Numerical approach to 2nd-order canonical perturbation theory

Application to exoplanets

Aya Alnajjarine
Work with J. Laskar and F. Mogavero

LTE, Observatoire de Paris, PSL

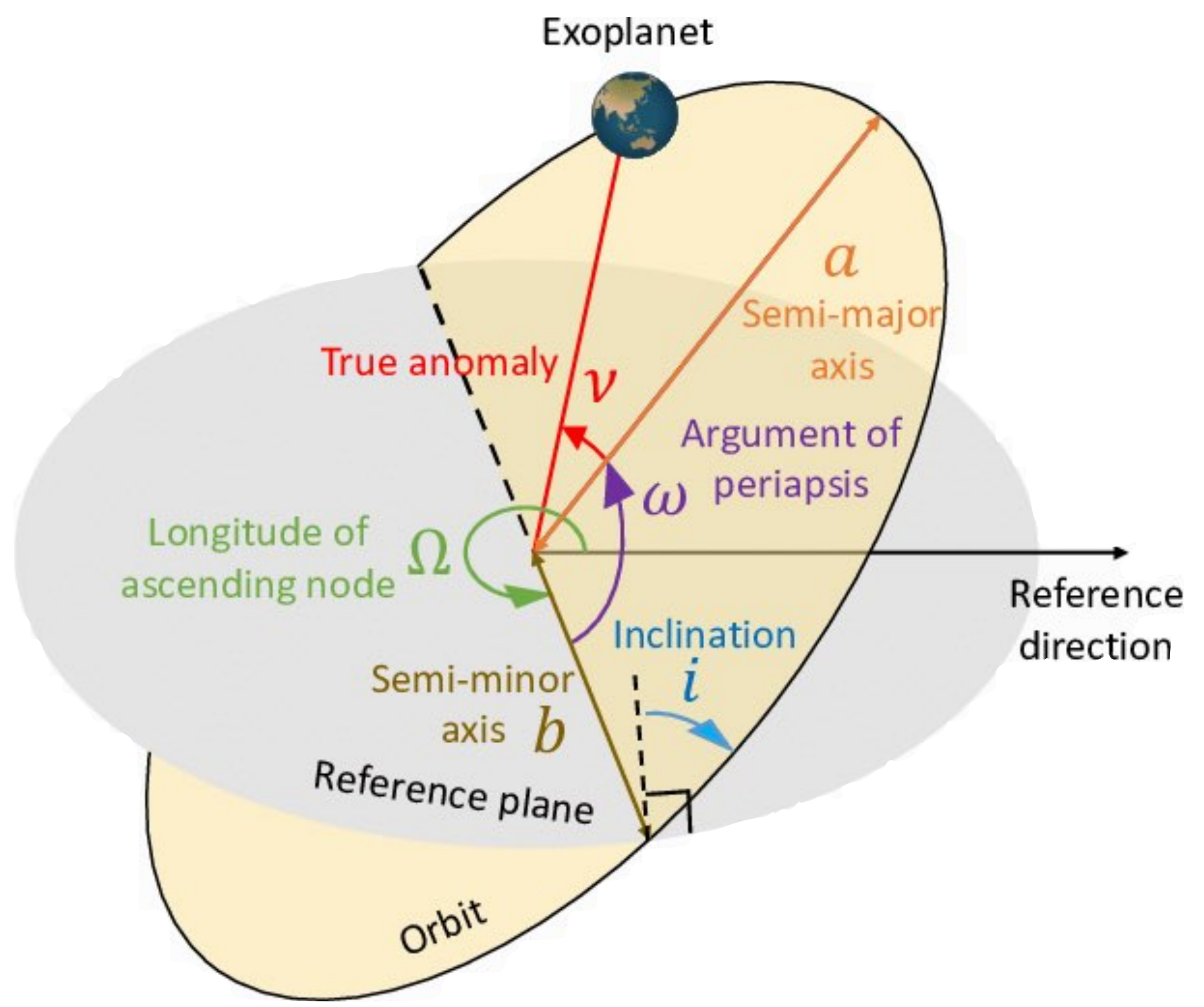
July 8, 2025

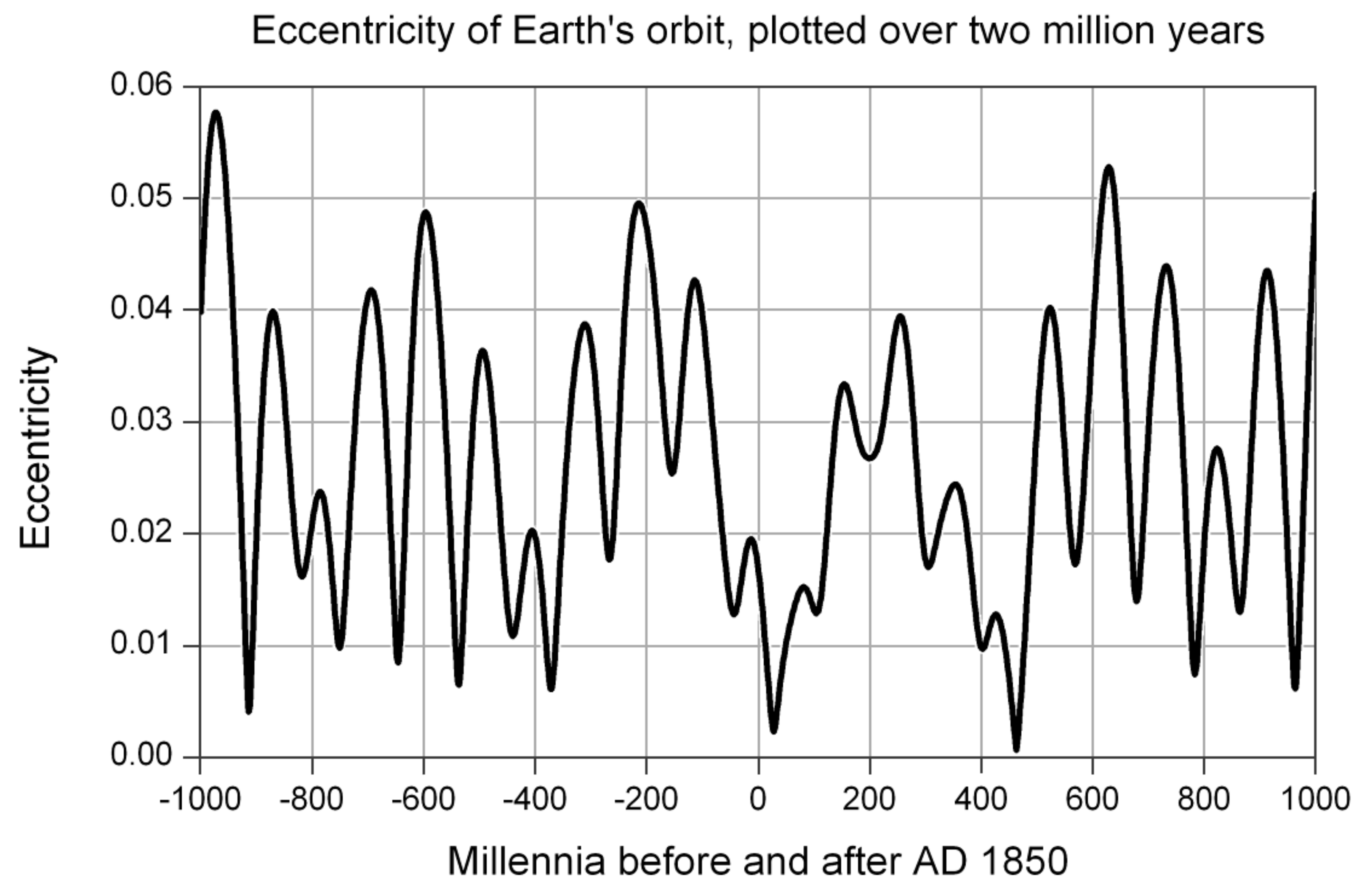
LTE Observatoire de Paris | PSL

Overview

- **Aim:** Study long-term orbital evolution of planetary systems, especially those near or within MMR

- **Significance of Secular Models:**
    - Simplifying dynamics by averaging short-term fluctuations.
    - Focusing on the main factors driving long-term changes.
