Hadron scatterings in small chemical potential

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in collaboration with

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Contents

- Introduction

- Hadron scatterings at small μ

- Demonstration in QC$_2$D

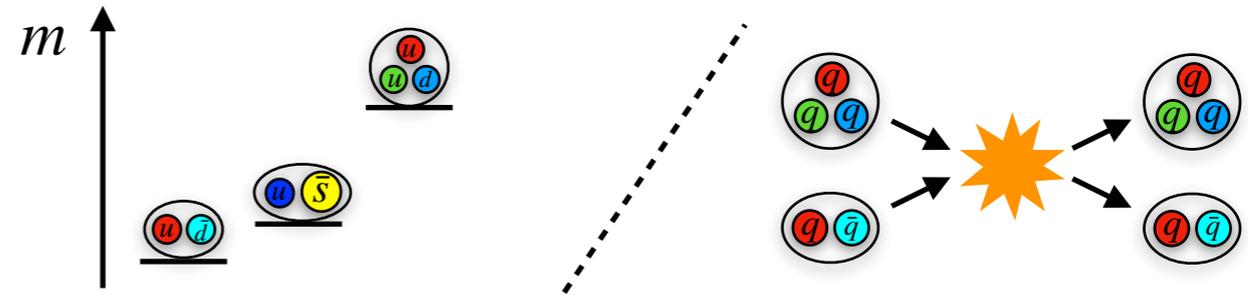- Usage of small-μ correlation function

- Conclusion/Future work

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QCD at very small μ and $T = 0$

- studies on the hadronic properties play an important role in understanding QCD in medium
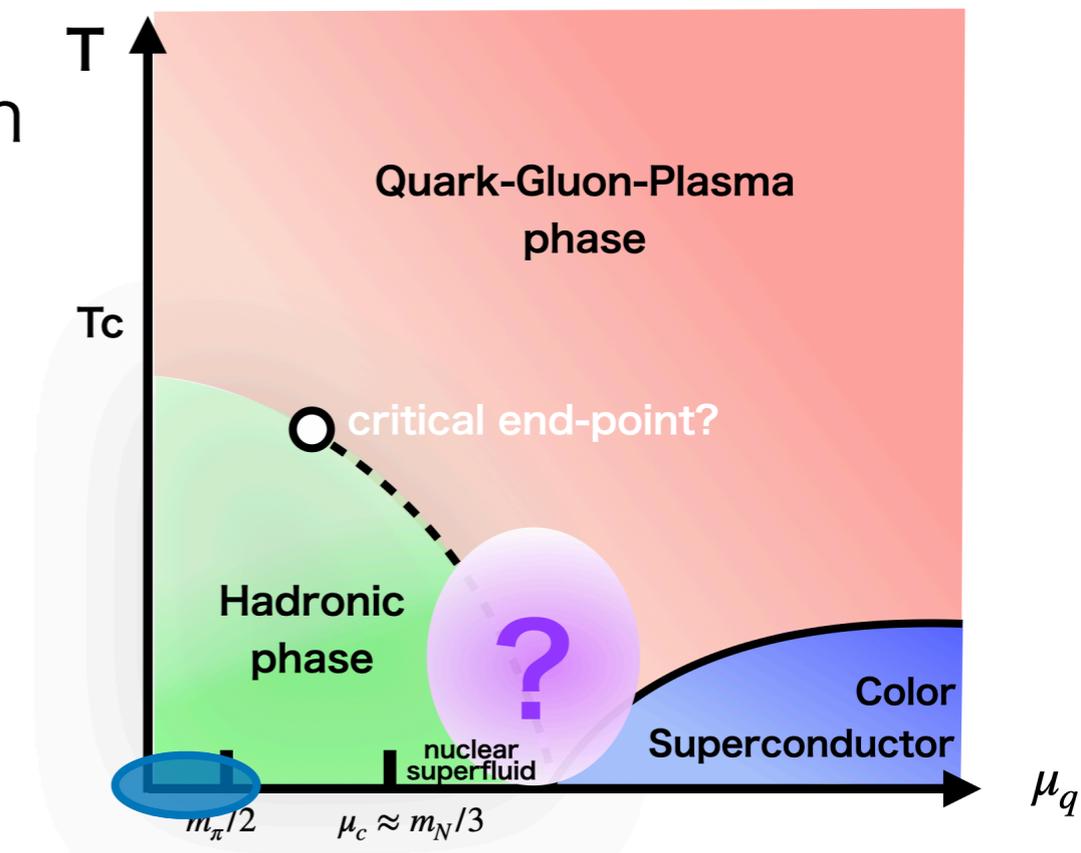
  - hadron mass spectrum

  - hadron interactions

- we focus on the region of small chemical potential μ and $T = 0$

- thermodynamics quantities in this region are unchanged under the change of μ

  (Silver Blaze phenomenon) [Cohen, 2003]

  ➡ hadronic properties are also independent of μ

Quark-Gluon-Plasma phase

Tc

critical end-point?

