# Recent progress in lattice QCD at non-zero temperature and density

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- Fluctuations of conserved charges and equation of state at non-zero baryon density
  HotQCD Collaboration, arXiv:2212.09043, PRD 105(2022) 074511,
  PRD 105 (2021 074512; D. Biswas, PP,  S. Sharma, work in progress

- Microscopic origin of universal scaling near the chiral transition
  H.T. Ding, W.-P. Huang, S. Mukherjee, PP, arXiv:2305.10916

- Heavy quark diffusion coefficient from lattice QCD
  HotQCD Collaboration, PRL 130 (2023)  231902

- Complex heavy quark potential at $T>0$
  HotQCD Collaboration, work in progress

xQCD 2023, Coimbra, Portugal, July 26-28, 2023

## QCD thermodynamics at non-zero chemical potential

Taylor expansion :

$$\frac{p(T, \mu_B, \mu_Q, \mu_S, \mu_C)}{T^4} = \sum_{ijkl} \frac{1}{i!j!k!l!} \chi_{ijkl}^{BQSC} \left(\frac{\mu_B}{T}\right)^i \left(\frac{\mu_Q}{T}\right)^j \left(\frac{\mu_S}{T}\right)^k \left(\frac{\mu_C}{T}\right)^l \quad \text{hadronic}$$

$$\frac{p(T, \mu_u, \mu_d, \mu_s, \mu_c)}{T^4} = \sum_{ijkl} \frac{1}{i!j!k!l!} \chi_{ijkl}^{udsc} \left(\frac{\mu_u}{T}\right)^i \left(\frac{\mu_d}{T}\right)^j \left(\frac{\mu_s}{T}\right)^k \left(\frac{\mu_c}{T}\right)^l \quad \text{quark}$$

$$\chi_{ijkl}^{abcd} = T^{i+j+k+l} \frac{\partial^i}{\partial\mu_b^i} \frac{\partial^j}{\partial\mu_b^j} \frac{\partial^k}{\partial\mu_c^i} \frac{\partial^l}{\partial\mu_d^l} \ln Z(T, V, \mu_a, \mu_b, \mu_c, \mu_d) \mid_{\mu_a = \mu_b = \mu_c = \mu_d = 0}$$

Taylor expansion coefficients give the fluctuations and correlations of conserved charges, e.g.

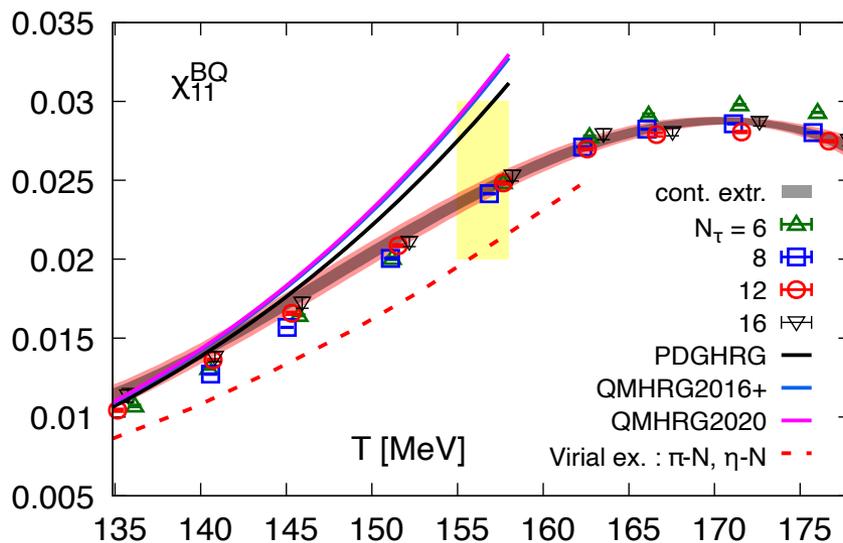$$\chi_2^X = \chi_X = \frac{1}{VT^3}(\langle X^2 \rangle - \langle X \rangle^2) \qquad \chi_{11}^{XY} = \frac{1}{VT^3}(\langle XY \rangle - \langle X \rangle \langle Y \rangle)$$
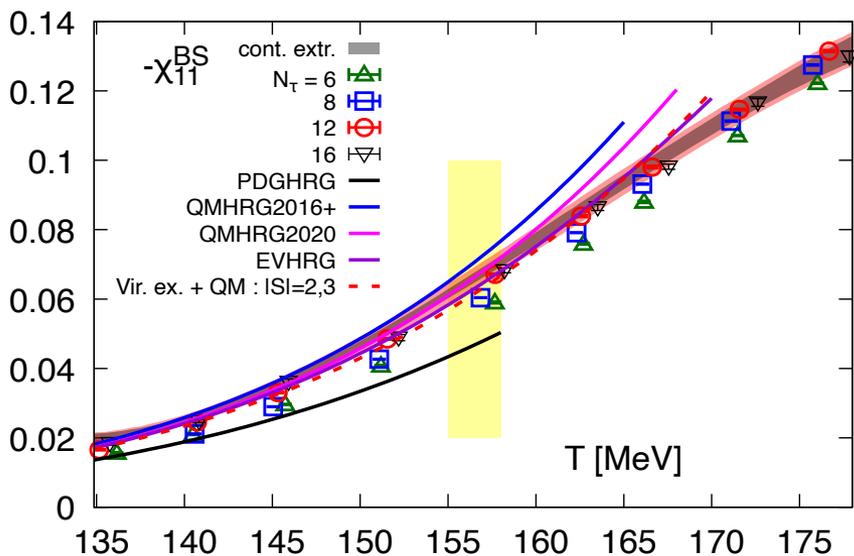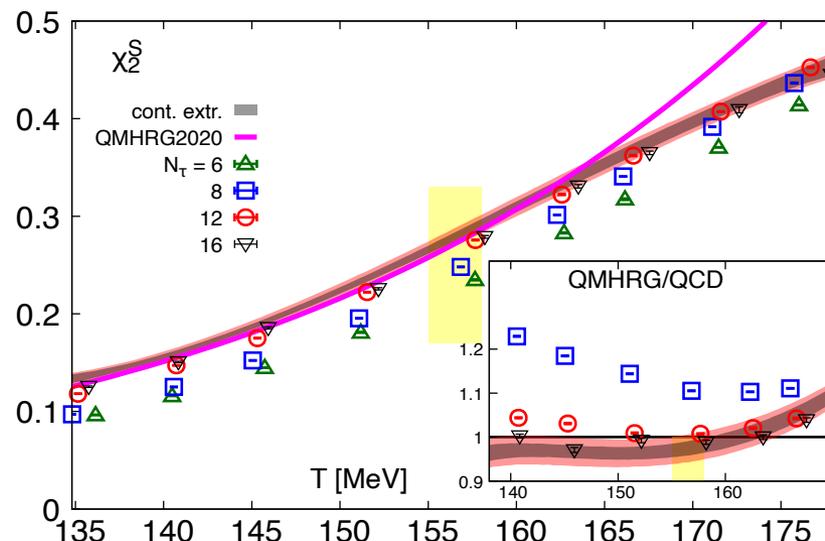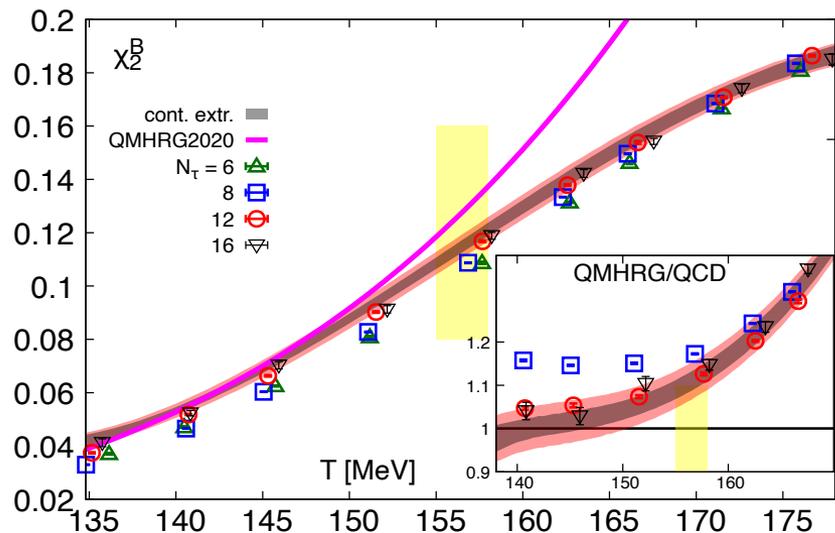
information about carriers of the conserved charges ( hadrons or quarks )
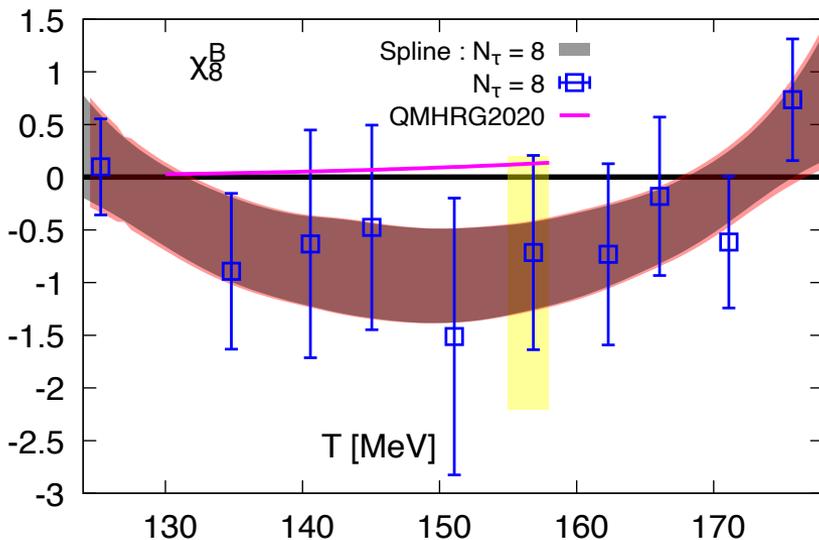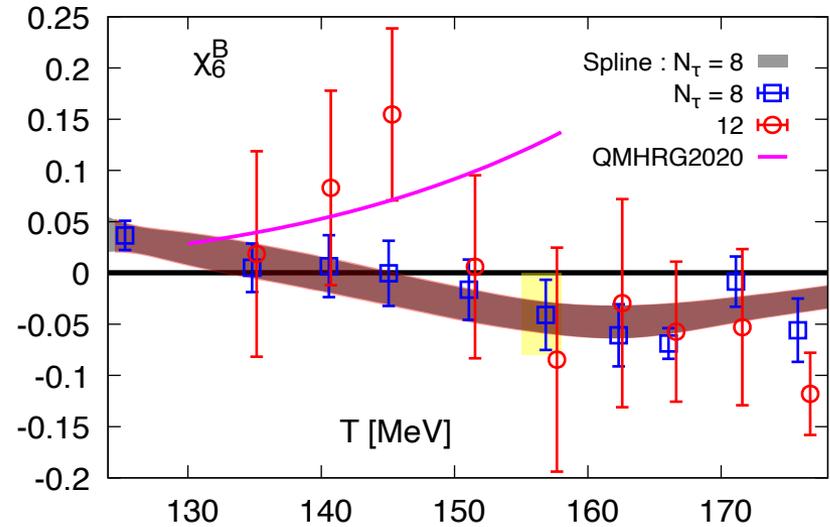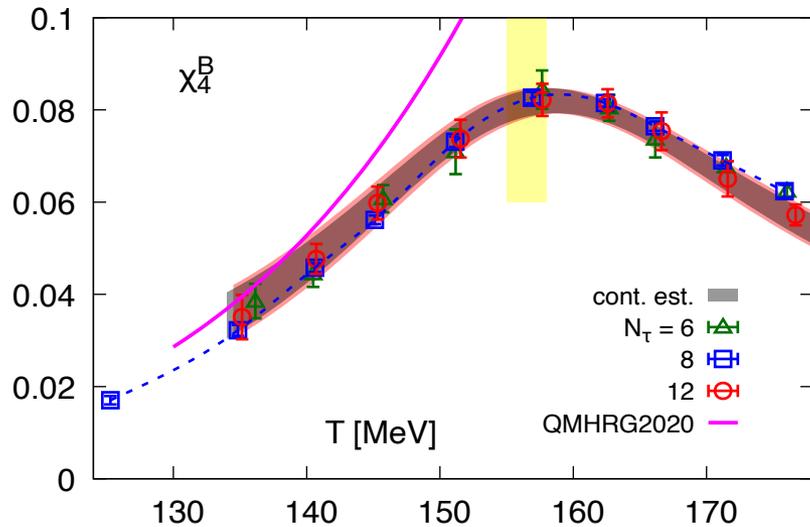
probes of deconfinement