    - Providing insights into the stability, formation, and overall architecture of planetary systems.



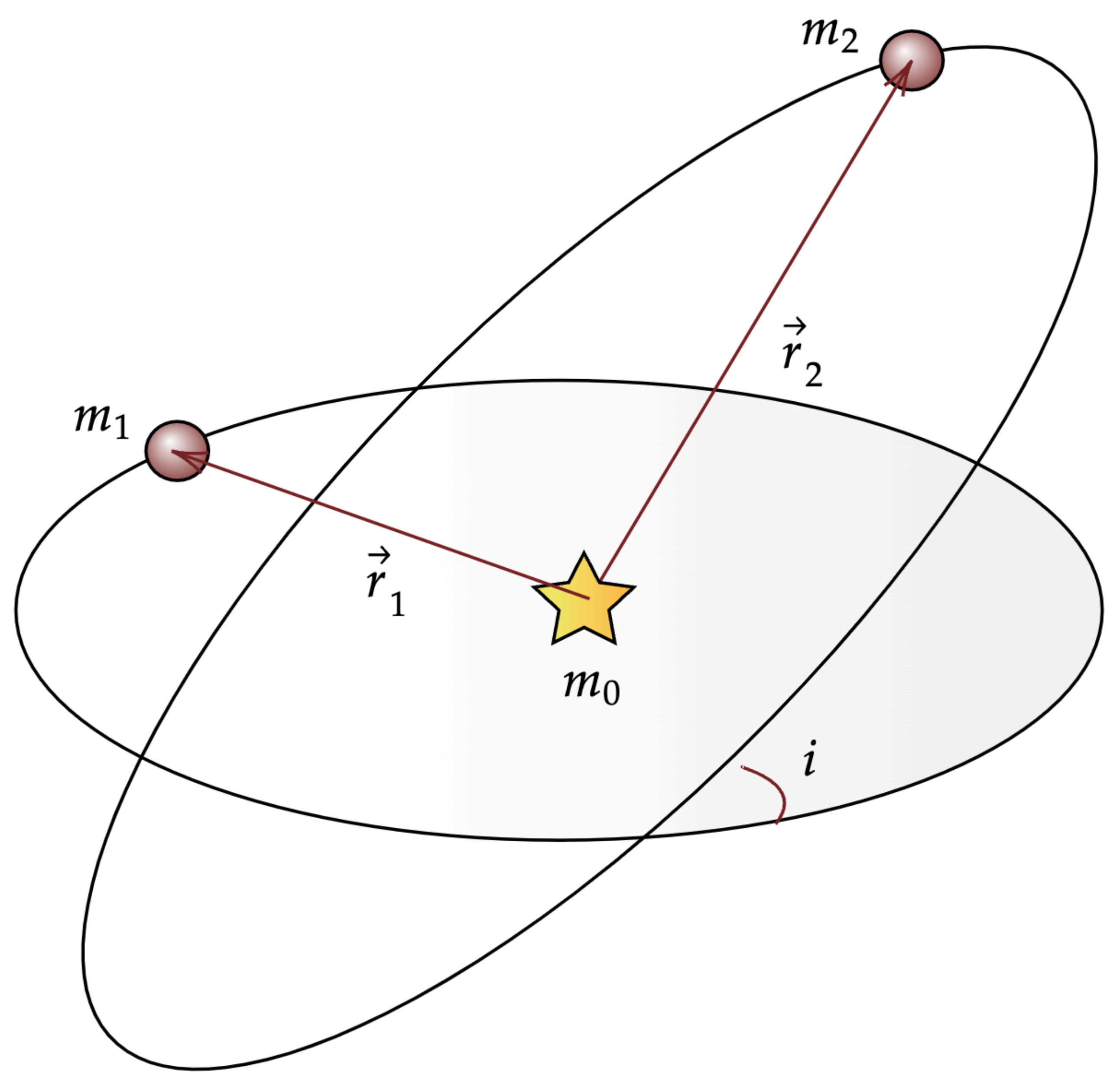


Hamiltonian Formulation

- In canonical heliocentric coordinates, Hamiltonian splits into

$\epsilon = \max(m_1, m_2)/m_0 \ll 1$

$$H = \underbrace{-\frac{\mu_1^2\beta_1^3}{2\Lambda_1^2} - \frac{\mu_2^2\beta_2^3}{2\Lambda_2^2}}_{H_0} + \underbrace{-\frac{Gm_1m_2}{||\boldsymbol{r}_1 - \boldsymbol{r}_2||} + \frac{\tilde{\boldsymbol{r}}_1 \cdot \tilde{\boldsymbol{r}}_2}{m_0}}_{\epsilon H_1}$$