Hadronic phase

?

nuclear superfluid

Color Superconductor

$m_\pi/2$ $\mu_c \approx m_N/3$ μ_q

Hadron scatterings at small μ

- temporal two-point correlation function itself can change even for small μ
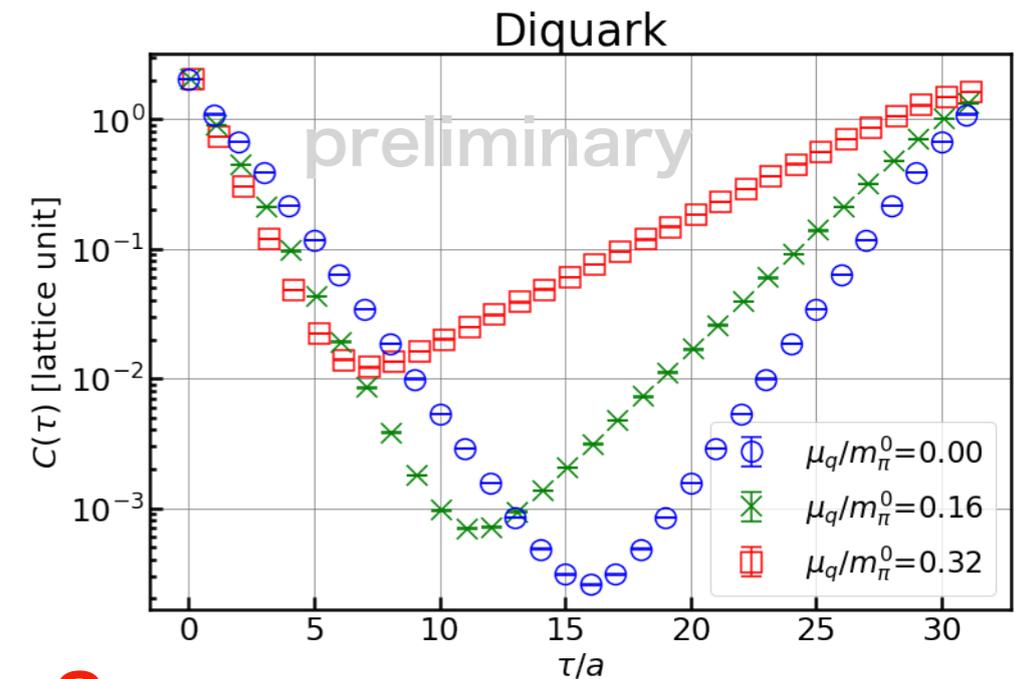
- the trivial modification of the dispersion relation

$$E(\mathbf{p}, \mu) = \sqrt{\mathbf{p}^2 + m^2} - \mu n_O$$

- Q. how can we see that the hadron scatterings are independent of small μ?

two-point function of
baryon (diquark) at $\mu \gtrsim 0$ in QC_2D

Diquark

Our work

- extend the method for studying hadron scatterings (HAL QCD method) to the $\mu \neq 0$ case

- see what are the μ-(in)dependent quantities step by step

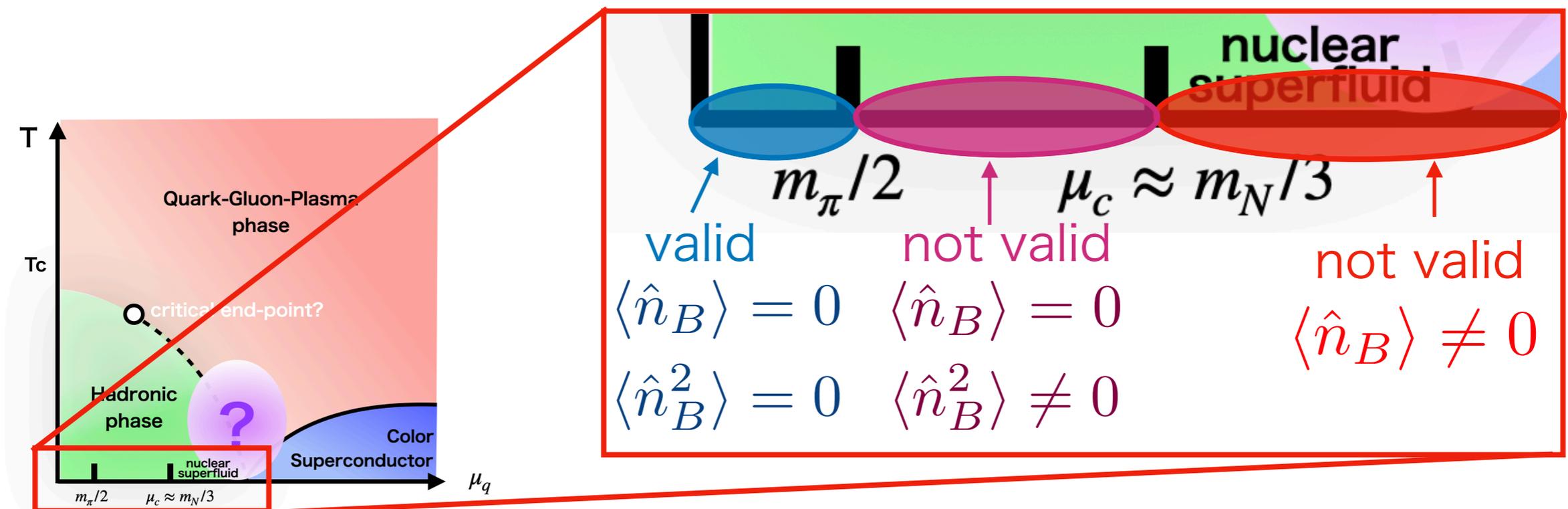- demonstrate our formulation in two-color QCD with nonzero quark chemical potential μ_q