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HISQ, $m_\pi^{phys}$, $a = 1/(TN_\tau)$



HRG works up to temperatures $\simeq 145\text{-}150$ MeV

# Higher order Taylor expansion coefficients and HRG

HISQ, $m_\pi^{phys}$, $a = 1/(TN_\tau)$



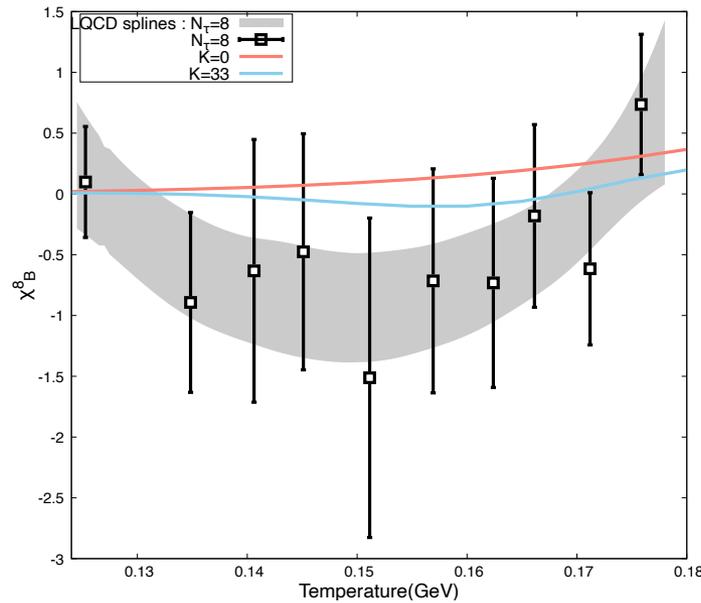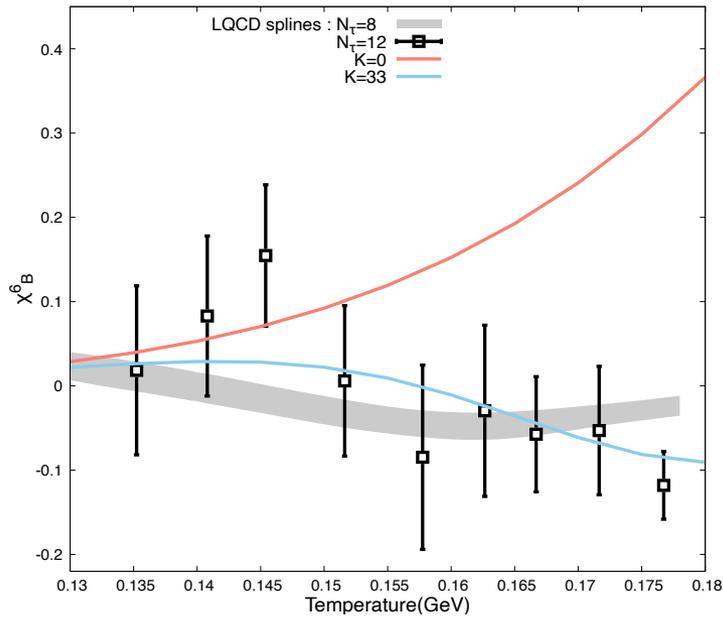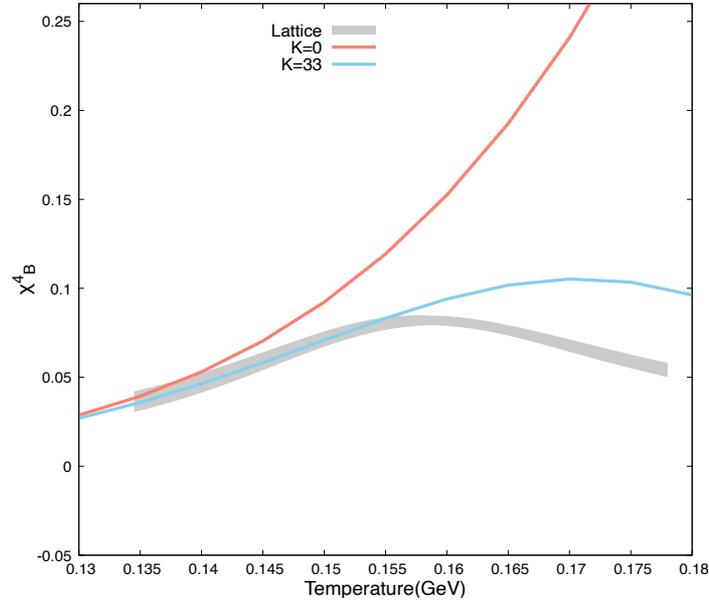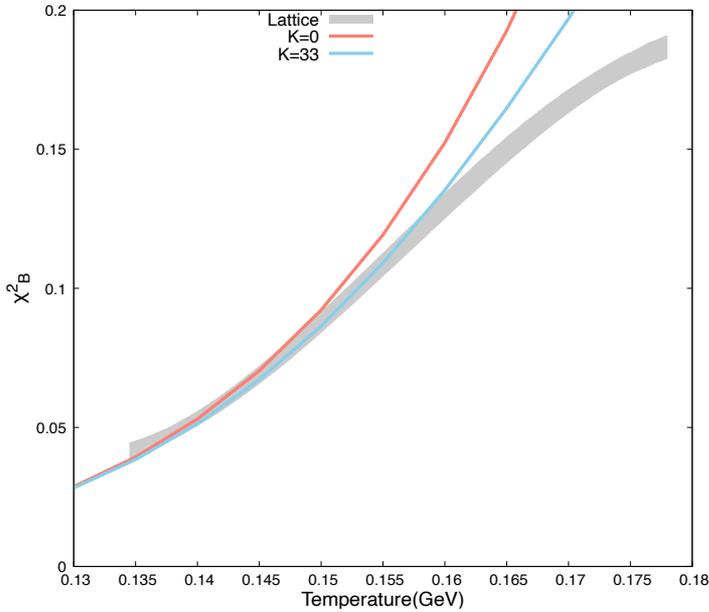For 4th order expansion coefficient HRG may work only for $T<140$ MeV

For 6th and 8th order expansion coefficients turn negative around $T_c$ HRG, only works for $T<135$ MeV

Possibly no singularity for real values of baryon chemical potential.

