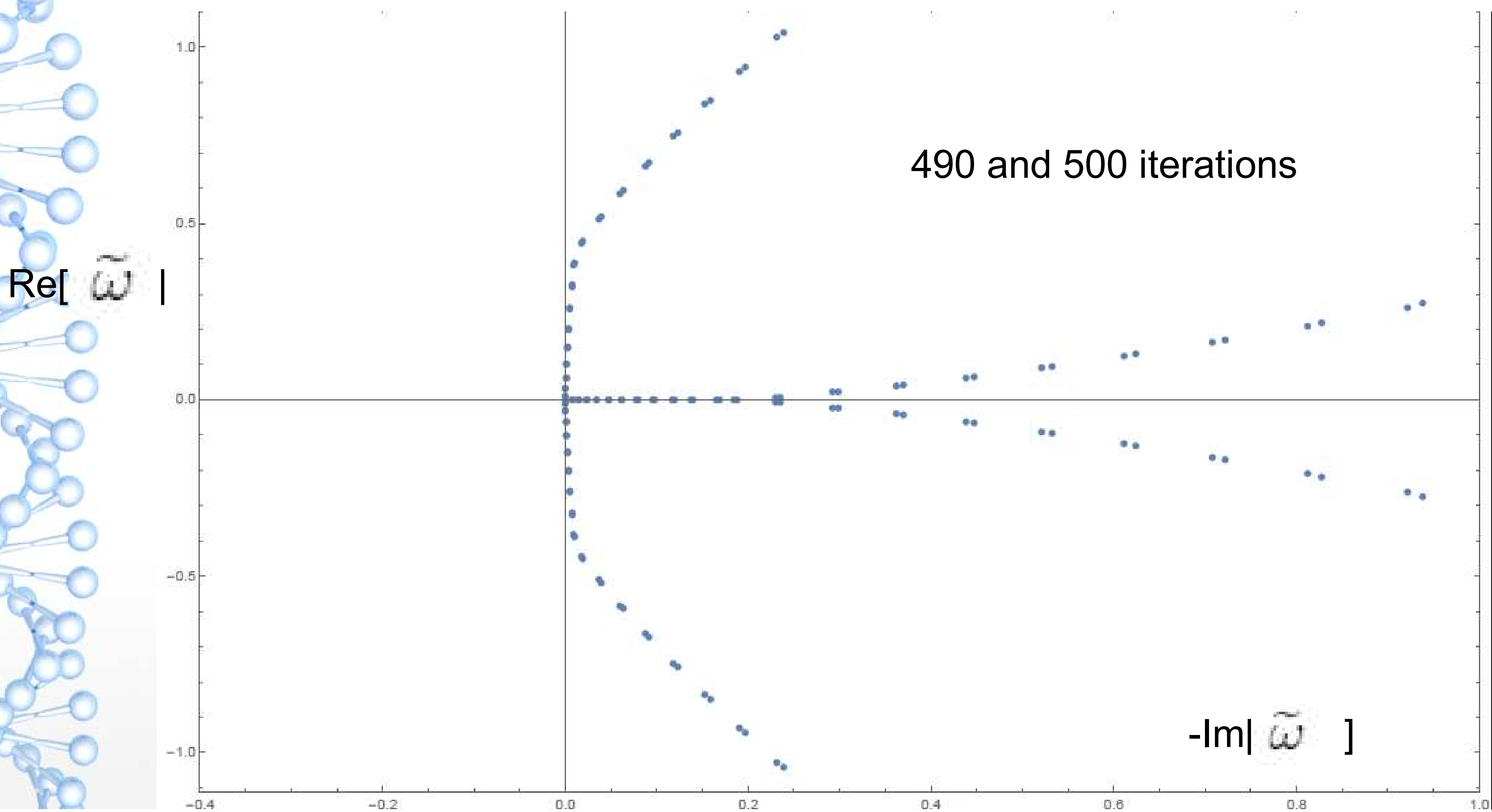
0.3737-i 0.0890	0.34676-i 0.2740	0.3011- i 0.4783	0.2515 -i 0.7051

Ordered according to raising real part:

0.0051-i 7.539	0.03061 -i 8.143	0.03144 -i 8.899	0.03476 -i 5.556

¡There are also modes with a negative imaginary part only!
→ Overdamped modes.

Axial modes: pcGR



Take $\omega_R - i \mid \omega_I \mid = (0.3061 - i\ 0.4783)/m_0$ as "experimental". The ratio is

$$
\begin{aligned}
\frac{\omega_R}{\mid \omega_I \mid} &= 0.640 \\
&= \frac{\widetilde{\omega}_R}{\mid \widetilde{\omega}_I \mid} \qquad (2)
\end{aligned}
$$

With $\widetilde{\omega}_R = m_0 \omega$, we obtain

$$
\begin{aligned}
m_0 &= \frac{0.3061}{5.24} 1000 = 58.75 \\
&\quad 39.2 \text{solar masses} \qquad (3)
\end{aligned}
$$

Now, look for a value in the extended theory with nearly the same ratio: $\widetilde{\omega} = 1.12196 - i\ 1.7895$. the ratio is 0.627, thus

$$
\begin{aligned}
m_0 &= \frac{1.12}{5.24} 1000 = 214. \\
&\quad 142.3 \text{solar masses} \qquad (4)
\end{aligned}
$$

approximately 3.6 times larger than in GR.

Polar modes (pcGR compared to GR)

Positive parity, l=2: (huge problems with interations for pcGR, L. Manfredi et al, PLB 779 (2018), 492: 400-500 iterations!)

We also went up tp 600 interations and no convergence.

One problem is: How to treat the pole in the potential???

Conclusions

- Stability of Schwarzschild solution, with variable mass m(r)
- Axial and polar perturbations
- Axial modes are stable, very small frequencies present and damped solutions. Polar modes do not converge due to the pole. Need to be treated!
- Axial modes indicate a larger mass, still of the same order as in GR.
- Work needed for Kerr solutions. There, the differences are much bigger and probably will provide in quality the same results as given in Hess, MNRAS 2016.