- $\boldsymbol{r}_i$ are position vectors and $\tilde{\boldsymbol{r}}_i$ are conjugated momenta, expressed in Poincaré elliptical variables $(\Lambda, x, y, \lambda, -i\bar{x}, -i\bar{y})$:

$$\underbrace{\Lambda = \beta\sqrt{\mu a}, \quad \lambda = M + \varpi}_{\text{fast variables}}, \qquad \underbrace{x = \sqrt{\Lambda}\sqrt{1-\sqrt{1-e^2}}\mathrm{E}^{\imath\varpi}, \quad y = \sqrt{2\Lambda}\left(1-e^2\right)^{\frac{1}{4}}\sin(I/2)\mathrm{E}^{\imath\Omega}}_{\text{secular variables}}$$

$$\beta_i = m_0 m_i/(m_0 + m_i)$$

$$\mu_i = G(m_0 + m_i)$$

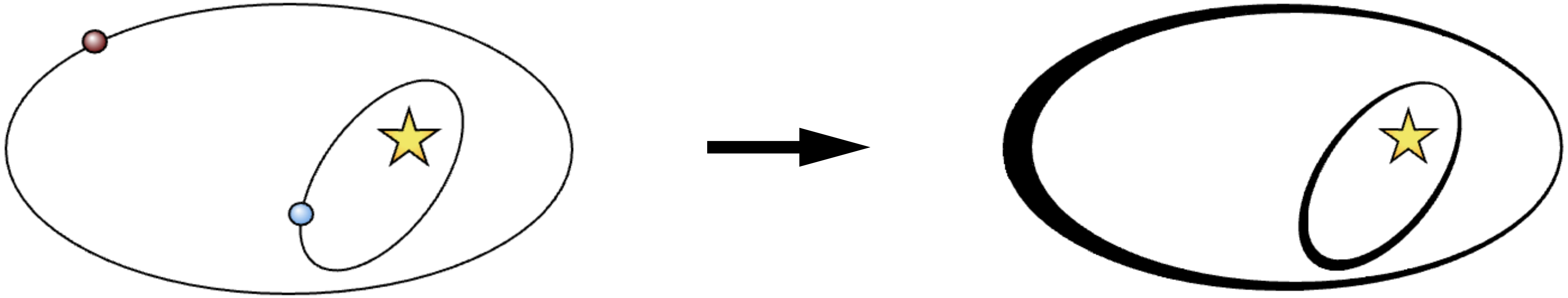
First Order Secular Models

Non-resonant systems[1]

$$< H_1 > = \frac{1}{(2\pi)^2} \int_0^{2\pi} \int_0^{2\pi} H_1 \; d\lambda_1 \; d\lambda_2$$

Systems near $k_1 : k_2$ MMR[2]

$$< H_1 > = \frac{1}{2\pi k_1} \int_0^{2\pi k_1} H_1 \; d\lambda_1$$

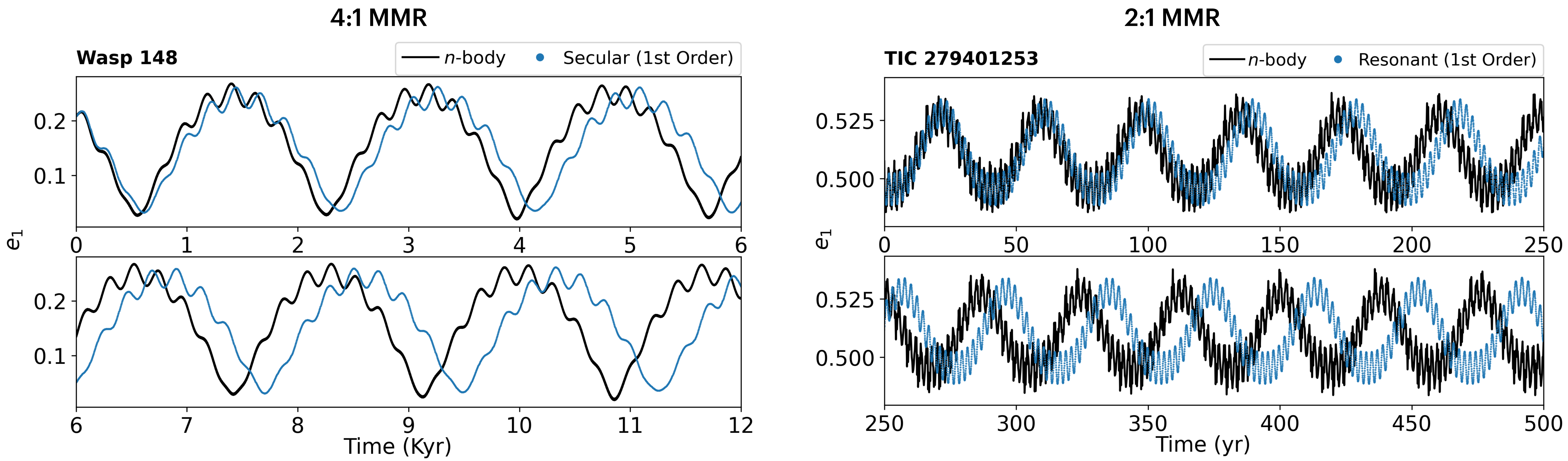


Gauss's Averaging Method (1818) : Averaging is equivalent to replacing each planet by its instantaneous Keplerian orbit

[1] **Laplace and Lagrange Theory** (18th century), **Multipole Theory** (Kozai 1962,Lidov 1962), **Model for Solar System**(Laskar 1988, 1989, 1990) **Analytical Model for Exoplanets** (Libert & Henrard 2005, 2006)

[2] **Second Fundamental model of Resonance** (Henrard & Lemaitre, 1983), **Analytical Model for Exoplanets** (Sansottera & Libert 2013)

Exoplanetary Systems



Eccentricity evolution of inner planets obtained using n-body integrations (black) and 1st Order averaging approximation (blue)

- Second-order averaging theories needed for more accurate dynamics.
- Ex: Solar System Stability (Laskar 1988, 1989, 1990) , Asteroidal MMRs (Milani et. al 1990), Planetary MMRs (Libert et. al 2013, Sansottera et. al 2019).
- **Aim is to perform second-order averaging numerically.**
- Numerical 1st Order: Secular Dynamics (Gauss 1818, Hill 1881), MMRs (Schubart 1964).