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- two-point correlation function at $T = 0$ and $\mu \neq 0$

meson: $\bar{q}(\tau)\Gamma q(\tau)$

baryon: $q(\tau)q(\tau)q(\tau)$

$$C(\mu; \tau) = \lim_{T \to 0} \frac{1}{Z} \mathrm{Tr}[e^{-\frac{1}{T}(\hat{H} - \mu\hat{N})} \hat{O}(\tau)\hat{O}^\dagger(0)]$$

- if we assume that $\hat{N}|0\rangle = 0$,

$$C(\mu; \tau) = \langle 0|\hat{O}(\tau)\hat{O}^\dagger(0)|0\rangle$$

$(\hat{n}_B = \hat{N}_B/V)$

valid
$\langle \hat{n}_B \rangle = 0$
$\langle \hat{n}_B^2 \rangle = 0$

not valid
$\langle \hat{n}_B \rangle = 0$
$\langle \hat{n}_B^2 \rangle \neq 0$

not valid
$\langle \hat{n}_B \rangle \neq 0$

- suppose $\hat{O}(\tau)$ has the quantum number for $\hat{N}$

$$[\hat{N}, \hat{O}(\tau)] = -n_O \hat{O}(\tau)$$

➡ Euclidean-time evolution

$$\hat{O}(\tau) = e^{+(\hat{H}-\mu\hat{N})\tau} \hat{O}(0) e^{-(\hat{H}-\mu\hat{N})\tau} = e^{\mu n_O \tau} e^{+\hat{H}\tau} \hat{O}(0) e^{-\hat{H}\tau}$$

➡ $C(\mu; \tau) = \underline{e^{\mu n_O \tau}} \langle 0|\hat{O}(0) e^{-\hat{H}\tau} \hat{O}^\dagger(0)|0\rangle$

$$\mathbf{1} = \sum_n |n\rangle\langle n|$$

$$= \sum_n |\langle 0|\hat{O}(0)|n\rangle|^2 e^{-\underline{(E_n - \mu n_O)\tau}}$$

$$E(\mathbf{p}, \mu) = \sqrt{\mathbf{p}^2 + m^2} - \mu n_O \quad \text{trivial shift of the Fermi surface}$$

- Note: dynamics are independent of μ despite of the shift **(see later)**

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HAL QCD method at $\mu = 0$

- calculate four-point correlation function given by

$$F(\mathbf{r}, \tau) = \lim_{T \to 0} \frac{1}{Z} \mathrm{Tr}[e^{-\hat{H}/T} \hat{O}(\mathbf{r}, \tau) \hat{O}(\mathbf{0}, \tau) \ \hat{O}^\dagger(0) \hat{O}^\dagger(0)]$$

$$= \langle 0 | \hat{O}(\mathbf{r}, \tau) \hat{O}(\mathbf{0}, \tau) \hat{O}^\dagger(0) \hat{O}^\dagger(0) | 0 \rangle$$

- evaluate the R-correlator defined by

$$R(\mathbf{r}, \tau) = \frac{F(\mathbf{r}, \tau)}{C(\tau) C(\tau)}$$

2-point correlation function

- (leading-order) interaction potential can be obtained by

$$V^{LO}(\mathbf{r}) = \frac{\frac{1}{2\tilde{m}} \left(\nabla^2 - m \frac{\partial}{\partial \tau} + \frac{1}{4} \frac{\partial^2}{\partial \tau^2} \right) R(\mathbf{r}, \tau)}{R(\mathbf{r}, \tau)}$$

HAL QCD method at $\mu \neq 0$

- four-point correlation function $\mu \neq 0$

$$F(\mu; \mathbf{r}, \tau) = \lim_{T \to 0} \frac{1}{Z} \mathrm{Tr}[e^{-\frac{1}{T}(\hat{H} - \mu\hat{N})} \hat{O}(\mathbf{r}, \tau)\hat{O}(\mathbf{0}, \tau)\, \hat{O}^\dagger(0)\hat{O}^\dagger(0)]$$

$$= \langle 0|\hat{O}(\mathbf{r}, \tau)\hat{O}(\mathbf{0}, \tau)\hat{O}^\dagger(0)\hat{O}^\dagger(0)|0\rangle$$

- R-correlator with $\mu \neq 0$

$$R(\mu; \mathbf{r}, \tau) = \frac{F(\mu; \mathbf{r}, \tau)}{C(\mu; \tau)C(\mu; \tau)}$$

2-point correlation function
with $\mu \neq 0$

- (leading-order) interaction potential with $\mu \neq 0$

$$V^{LO}(\mathbf{r}) = \frac{\frac{1}{2\tilde{m}}\left(\nabla^2 - m\frac{\partial}{\partial\tau} + \frac{1}{4}\frac{\partial^2}{\partial\tau^2}\right)R(\mu; \mathbf{r}, \tau)}{R(\mu; \mathbf{r}, \tau)}$$

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μ-independence of the potential

- μ-dependence of $C(\mu; t)$ and $F(\mu; \mathbf{r}, \tau)$

$$C(\mu; \tau) = \underline{e^{\mu n_O \tau}} \langle 0| \hat{O}(0) e^{-\hat{H}\tau} \hat{O}^\dagger(0) |0\rangle$$

$$F(\mu; \mathbf{r}, \tau) = \underline{e^{2\mu n_O \tau}} \langle 0| \hat{O}(\mathbf{r}, 0) \hat{O}(\mathbf{0}, 0) e^{-\hat{H}\tau} \hat{O}^\dagger(0) \hat{O}^\dagger(0) |0\rangle$$