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# HRG with repulsive mean field



D. Biswas, PP,
S. Sharma,
work in progress

Improved
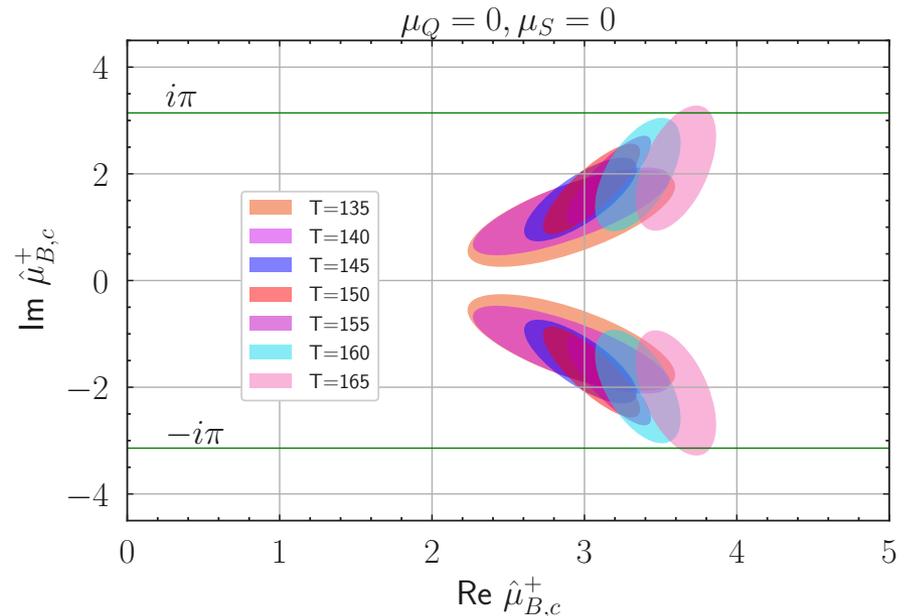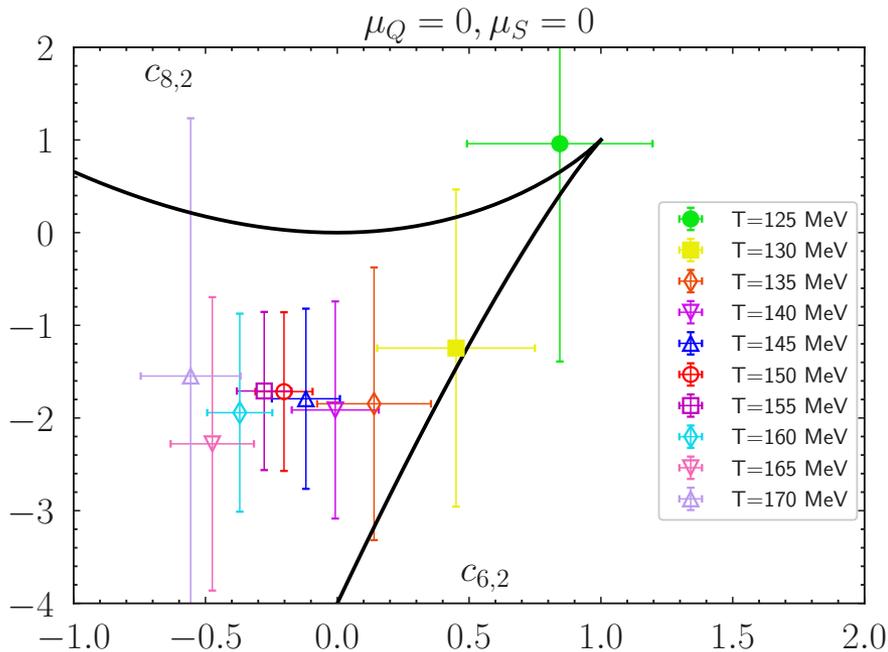agreement
between lattice
and HRG.

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$$\Delta P(T, \mu_B) = P(T, \mu_B) - P(T, 0) = \sum_{k=1}^{\infty} P_{2k} \mu_B^{2k} \qquad \bar{x} = (\mu_B/T) \cdot \sqrt{P_4/P_2}$$

$$\frac{\Delta P(T, \mu_B)}{T^4} = \frac{P_2^2}{P_4} \sum_{k=1}^{\infty} c_{2k,2} \bar{x}^{2k} = \frac{P_2^2}{P_4} (\bar{x}^2 + \bar{x}^4 + c_{6,2} \bar{x}^6 + c_{8,2} \bar{x}^8 + ...) \to \frac{P_2^2}{P_4} P_{[4,4]}$$

$$c_{6,2} = \frac{P_6 P_2}{P_4^2} = \frac{2}{5} \frac{\chi_6^B \chi_2^B}{(\chi_4^B)^2} \, , \qquad P_{[4,4]} = \frac{(1 - c_{6,2}) \bar{x}^2 + (1 - 2c_{6,2} + c_{8,2}) \bar{x}^4}{(1 - c_{6,2}) + (c_{8,2} - c_{6,2}) \bar{x}^2 + (c_{6,2}^2 - c_{8,2}) \bar{x}^4} .$$
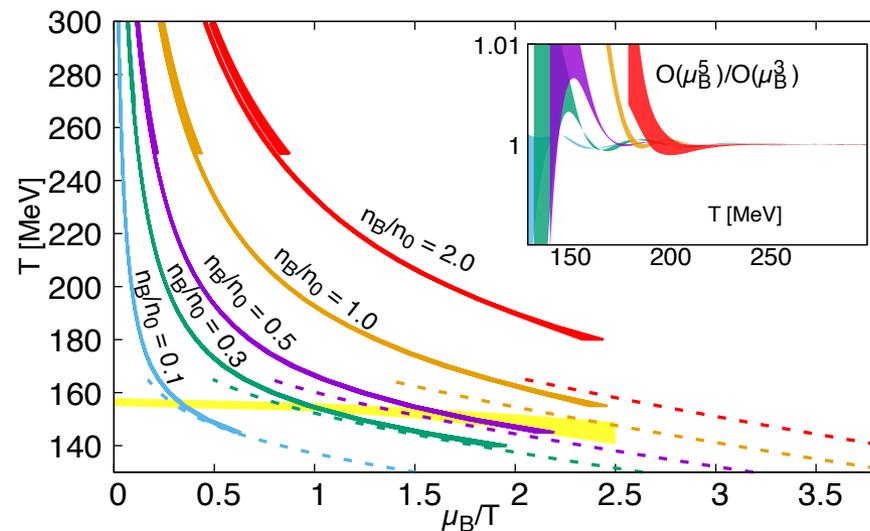
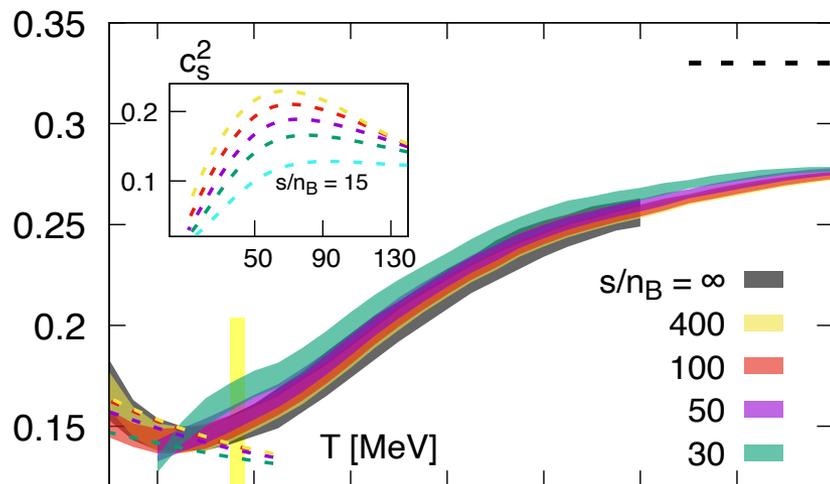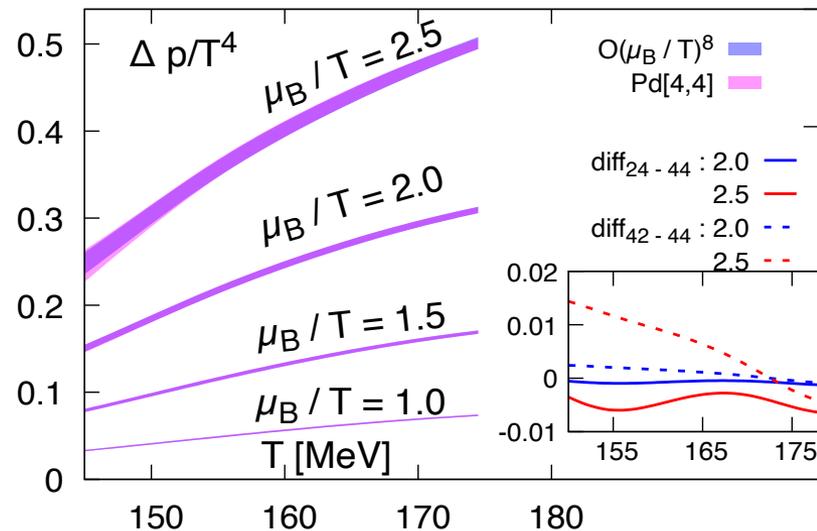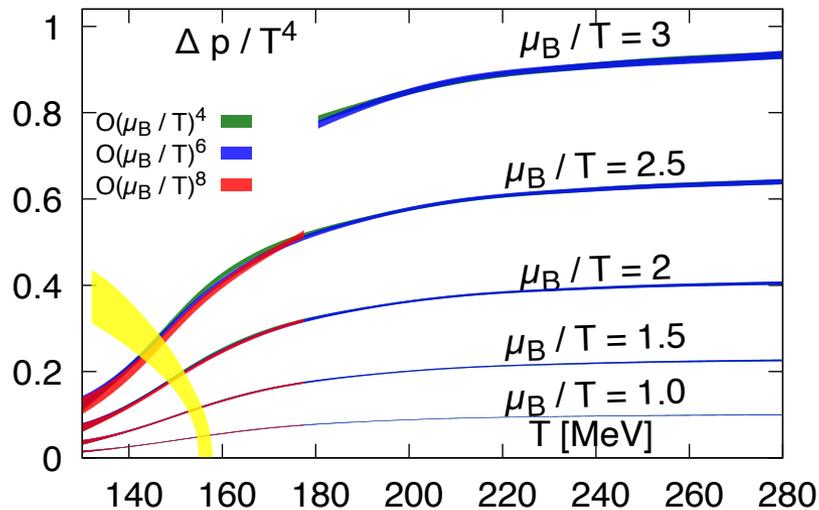$$c_{8,2} = \frac{P_8 P_2^2}{P_4^3} = \frac{3}{35} \frac{\chi_8^B (\chi_2^B)^2}{(\chi_4^B)^3} \, .$$