Lie Series Formalism

- Lie Series Transformation: $(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{\Lambda}, \boldsymbol{\lambda}) = \mathrm{E}^{\{S,\,\cdot\}}(\hat{\boldsymbol{x}}, \hat{\boldsymbol{y}}, \hat{\boldsymbol{\Lambda}}, \hat{\boldsymbol{\lambda}})\,;\ \widehat{H} = E^{\{S,\,\cdot\}}\,H$
- Secular Hamiltonian $\widehat{H} = \widehat{H_0}\,(\hat{\boldsymbol{\Lambda}}) + \epsilon\,\widehat{H_1}\,(\hat{\boldsymbol{x}}, \hat{\boldsymbol{y}}; \hat{\boldsymbol{\Lambda}}) + \epsilon^2\,\widehat{H_2}\,(\hat{\boldsymbol{x}}, \hat{\boldsymbol{y}}; \hat{\boldsymbol{\Lambda}}) + \mathcal{O}(\epsilon^3)$
- Transformation constructed by generating function $S = \sum_{n=1}^{\infty} S_n \epsilon^n$
- Eliminate systematically fast angles, by choosing at each order appropriate S_i

$$\widehat{H}_0 = H_0$$

$$\widehat{H}_1 = H_1 + \{S_1, H_0\}$$

$$\widehat{H}_2 = \{S_1, H_1\} + \frac{1}{2}\{S_1, \{S_1, H_0\}\} + \{S_2, H_0\}$$

$$\{f, g\} = \frac{\partial f}{\partial \boldsymbol{p}} \cdot \frac{\partial g}{\partial \boldsymbol{q}} - \frac{\partial f}{\partial \boldsymbol{q}} \cdot \frac{\partial g}{\partial \boldsymbol{p}}$$

$$\boldsymbol{x} = (x_1, x_2)$$

First-order Averaging

- First-Order Averaging: $\widehat{H}_1 = H_1 + \epsilon\ \{S_1, H_0\} \equiv < H_1 >$; $H_1 = \sum_{\boldsymbol{k}\in\mathbb{Z}^2} h_1^{\boldsymbol{k}}(\hat{\boldsymbol{x}}, \hat{\boldsymbol{y}}; \hat{\boldsymbol{\Lambda}})\ \mathrm{E}^{\imath \boldsymbol{k}\cdot\hat{\lambda}}$
- Secular Hamiltonian: $\widehat{H} = H_0 + \epsilon\,\widehat{H_1} + O(\epsilon^2)$ with $\widehat{H_1} = h_1^{\boldsymbol{0}}(\hat{\boldsymbol{x}}, \hat{\boldsymbol{y}}; \hat{\boldsymbol{\Lambda}})$
- Solving Homological Equation: $S_1 = -\imath \sum_{\boldsymbol{k}\in\mathbb{Z}^2\setminus\{\boldsymbol{0}\}} \frac{h_1^{\boldsymbol{k}}}{\boldsymbol{k}\cdot\boldsymbol{n}_0}\ \mathrm{E}^{\imath \boldsymbol{k}\cdot\hat{\lambda}}$; $\boldsymbol{k}\cdot\boldsymbol{n}_0 \sim 0 \rightarrow$ small divisor problem
- $\boldsymbol{n}_0 = \partial H_0/\partial\hat{\boldsymbol{\Lambda}}$ is the mean motion vector of planets.
- Near $k_1 : k_2$ MMR $\rightarrow\ \widehat{H_1} = \sum_{\boldsymbol{l}\in\mathscr{R}} h_1^{\boldsymbol{l}}\ \mathrm{E}^{\imath \boldsymbol{l}\cdot\hat{\lambda}}$; $\mathscr{R}$ is the set of zero and resonant harmonics $(k_1\lambda_1 - k_2\lambda_2)$

Second-order Averaging

- Second-Order terms: $\widehat{H}_2 = \frac{1}{2}\{S_1, H_1\} + \frac{1}{2}\{S_1, h_1^{(0,0)}\} + \{S_2, H_0\}$. (with $\{S_1, h_1^{(0,0)}\} \to 0$)
- Second-Order Averaging: $\widehat{H}_2 \equiv \frac{1}{2}\left\langle \{S_1, H_1\} \right\rangle$.
- Substituting S_1 and H_1 :

$$\{S_1, H_1\} = -\sum_{\boldsymbol{l}\in\mathbb{Z}^2} \mathrm{E}^{\imath \boldsymbol{l}\cdot\hat{\boldsymbol{\lambda}}}\left[\sum_{\boldsymbol{k}\in\mathbb{Z}^2\setminus\{\boldsymbol{0}\}} \frac{\imath\{h_1^{\boldsymbol{k}}, h_1^{\boldsymbol{l}-\boldsymbol{k}}\}^\star}{\boldsymbol{k}\cdot\boldsymbol{n}_0} + (\boldsymbol{k}-\boldsymbol{l})\cdot\frac{\partial}{\partial\hat{\boldsymbol{\Lambda}}}\left(\frac{h_1^{\boldsymbol{k}}}{\boldsymbol{k}\cdot\boldsymbol{n}_0}\right)h_1^{\boldsymbol{l}-\boldsymbol{k}} + \boldsymbol{k}\cdot\frac{\partial h_1^{\boldsymbol{l}-\boldsymbol{k}}}{\partial\hat{\boldsymbol{\Lambda}}}\frac{h_1^{\boldsymbol{k}}}{\boldsymbol{k}\cdot\boldsymbol{n}_0}\right]$$