➡️ $R(\mu; \mathbf{r}, \tau) = \dfrac{F(\mu; \mathbf{r}, \tau)}{C(\mu; \tau) C(\mu; \tau)}$: independent of μ

➡️ $V^{LO}(\mathbf{r}) = \dfrac{\frac{1}{2\tilde{m}}\left(\nabla^2 - m\frac{\partial}{\partial \tau} + \frac{1}{4}\frac{\partial^2}{\partial \tau^2}\right) R(\mu; \mathbf{r}, \tau)}{R(\mu; \mathbf{r}, \tau)}$: independent of μ

➡️ **the scattering phase shifts are independent of μ (Silver Blaze phenomenon)**

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S-wave scatterings of $\pi\pi$ and DD in QC$_2$D

- we demonstrate our formulation and theoretical results by the lattice simulation in QC$_2$D in the hadronic phase

- we examine S-wave scatterings using the HAL QCD method
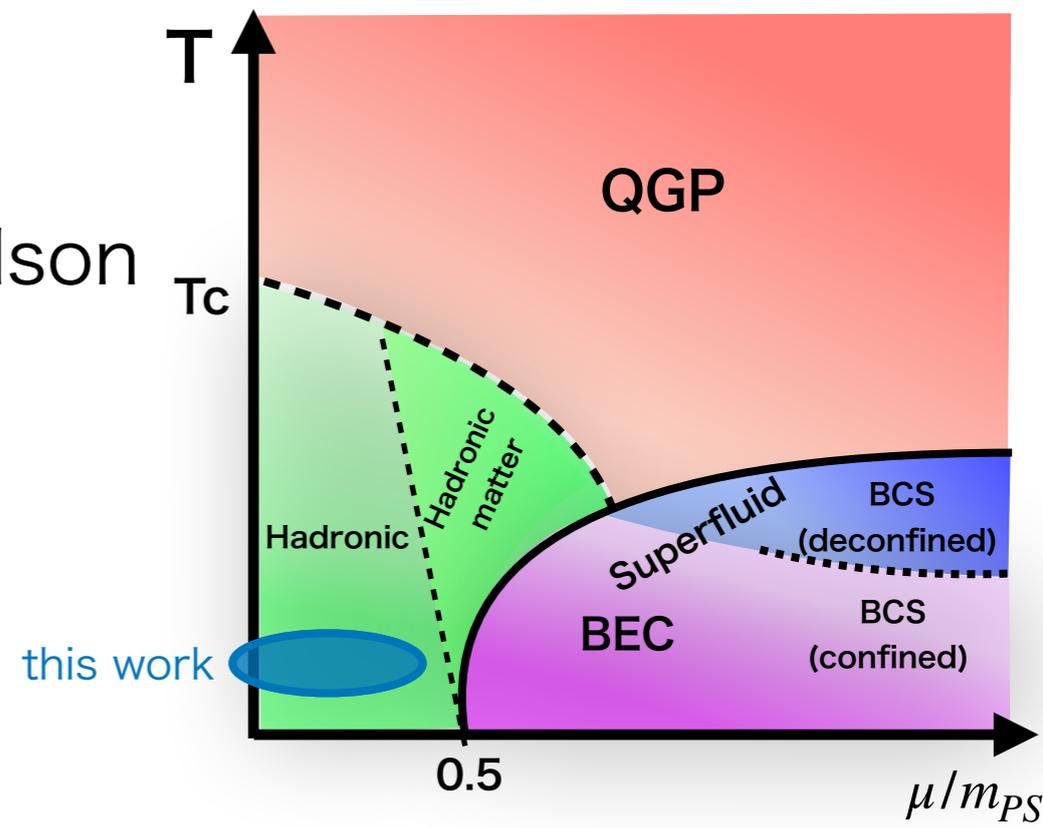
  target

  - two pions with isospin $I = 2$ ("$\pi\pi$")

  - two scalar diquarks ("DD") $(D \sim ud)$

- Conf.: Iwasaki gauge action + 2-flavor Wilson quark action in QC$_2$D in finite μ_q