Padé poles: Mercer-Roberts estimators of radius of convergence



For 135 MeV $< T <$ 165 MeV only complex poles for $|\mu_B/T| > 2.5$

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$$\epsilon(T_{\mathrm{pc}}(\mu_B)) = \begin{cases} 370(40)(30) \text{ MeV/fm}^3 \ , \ \mu_B/T = 0 \\ 330(28)(53) \text{ MeV/fm}^3 \ , \ \mu_B/T = 2.5 \end{cases}$$

# Chiral transition and spectrum of Dirac eigenvalues

Macroscopic

$$\bar{\psi}\psi(m) \equiv 2\mathrm{Tr}(\not{D}[\mathcal{U}] + m)^{-1}$$

Microscopic

$$= 2\sum_j (i\lambda_j + m)^{-1}$$

$$P_{\mathcal{U}}(\lambda; m) = \frac{4m\rho_{\mathcal{U}}(\lambda)}{\lambda^2 + m^2}, \text{ and } \rho_{\mathcal{U}}(\lambda) = \sum_j \delta(\lambda - \lambda_j)$$

$$\mathbb{K}_1(\bar{\psi}\psi) = \frac{T}{V}\langle(\bar{\psi}\psi)\rangle$$

$$= \int_0^\infty K_1[P_{\mathcal{U}}(\lambda; m_l)] \, \mathrm{d}\lambda = \frac{T}{V}\int_0^\infty \mathrm{d}\lambda \frac{4m_l\langle\rho_{\mathcal{U}}(\lambda)\rangle}{\lambda^2 + m_l^2} =$$

$$= \int_0^\infty P_1(\lambda)d\lambda$$

$$\mathbb{K}_2(\bar{\psi}\psi) = \frac{T}{V}\langle[(\bar{\psi}\psi) - \langle\bar{\psi}\psi\rangle]^2\rangle$$

$$= \int_0^\infty K_1[P_{\mathcal{U}}(\lambda_1; m_l), P_{\mathcal{U}}(\lambda_2; m_l)] \, \mathrm{d}\lambda_1 \mathrm{d}\lambda_2$$

**Chiral susceptibility**

$$= \frac{T}{V}\int_0^\infty \mathrm{d}\lambda_1 \mathrm{d}\lambda_2 \frac{(4m_l)^2}{(\lambda_1^2 + m_l^2)(\lambda_2^2 + m_l^2)} \times$$

$$[\langle\rho_{\mathcal{U}}(\lambda_1)\rho_{\mathcal{U}}(\lambda_2)\rangle - \langle\rho_{\mathcal{U}}(\lambda_1)\rangle\langle\rho_{\mathcal{U}}(\lambda_2)\rangle] = \int_0^\infty P_2(\lambda)d\lambda$$

$$\mathbb{K}_3(\bar{\psi}\psi) = \frac{T}{V}\langle[(\bar{\psi}\psi) - \langle\bar{\psi}\psi\rangle]^3\rangle$$

$$= \int_0^\infty P_3(\lambda)d\lambda$$

**Binder cumulant**

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$$m_l \to 0: \quad \frac{m}{\lambda^2 + m^2} \to \pi\delta(\lambda)$$

$$\lim_{m_l \to 0} \mathbb{K}_1(\bar{\psi}\psi) = \lim_{m_l \to 0} \frac{T}{V}\langle(\bar{\psi}\psi)\rangle = 2\pi\mathbb{K}_1[\rho_{\mathcal{U}}(0)] = 2\pi\langle\rho_{\mathcal{U}}(0)\rangle \qquad \text{Banks-Casher relation}$$

$$\lim_{m_l \to 0} \mathbb{K}_n(\bar{\psi}\psi) = \lim_{m_l \to 0} = (2\pi)^n \mathbb{K}_n[\rho_{\mathcal{U}}(0)]$$

Universal *O(N)* scaling of the cumulants of the chiral condensate:

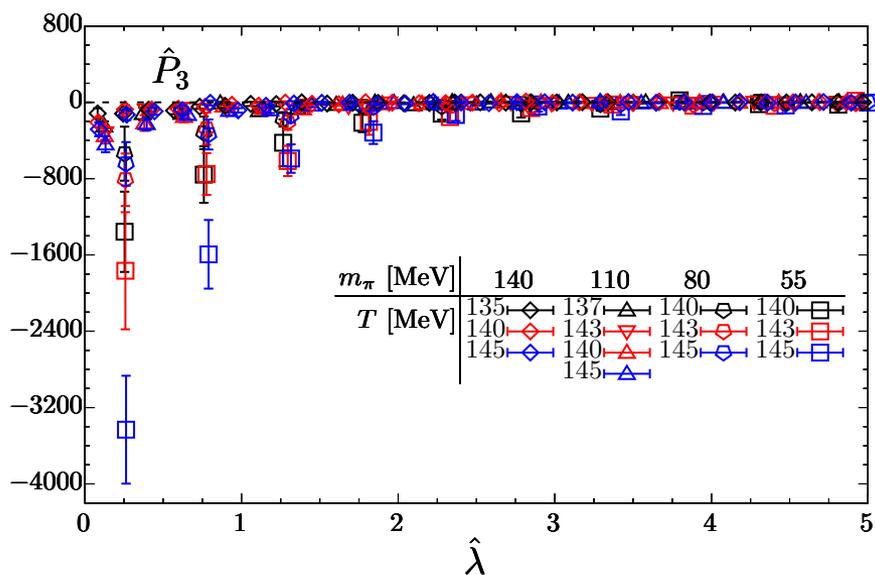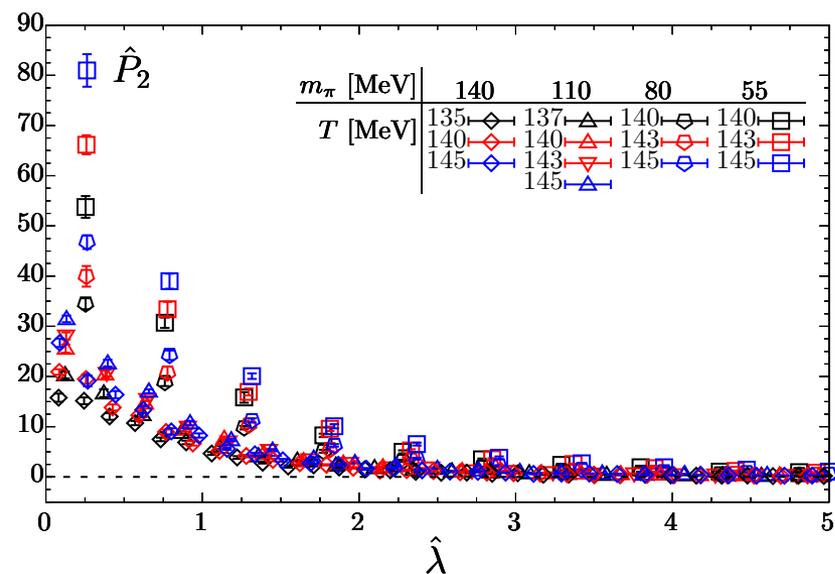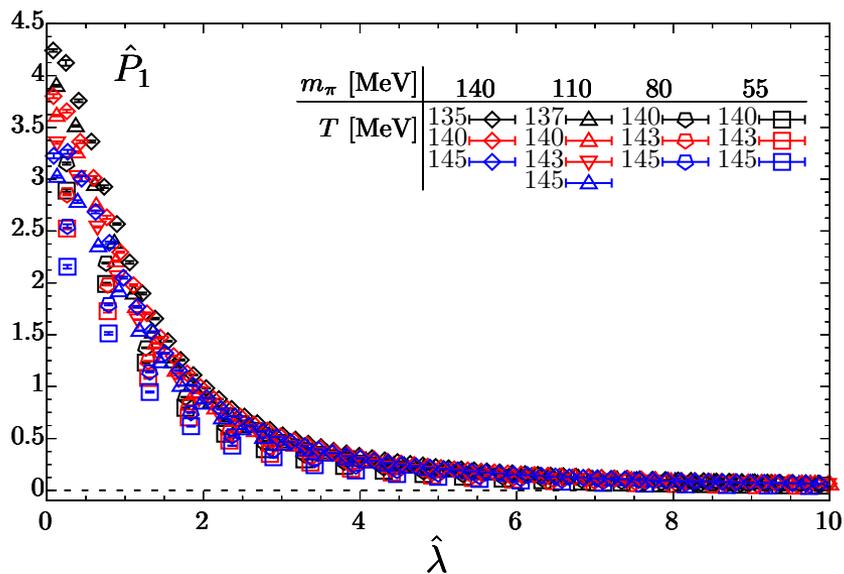$$\mathbb{K}_n[\bar{\psi}\psi] = \int_0^\infty P_n(\lambda)d\lambda \sim m_l^{1/\delta - n + 1} f_n(z)$$