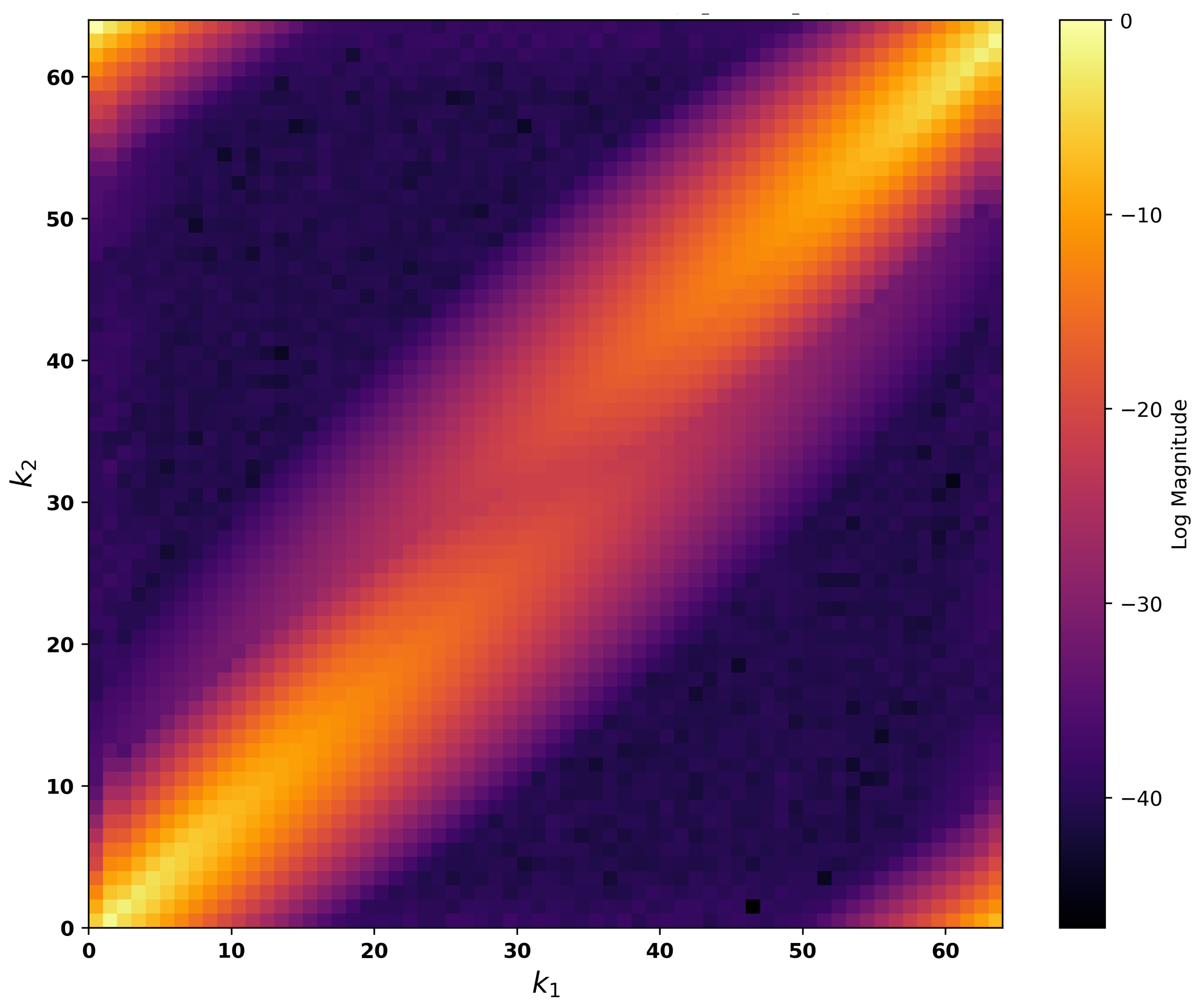
- $\{\,.\,,\,.\,\}^\star$ is the Poisson bracket with respect to the secular variables x and y.

Numerics: FFT

- Compute the Fourier Coefficients using 2D FFT :

$$h_1^{\boldsymbol{k}},\ \partial h_1^{\boldsymbol{k}}/\partial x,\ \partial h_1^{\boldsymbol{k}}/\partial y,\ \partial h_1^{\boldsymbol{k}}/\partial \Lambda,\ h_1^{\boldsymbol{k}}/\partial \lambda$$

- At first order, extract zero frequency harmonic + harmonics of resonance $\boldsymbol{l} \in \mathscr{R}$
- $$\widehat{H} = H_0 + \epsilon \sum_{\boldsymbol{l} \in \mathscr{R}} h_1^{\boldsymbol{l}}\, \mathrm{E}^{\imath \boldsymbol{l} \cdot \hat{\lambda}}$$
- $$\dot{x}_i = -\imath \sum_{\boldsymbol{l} \in \mathscr{R}} \frac{\partial h_1^{\boldsymbol{l}}}{\partial \bar{x}_i}\, \mathrm{E}^{\imath \boldsymbol{l} \cdot \hat{\lambda}}\ ;\ \dot{y}_i = -\imath \sum_{\boldsymbol{l} \in \mathscr{R}} \frac{\partial h_1^{\boldsymbol{l}}}{\partial \bar{y}_i}\, \mathrm{E}^{\imath \boldsymbol{l} \cdot \hat{\lambda}}$$



FFT Magnitude Spectrum $|h_1^{k_1,k_2}/h_1^{0,0}|$

Numerics: 2nd Order terms

- At second order, perform summation over order of harmonic $|\boldsymbol{k}| = |k_1| + |k_2|$.