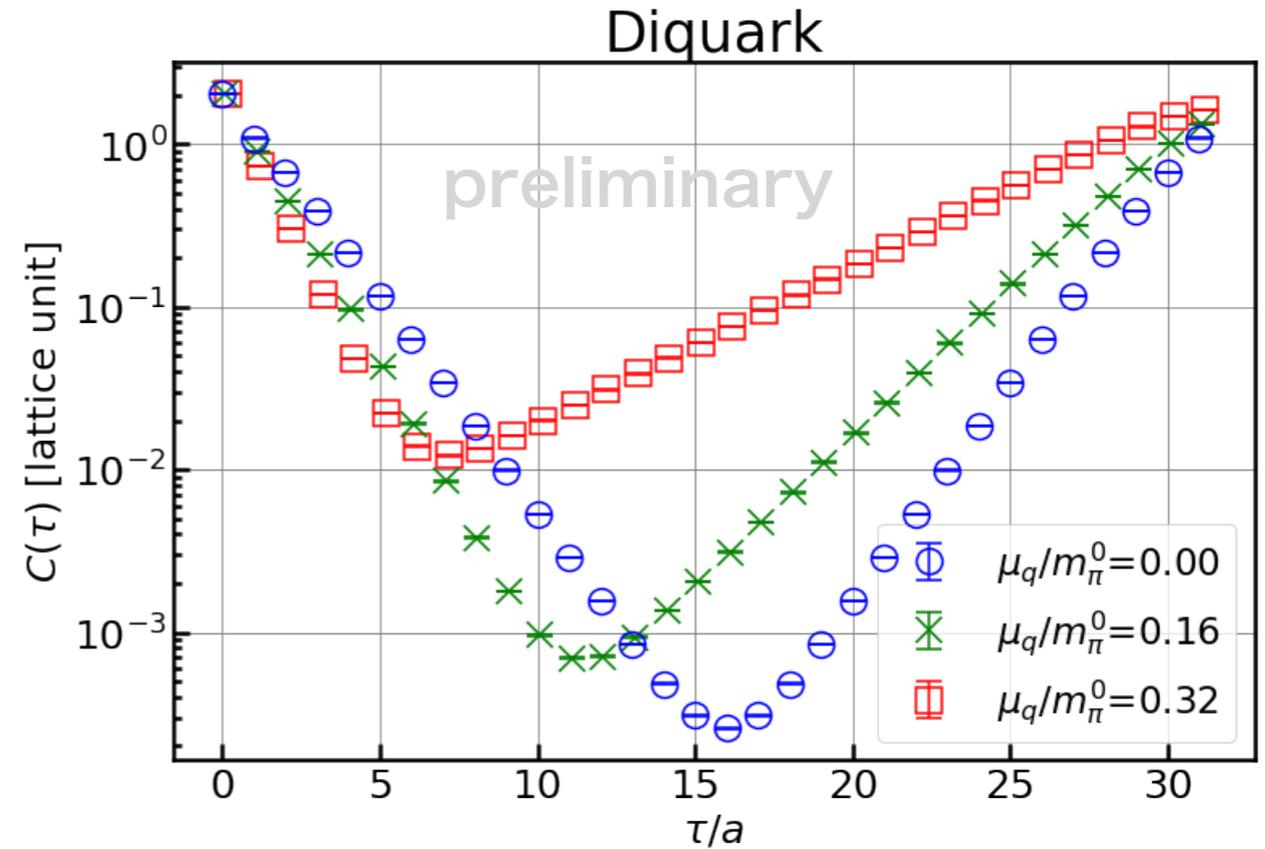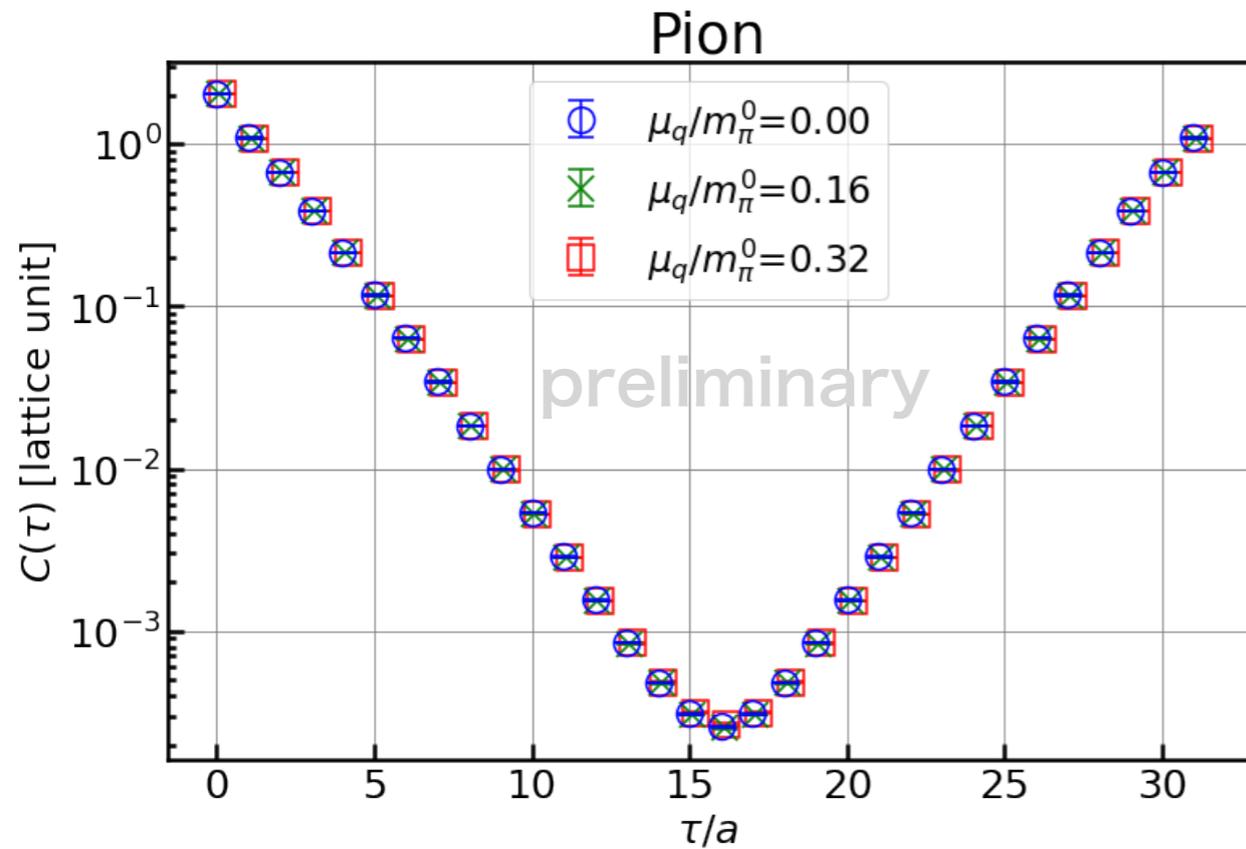- $\beta = 0.8$, 32^4 lattices, $m_\pi/m_\rho \approx 0.81$