*O(N)* scaling function

$$z \propto z_0 m_l^{-1/\beta\delta}(T - T_c)/T_c$$

Conjecture: $\quad P_n(\lambda) = m_l^{1/\delta - n + 1} f_n(z)g_n(\lambda)$
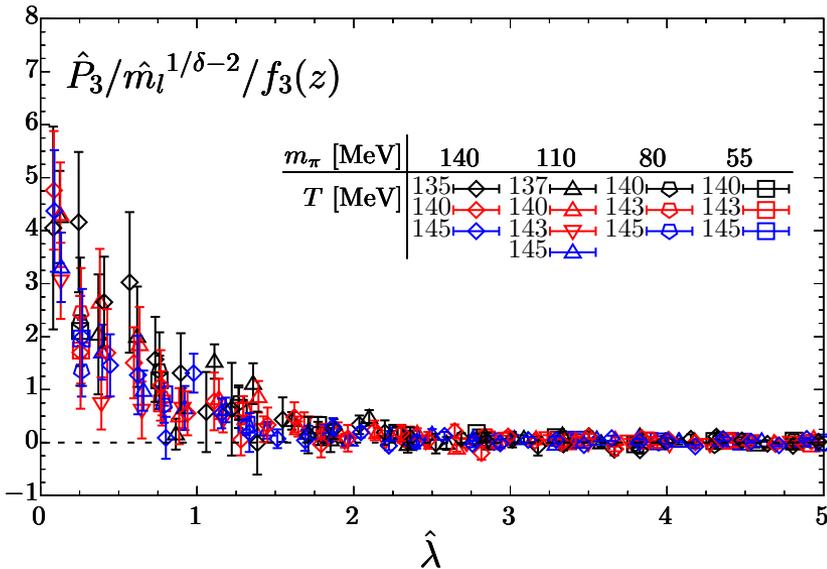
non-universal scaling function
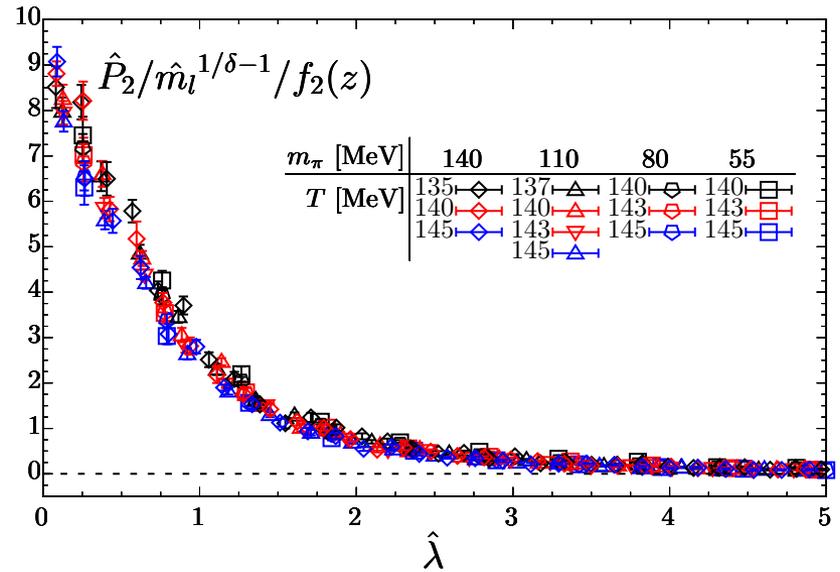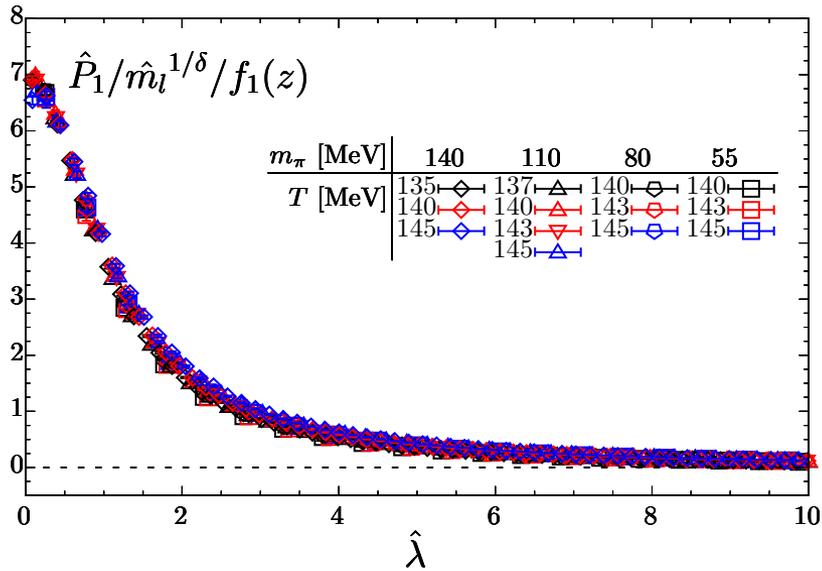
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HISQ, $m_\pi^{phys}$, $N_\tau = 8$, $N_\sigma = 32 - 56$
$m_l/m_s = 1/27,\ 1/40,\ 1/80,\ 1/160$
$m_\pi = 140,\ 110,\ 80,\ 55$ MeV
$T = 135 - 176$ MeV

$$\hat{\lambda} = \lambda/m_l$$

Strong quark mass and temperature dependence

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Scaling works for $T<T_c \simeq 144$ MeV !

Dirac eigenvalues (energy levels of quarks) know about universality class of the QCD chiral transition

# Heavy quark diffusion and lattice QCD

Obtain the momentum heavy quark transport coefficient through the force correlator

$$\langle f_i(t) f_j(t) \rangle = \langle E_i(t) E_j(t') \rangle + \frac{1}{3} \langle \mathbf{v}^2 \rangle \langle \delta_{ij} B_k(t) B_k(t') - B_i(t') B_j(t) \rangle \qquad \langle \mathbf{v}^2 \rangle = \frac{3T}{M}$$

$$t \to i\tau$$

### Can be rigorously derived in Heavy Quark Effective Theory

Casalderrey-Solana, Teaney, PRD 74 (2006) 085012; Caron-Huot, Laine, Moore, JHEP 0904 ('09) 053

Bouttefeux, Laine, JHEP 12 (2020) 150

$$G_E(\tau) = -\frac{1}{3} \sum_{i=1}^{3} \frac{\left\langle \mathrm{ReTr}\left[ U(\beta, \tau)\, g E_i(\tau, \vec{0})\, U(\tau, 0)\, g E_i(0, \vec{0}) \right] \right\rangle}{\left\langle \mathrm{ReTr}[U(\beta, 0)] \right\rangle} \qquad \kappa_E = \lim_{\omega \to 0} \frac{2T}{\omega} \rho_E(\omega)$$

$$G_B(\tau) = -\frac{1}{3} \sum_{i=1}^{3} \frac{\left\langle \mathrm{ReTr}\left[ U(\beta, \tau)\, g B_i(\tau, \vec{0})\, U(\tau, 0)\, g B_i(0, \vec{0}) \right] \right\rangle}{\left\langle \mathrm{ReTr}[U(\beta, 0)] \right\rangle} \qquad \kappa_B = \lim_{\omega \to 0} \frac{2T}{\omega} \rho_B(\omega)$$
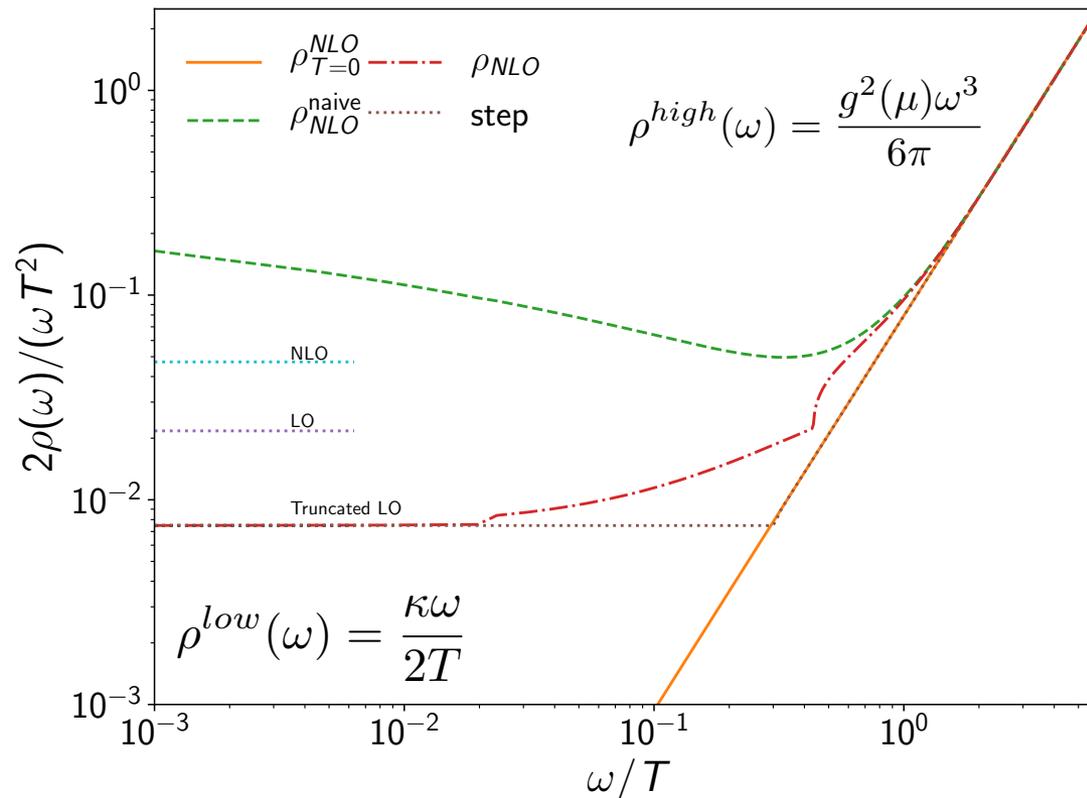
$$G_{E,B}(\tau) = \int_0^\infty \frac{\mathrm{d}\omega}{\pi} \rho_{E,B}(\omega) \frac{\cosh\left(\tau - \frac{1}{2T}\right)\omega}{\sinh\frac{\omega}{2T}}$$

$$\kappa = \kappa_E + \frac{2}{3} \langle \mathbf{v}^2 \rangle \kappa_B$$

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Challenge 1:  obtain precise results for chromo-electric  and chromo-magnetic  (very noisy)