$$\widehat{h}_2^{\boldsymbol{l}} = -\frac{1}{2}\sum_{\boldsymbol{k}\in\mathbb{Z}^2\setminus\mathscr{R}}\left[\frac{\imath\{h_1^{\boldsymbol{k}}, h_1^{\boldsymbol{l}-\boldsymbol{k}}\}^\star}{\boldsymbol{k}\cdot\boldsymbol{n}_0} + \boldsymbol{k}\cdot\frac{\partial}{\partial\hat{\boldsymbol{\Lambda}}}\left(\frac{h_1^{\boldsymbol{k}}h_1^{\boldsymbol{l}-\boldsymbol{k}}}{\boldsymbol{k}\cdot\boldsymbol{n}_0}\right) - \boldsymbol{l}\cdot\frac{\partial}{\partial\hat{\boldsymbol{\Lambda}}}\left(\frac{h_1^{\boldsymbol{k}}}{\boldsymbol{k}\cdot\boldsymbol{n}_0}\right)h_1^{\boldsymbol{l}-\boldsymbol{k}}\right]$$

- We take derivative of second-order terms numerically

$$\frac{\partial\,\widehat{H_2}}{\partial\bar{x}_i},\quad \frac{\partial\,\widehat{H_2}}{\partial\bar{y}_i},\quad \frac{\partial\,\widehat{H_2}}{\partial\Lambda_i}$$

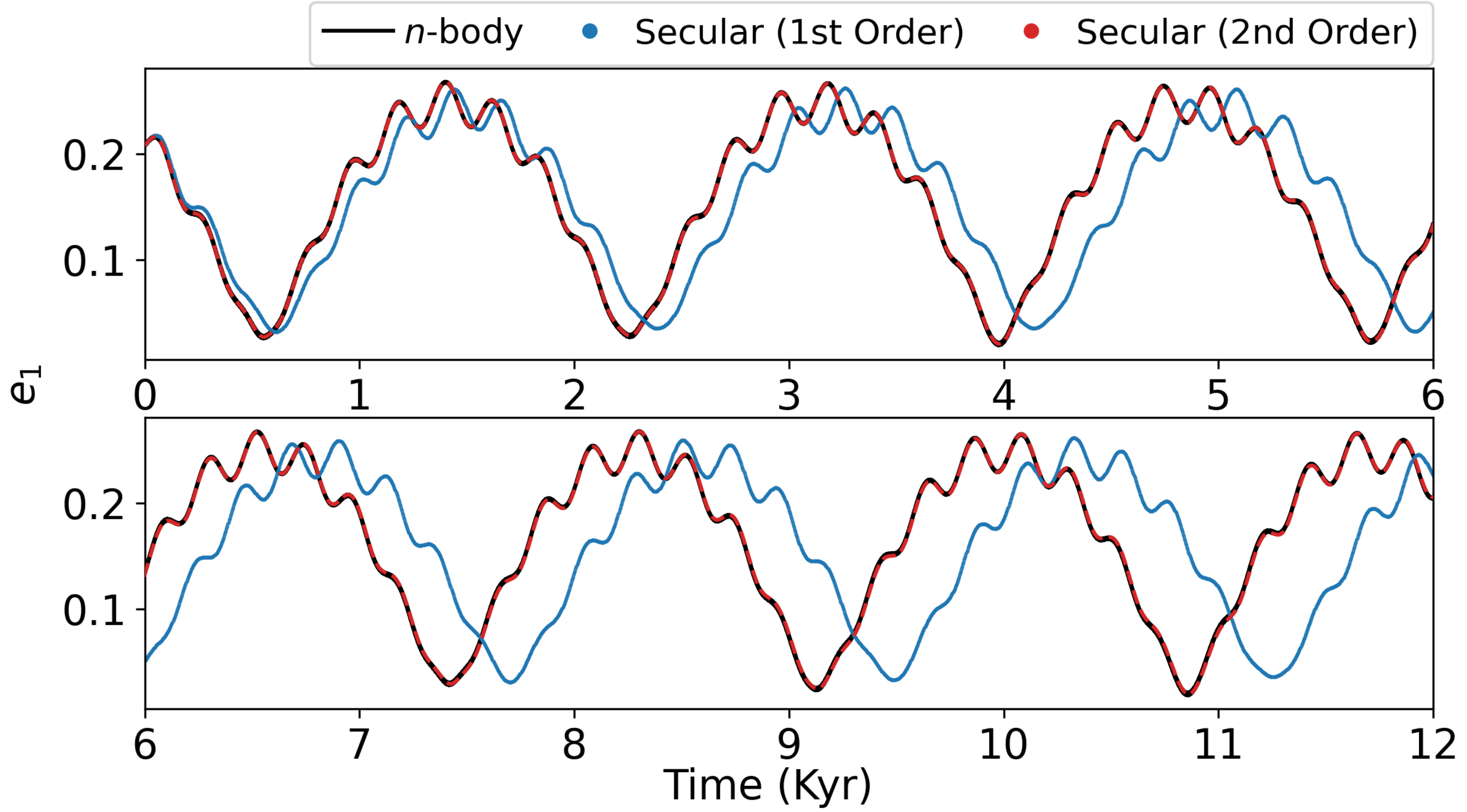
- Integrate EOMS using Adams PECE method of order 12

Wasp 148 System (Hebrard et al. 2020)

Masses: 0.31 and 0.41 M_{Jup}

Periods: 8.80 and 34.53 days

Eccentricities: 0.21 and 0.18



Relative Error (%)	g_1	g_2	s_2
Secular 1st	12.2	6.5	0.1
Resonant 1st	0.7	0.5	0.1
Secular 2nd	0.01	0.03	0.003
Resonant 2nd	0.002	0.0002	0.001