$\mu_q/m_\pi \simeq 0,\ 0.16,\ 0.32,\ \underline{T \approx 0.19T_c}$

$\longrightarrow$ finite-T (periodicity) effect appears

K. Iida, E. Itou and T. -G. Lee, PTEP **2021**, no.1, 013B05 (2021)

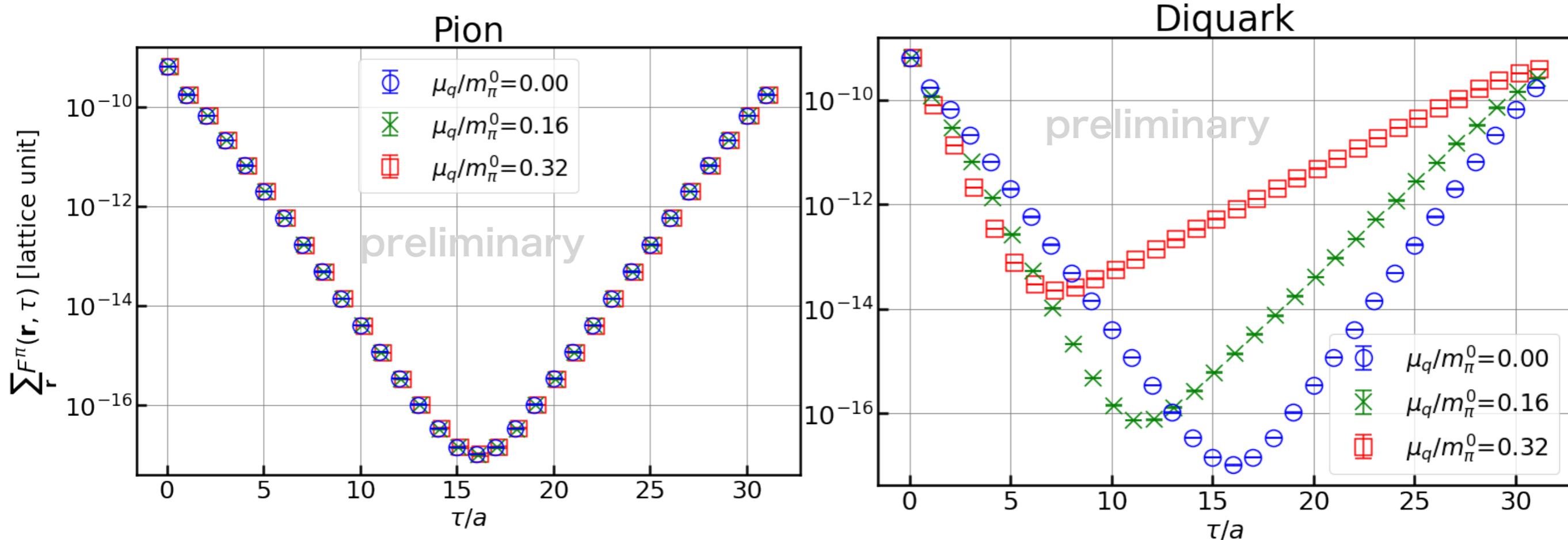# Two-point correlation functions of pion and diquark

- pion: symmetric under time reversal, independent on μ_q

- diquark: become more asymmetric for nonzero μ_q

- we define

  - diquark propagator: $C^D(aN_\tau - \tau)$

  - antidiquark propagator: $\underline{C^D(\tau)}$ ← suffer from the periodicity

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Four-point correlation functions of pion and diquark