$\Rightarrow$ Noise reduction via multi-level algorithm, applicable to quenched QCD (pure glue plasma)

$\Rightarrow$ Noise reduction by gradient flow method (new development !) , also applicable in full QCD

Challenge 2:  reconstruct the spectral function  from the Euclidean time lattice correlator



$$\rho^{high}(\omega) = \frac{g^2(\mu)\omega^3}{6\pi}$$

$$\rho^{low}(\omega) = \frac{\kappa\omega}{2T}$$

$\Rightarrow$ use known  large and small energy  behavior of the spectral

Parameterize $\rho(\omega, T)$ as smooth interpolation between $\rho^{low}(\omega, T)$ and $\rho^{high}(\omega)$, and treat $\kappa$ as well as the additional nuisance parameters of interpolation as fit parameters

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2+1 flavor QCD with $m_l = m_s/5$ ($m_\pi = 320$ MeV), $T = 195 - 354$ MeV, $96^3 \times N_\tau$ lattices with $N_\tau = 36, 32, 28, 24, 20$; additional $64^3 \times N_\tau$ lattices with $N_\tau = 20, 22, 24 \Rightarrow 3$ lattice spacings at each $T$; Gradient flow for noise reduction
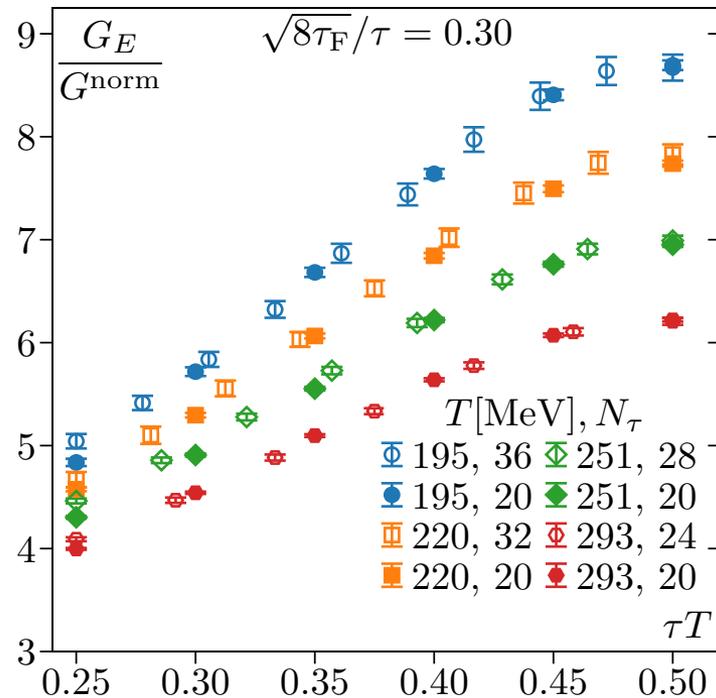
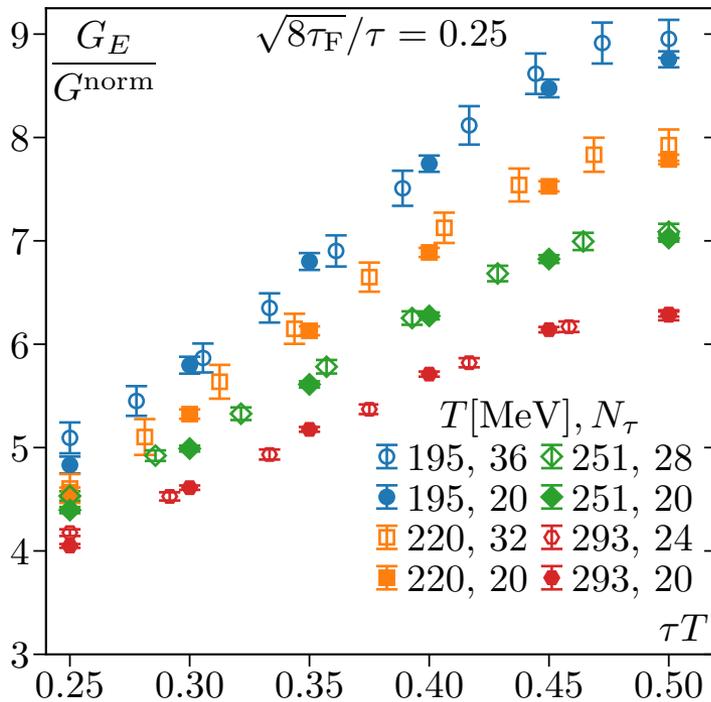$A_\mu(x) \to B_\mu(\tau_F, x)$     $\partial_{\tau_F} B_\mu(\tau_F, x) = -g_0^2 \dfrac{\delta S_{\mathrm{YM}}[B]}{\delta B_\mu(\tau_F, x)}$     Symanzik gauge action and Zeuthen flow

$B_\mu(0, x) = A_\mu(x)$

Gauge fields are smeared in the radius $\sqrt{8\tau_F}$     $a < \sqrt{8\tau_F} < \tau/3$



We see small cutoff effects thanks improved actions

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## Analysis of the chromo-electric correlator:

- Extrapolate the lattice results on the chromo-electric correlator to the continuum limit
- Perform the zero flow time extrapolation

## Fits to model spectral function:

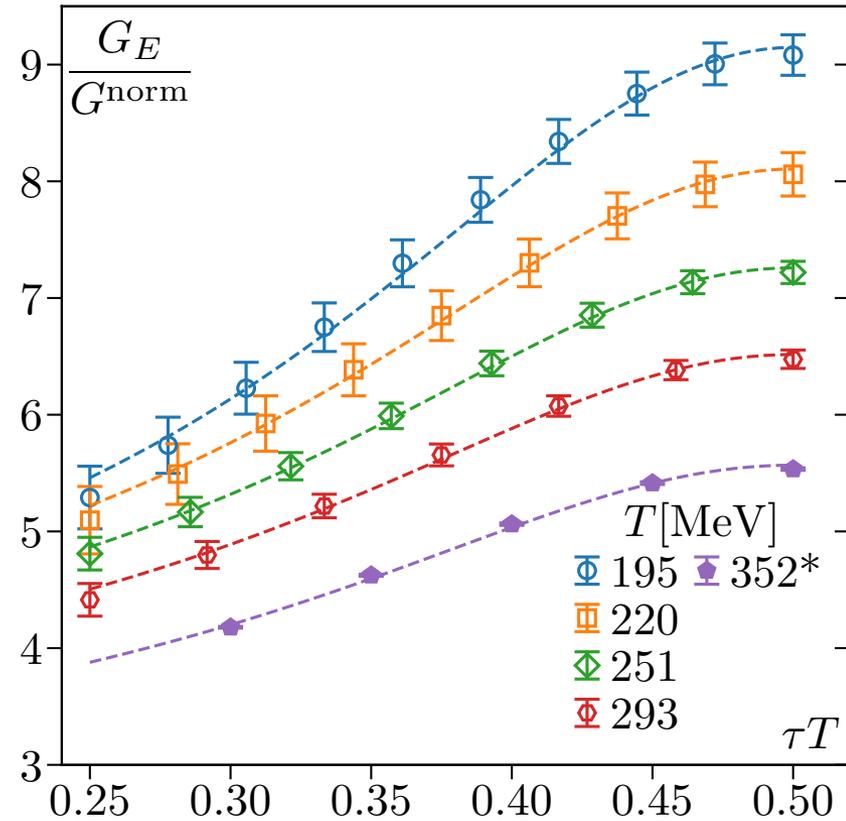$$\rho^{low}(\omega, T) = \frac{\kappa\omega}{2T} \qquad \rho^{high}(\omega) = \rho^{LO,NLO}(\omega)$$