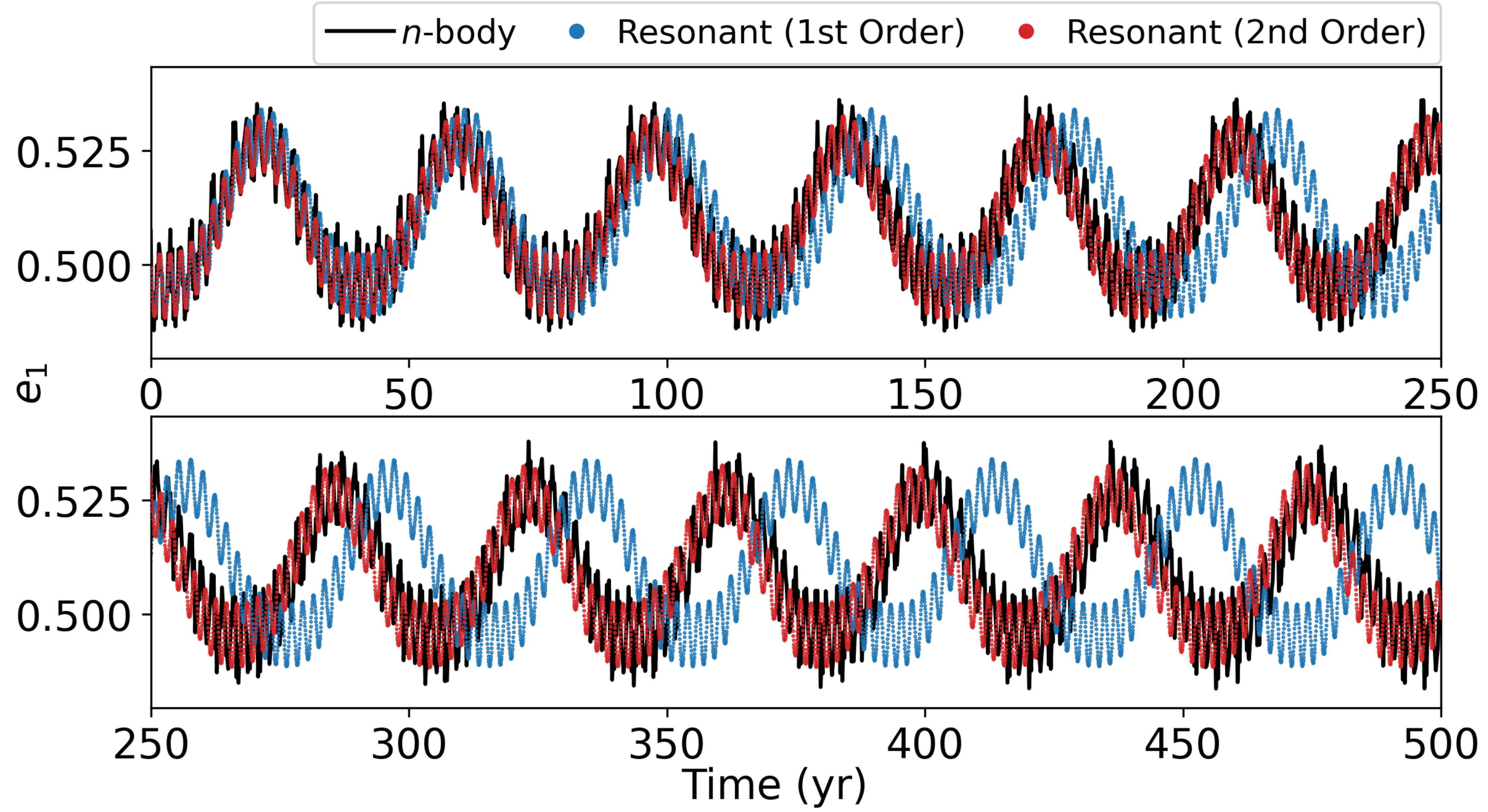
Relative Error (%) w.r.t to n-body

TIC 279401253 System (Bozhilov et al. 2023)