- see temporal dependence of $\sum_{\mathbf{r}} F(\mu_q; \mathbf{r}, \tau)$

- similar behavior to that of two-point correlation functions

- Note: contribution from $\begin{cases} DD \text{ system: } F^D(\mu_q, \mathbf{r}, aN_\tau - \tau) \\ \bar{D}\bar{D} \text{ system: } F^D(\mu_q, \mathbf{r}, \tau) \end{cases}$
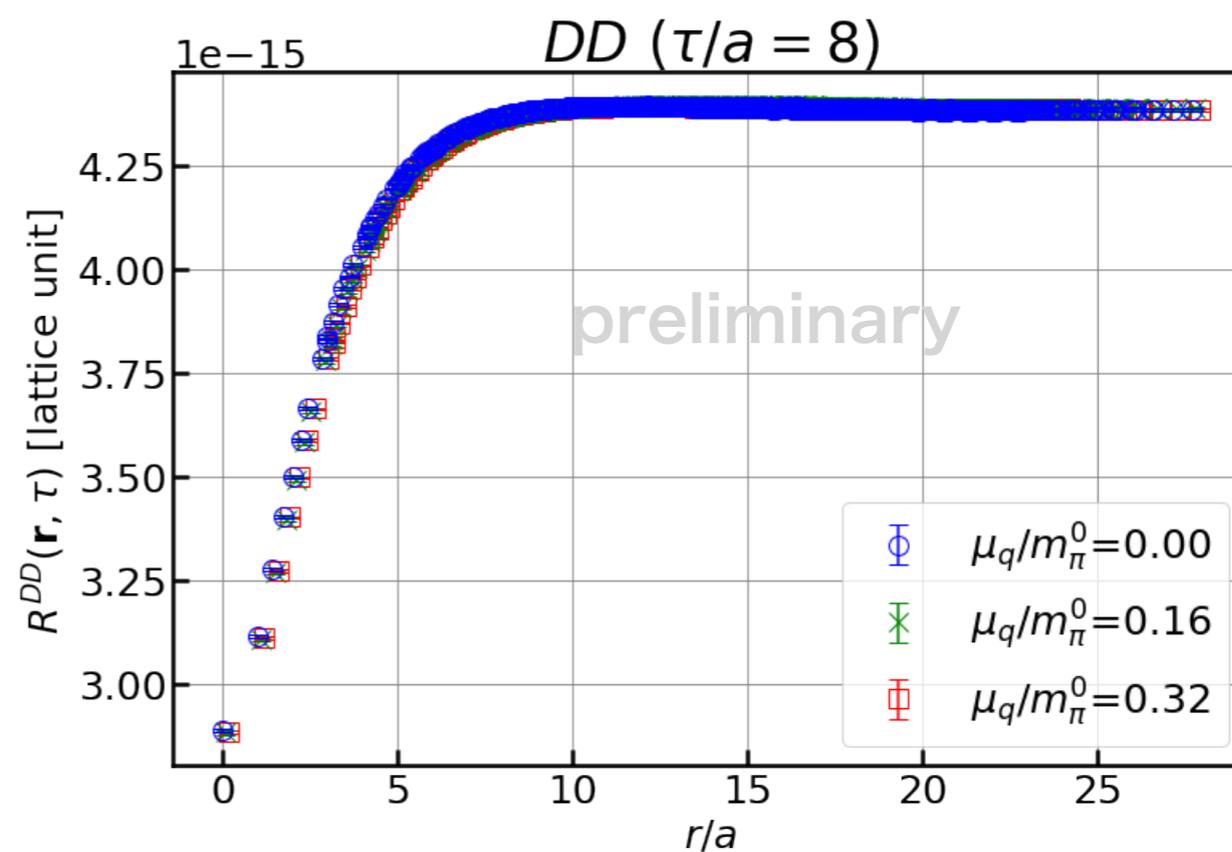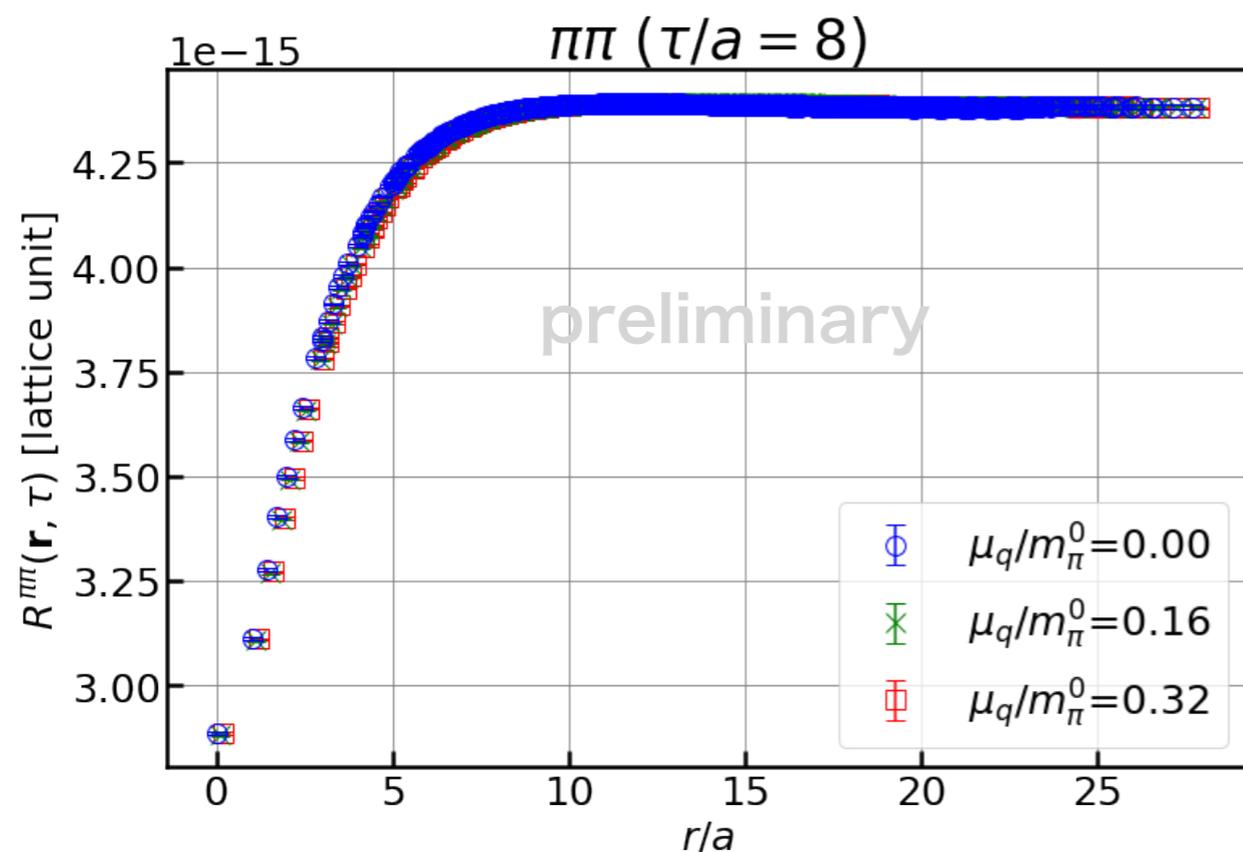
  suffer from the periodicity