$$\rho^{max}(\omega, T) = max(\rho^{low}(\omega, T), \rho^{high}(\omega))$$
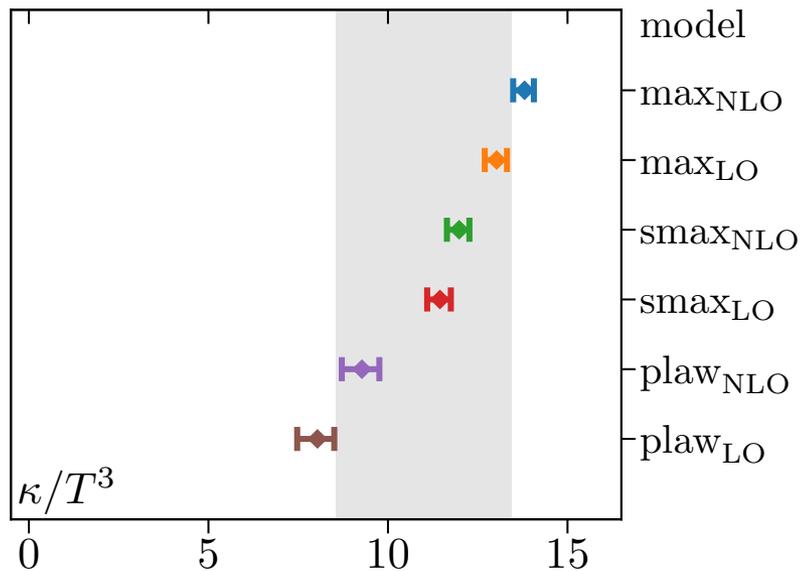
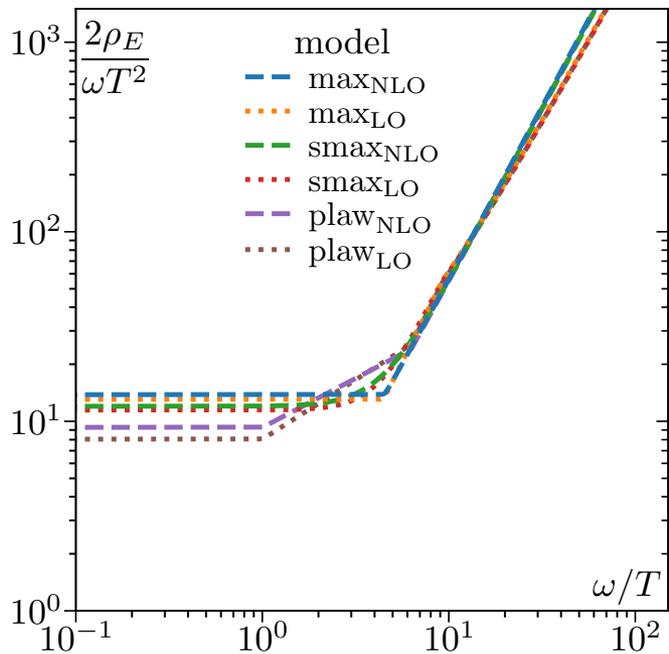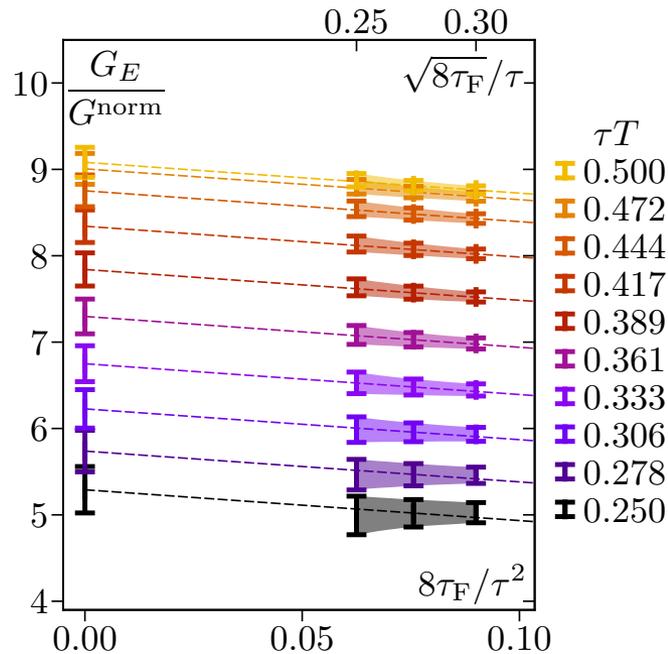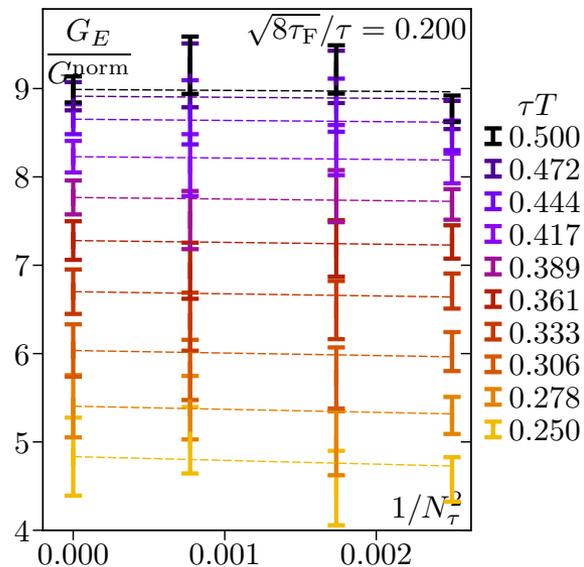$$\rho^{smax}(\omega, T) = \sqrt{(\rho^{low})^2 + (\rho^{high})^2}$$

$$\rho^{pow}(\omega, T) = \rho^{low}(\omega, T), \ \omega \leq \omega_{IR}$$

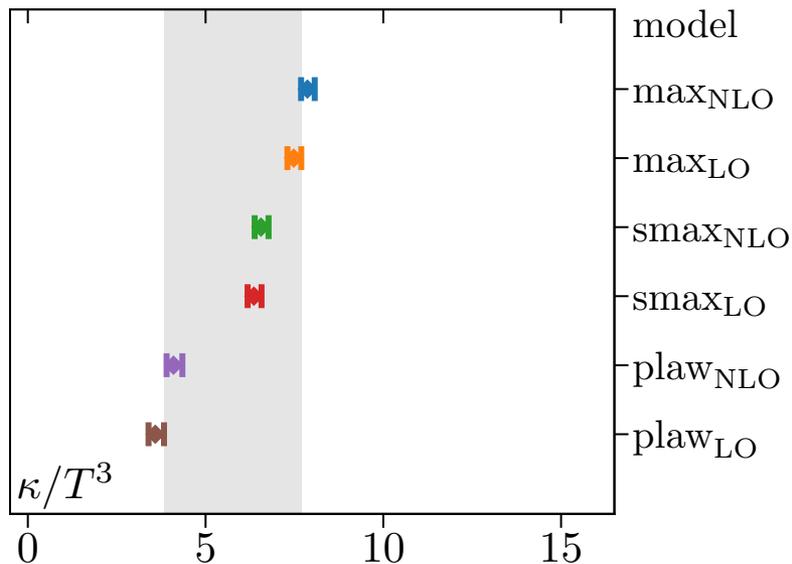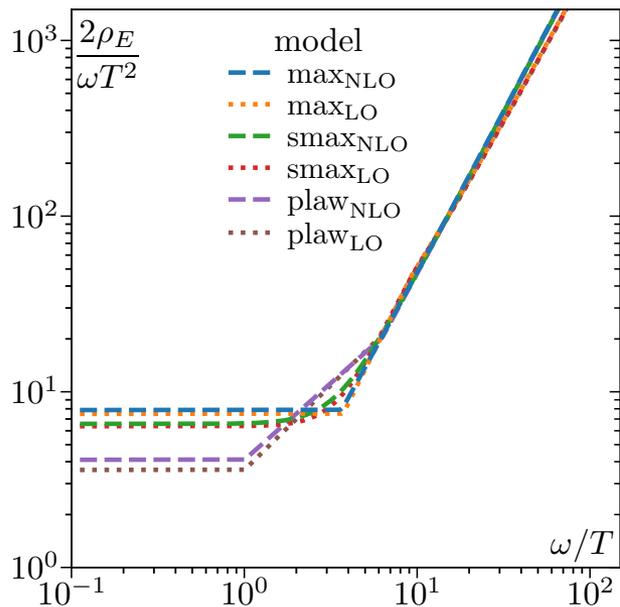$$\rho^{pow}(\omega, T) = A\omega^{\alpha}, \ \omega_{IR} < \omega < \omega_{UV}$$
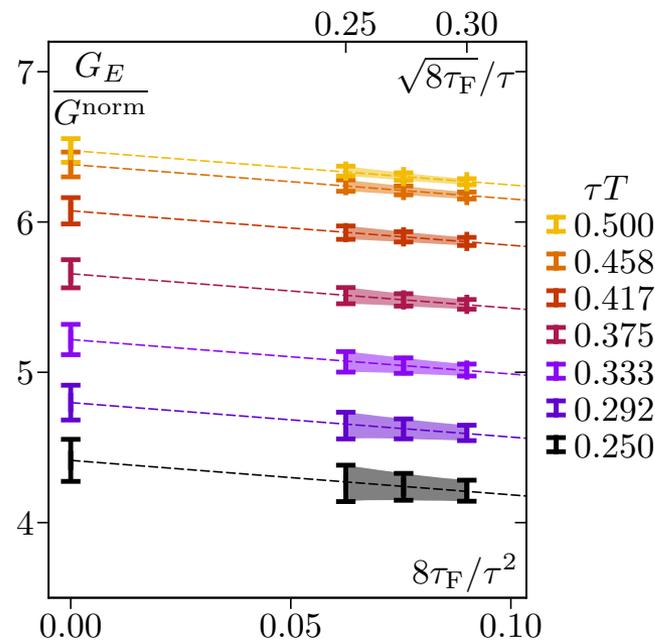
$$\rho^{pow}(\omega) = \rho^{high}(\omega), \ \omega \geq \omega_{UV}$$



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**T=195 MeV:**

T=293 MeV:

- $\kappa/T^3$ has significant temperature dependence

- $D_s$ is significantly smaller in 2+1 flavor QCD than in quenched QCD and is close to the AdS/CFT limit

$$D_s = \frac{2T^2}{\kappa}$$

$1/M$ correction to the momentum heavy quark diffusion coefficient $\Rightarrow G_B(\tau, T)$
$G_B(\tau, T)$ has anomalous dimension $\Rightarrow$ additional matching to $\overline{MS}$ is needed

## Quenched QCD

Gradient flow + incomplete 1-loop matching



Multi-level algorithm +
non-perturbative matching
via Schrödinger functional

$$1.5T_c : \kappa_B = (1.23 - 2.54)T^3, \qquad\qquad \kappa_B = (1.0 - 2.1)T^3$$

Brambilla, Leino, Mayer-Steudte, PP
(TUMQCD), PRD 107 (2023) 054508

Banerjee, Datta, Laine JHEP 08 (2022) 128

$<v^2>$ is taken from PP, EPJC 62 (2009) 85

10-20% correction for bottom quark, ~30% correction for charm quark

## Quark anti-quark potential at *T>0*

Conjecture, Matsui and Satz, PLB 178 (86) 416 $\quad -\dfrac{4}{3}\dfrac{\alpha_s}{r} + \sigma r \rightarrow -\dfrac{4}{3}\dfrac{\alpha_s}{r}e^{-m_D r}, T > T_c$

Extending pNRQCD to *T>0*: the potential is complex, the real part can have thermal correction but is not necessarily screened, except when *r ~ 1/m$_D$*

Based on weak coupling

Laine, Philipsen, Romatschke, Tassler, JHEP 03 (06) 054
Brambilla, Ghiglieri, PP, Vairo, PRD 78 (08) 014017