Masses: 6.38 and 8.13 M_{Jup}

Periods: 76.8 and 155.2 days

Eccentricities: 0.45 and 0.22

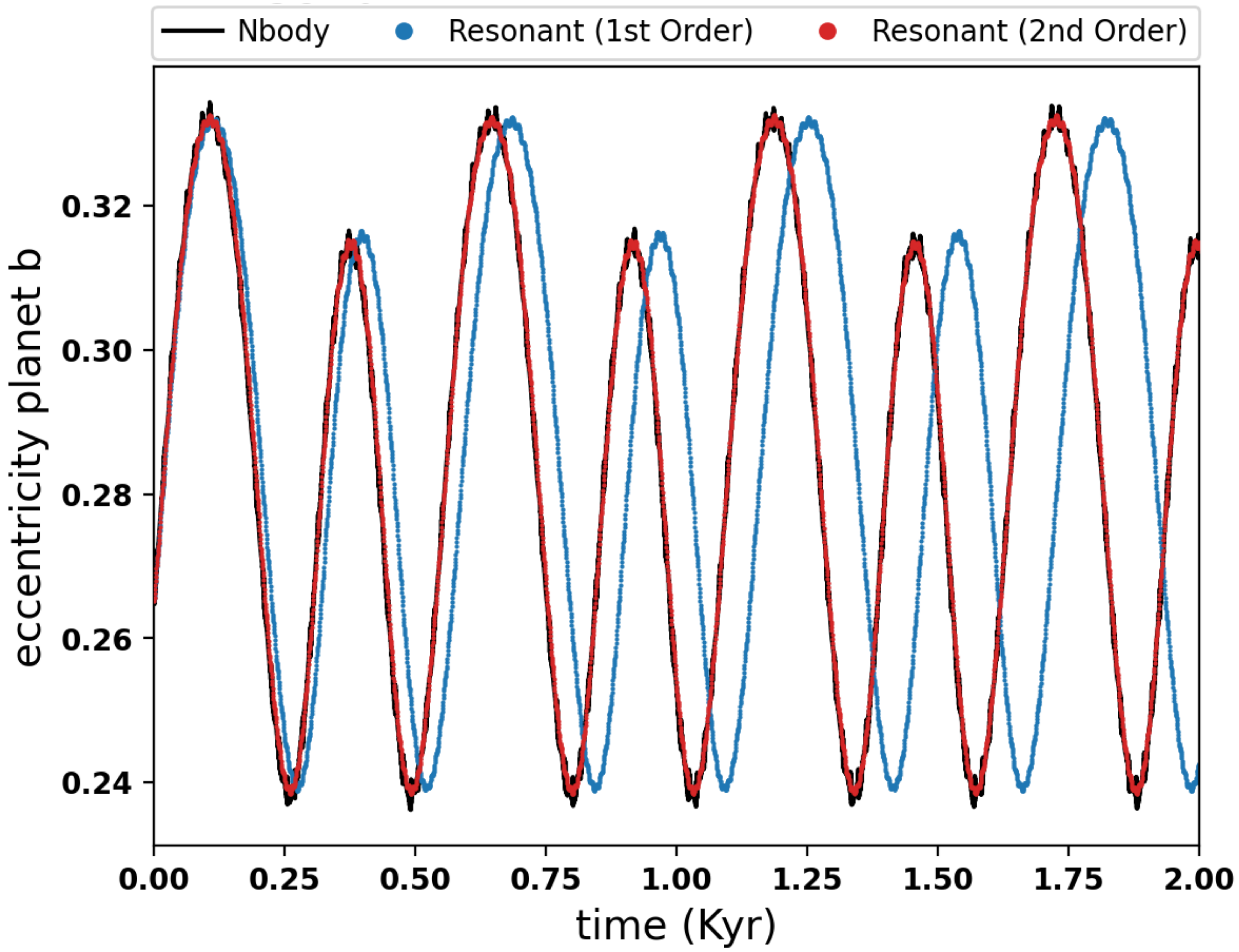


Relative Error (%)	g_1	g_2
Resonant 1st	0.2	5.1
Resonant 2nd	1.2	0.3

Relative Error (%) w.r.t to n-body

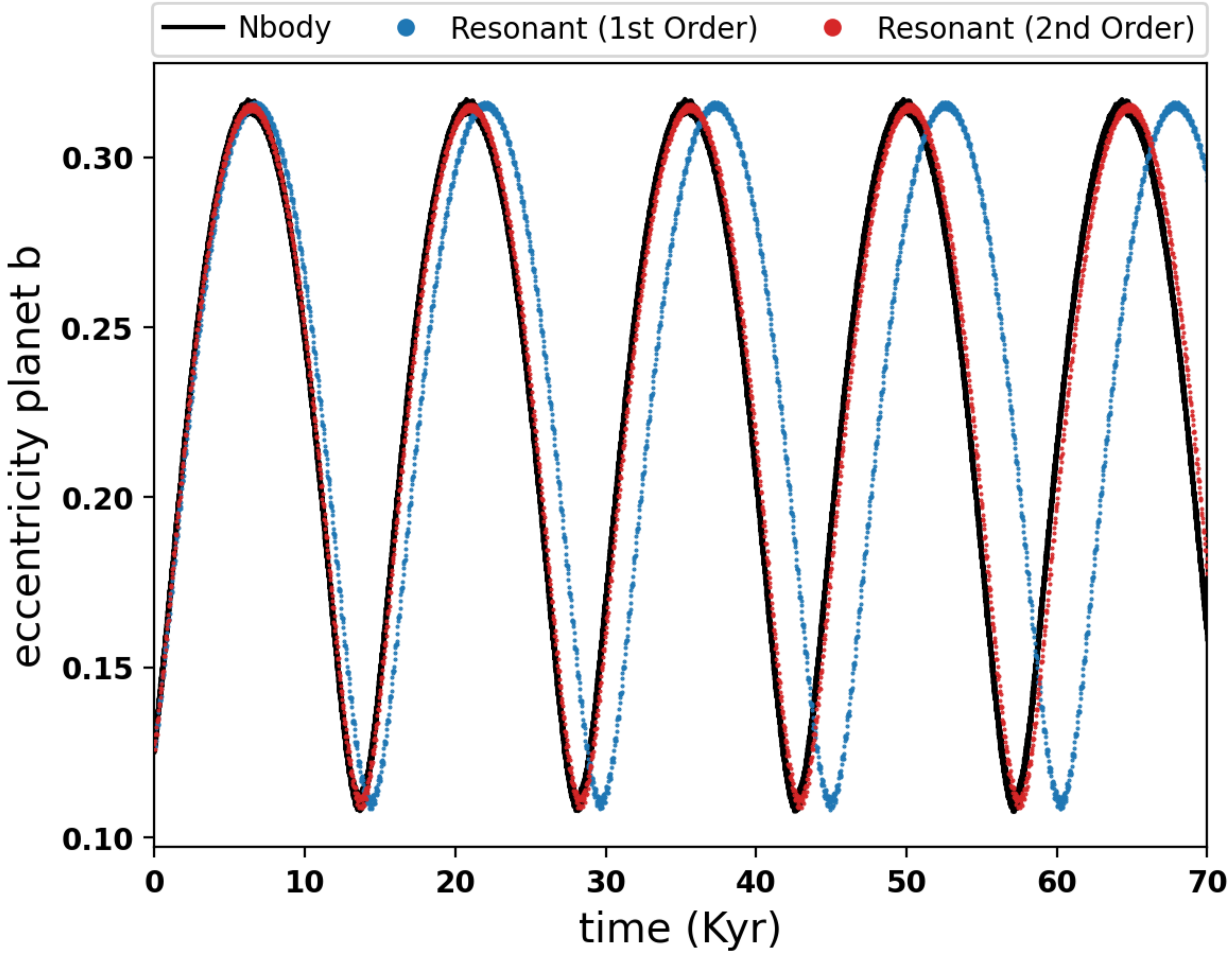
HD 73526 System (Tinney et al. 2006)

Masses: 2.35 and 2.19 M_{Jup}
Periods: 189.7 and 376.9 days
Eccentricities: 0.27 and 0.20



HD 108874 System (Wright et al. 2009)

Masses: 1.34 and 1.06 M_{Jup}
Periods: 394 and 1680 days
Eccentricities: 0.13 and 0.27



Conclusion

- Deployed a second-order secular and resonant perturbation theory for planetary systems using a complete numerical approach.

- Improved accuracy in modelling long-term dynamics of planetary systems, especially near mean motion resonances.

- This is crucial to determine the chaotic character of the secular dynamics, as is the case in the Solar System

- For more details:
  - A. Alnajjarine, F. Mogavero, J. Laskar, “Numerical approach to second-order canonical perturbation theory in the planetary 3-body problem. Application to exoplanets ”, Physical Review D, Accepted (2025).