Calculate the potential non-perturbatively on the lattice by considering Wislon loops of size *r x τ* at *T>0*

$$W(r,\tau,T) = \int_{-\infty}^{\infty} \rho_r(\omega, T)e^{-\omega\tau}$$

If potential at $T > 0$ exists the $\rho_r(\omega, T)$ should have a well defined peak at $\omega \simeq \mathrm{Re}V(r,T)$, and the width of the peak is $\mathrm{Im}V(r,T)$

Rothkopf, Hatsuda, Sasaki, PRL 108 (2012) 162001

Challenge: reconstruct $\rho_r(\omega, T)$

Hybrid potentials, pairs of static-light mesons ...

$$\rho_r(\omega, T=0) = \delta(\omega - V(r)) + \sum_n \delta(\omega - E_n(r))$$

$2 + 1$ f QCD, $m_\pi = 300$ MeV $\quad T = 126, \ 196, \ 220, \ 252, \ 294, \ 354$ MeV

$a = 0.028$ fm, $96^3 \times N_\tau, N_\tau = 56, \ 36, \ 32, \ 28, \ 24, \ 20$
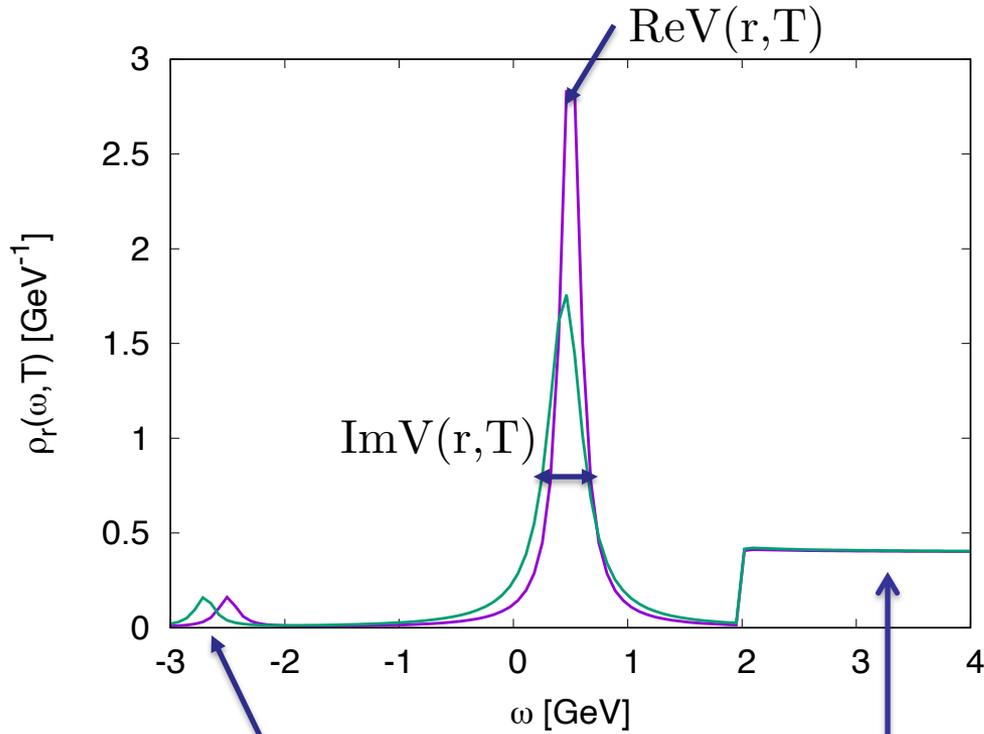
Gradient flow for noise reduction: $\quad \sqrt{8\tau_F}T = 0.04 - 0.05$

$$m_{eff}(r, \tau, T) = -\partial_\tau \log(W(\tau, r, T)) \simeq \frac{1}{a} \ln \frac{W(r, \tau, T)}{W(r, \tau + a, T)}$$



- No plateau at $T > 0$ in $m_{eff}$ at $T > 0$

- Only tiny $T$-dependence for small $\tau$

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ReV(r,T)

ImV(r,T)

$$W^{high}(r, \tau) = \int_{-\infty}^{\infty} d\omega \rho_r^{high}(\omega) e^{-\omega\tau}$$

On the lattice:

$$W^{high}(r, \tau) = W(r, \tau, T = 0) - A_0 \exp(-V(r)\tau)$$

$$W^{sub}(r, \tau, T) = W(r, \tau, T) - W^{high}(r, \tau)$$

$$\rho_r(\omega, T) = \rho_r^{tail}(\omega, T) + \rho_r^{peak}(\omega, T) + \rho_r^{high}(\omega)$$

See, Bala et al (HotQCD), PRD 105 (2022) 054513

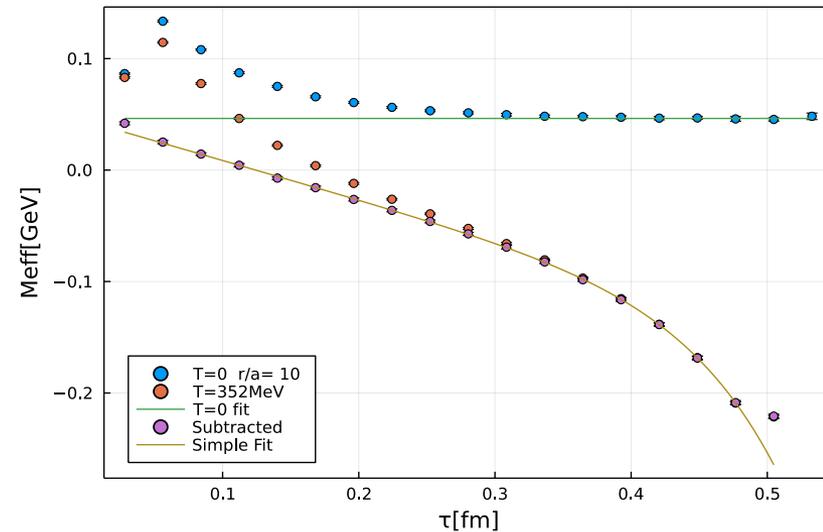Cumulants of $W^{sub}(r, \tau, T)$ carry information about $T$-dependent part of the spectral function

$m_{eff}$ for the subtracted correlator has milder $\tau$-dependence, which is approximately linear
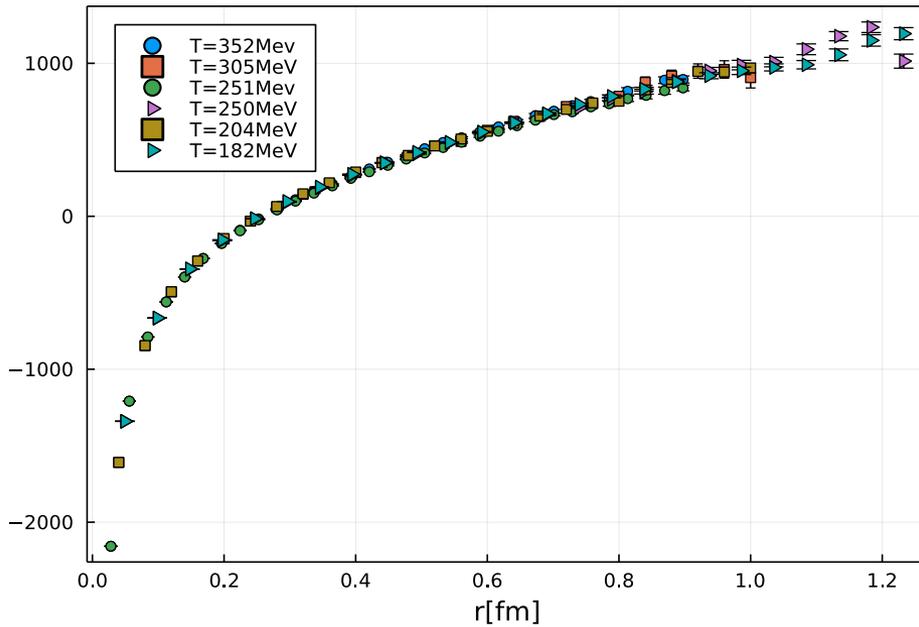
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# Model spectral function and the complex potential

$$\rho_r^{peak}(\omega, T) = \frac{A}{\pi} \frac{\Gamma(\omega, T)}{(\omega - \text{Re}V(r, T))^2 + \Gamma(\omega, T)}. \qquad \Gamma(\omega, r, T) = \begin{cases} \Gamma_0(r, T) & -2\Gamma_0 < \omega < 2\Gamma_0 \\ 0 & n \text{ otherwise} \end{cases}$$

$$\rho_r^{tail}(\omega, T) = A^{tail}\delta(\omega - E^{tail})$$



$\text{Re}V(r, T)$ shows tiny temperature dependence and no hint of screening

$\text{Im}V(r, T)$ increases with $rT$ and is proportional to $T$

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## Summary

- Hadron resonance gas (HRG) model can describe fluctuations of conserved charges in the low temperature region; The range of validity of HRG can be extended by including repulsive baryon-baryon interactions.

- Thermodynamic quantity can be obtained using Taylor expansion for $\mu_B/T < 2.5$ and no hint for critical point can be seen for these values of $\mu_B$.

- Universal aspect of the chiral transition and O(N) scaling can be seen in terms of spectral density of the eigenvalues of the Dirac operator, since cumulants of spectral density are related to the cumulants of the chiral condensate

- First full QCD calculation of the heavy quark diffusion coefficient become available now and indicate that $\kappa/T^3$ is larger than un quenched QCD and close to the AdS/CFT bound

- The quark mass suppressed effects in the heavy quark diffusion coefficient have been estimated and turn out to be $10 - 20\%$ for $m_b$ $30\%$ for $m_c$

- $\mathrm{Re}V(r,T)$ is not screened but $\mathrm{Im}V(r,T)$ increases with $T$ and $r$