$\pi\pi$ and DD R-correlators

- $\pi\pi$ and DD R-correlators are defined by

$$R^{\pi\pi}(\mu_q; \mathbf{r}, \tau) = \frac{F^\pi(\mu_q; \mathbf{r}, \tau)}{C^\pi(\mu_q; \tau)C^\pi(\mu_q; \tau)},$$

$$R^{DD}(\mu_q; \mathbf{r}, \tau) = \frac{F^D(\mu_q; \mathbf{r}, aN_\tau - \tau)}{C^D(\mu_q; aN_\tau - \tau)C^D(\mu_q; aN_\tau - \tau)}$$
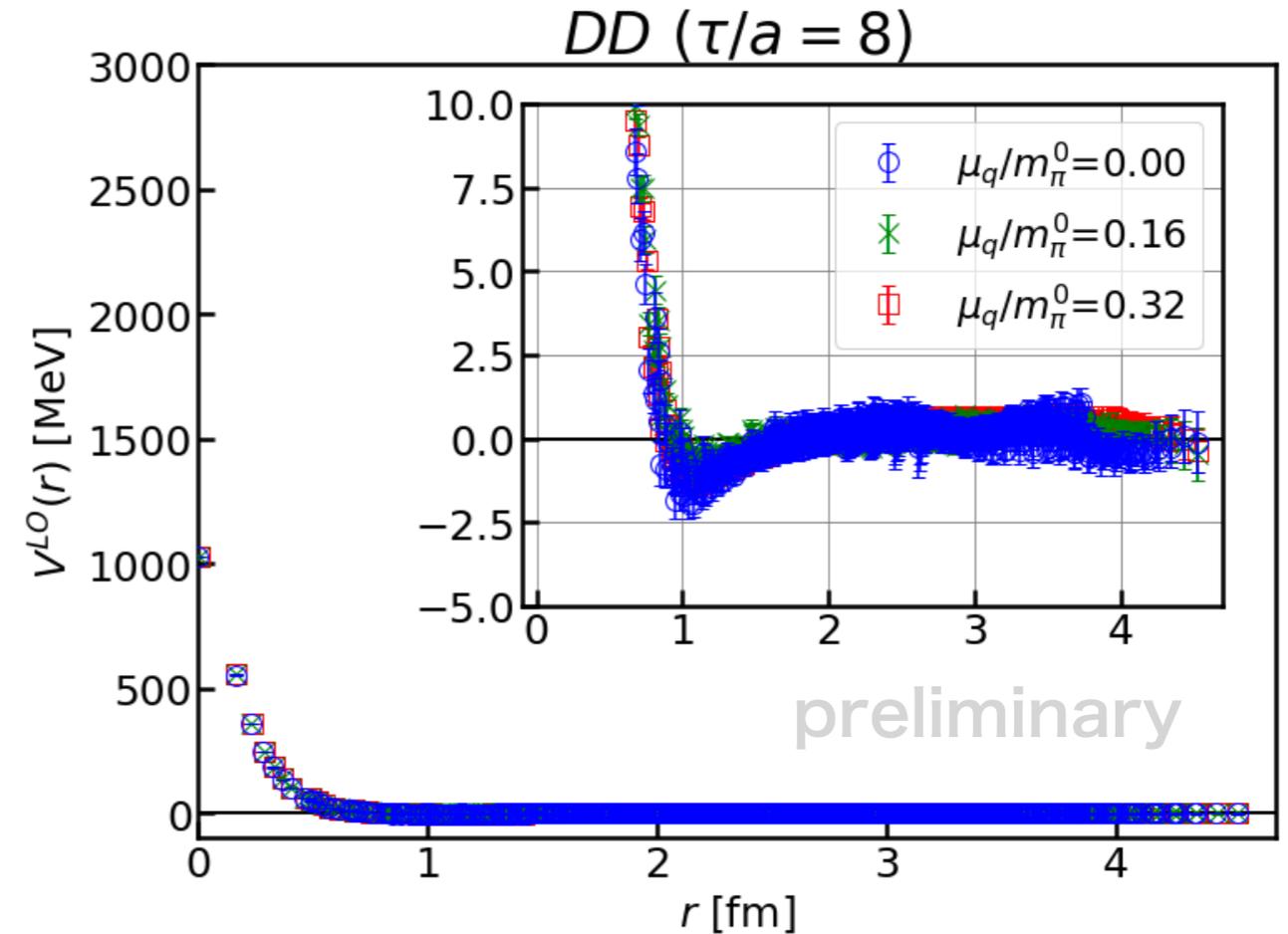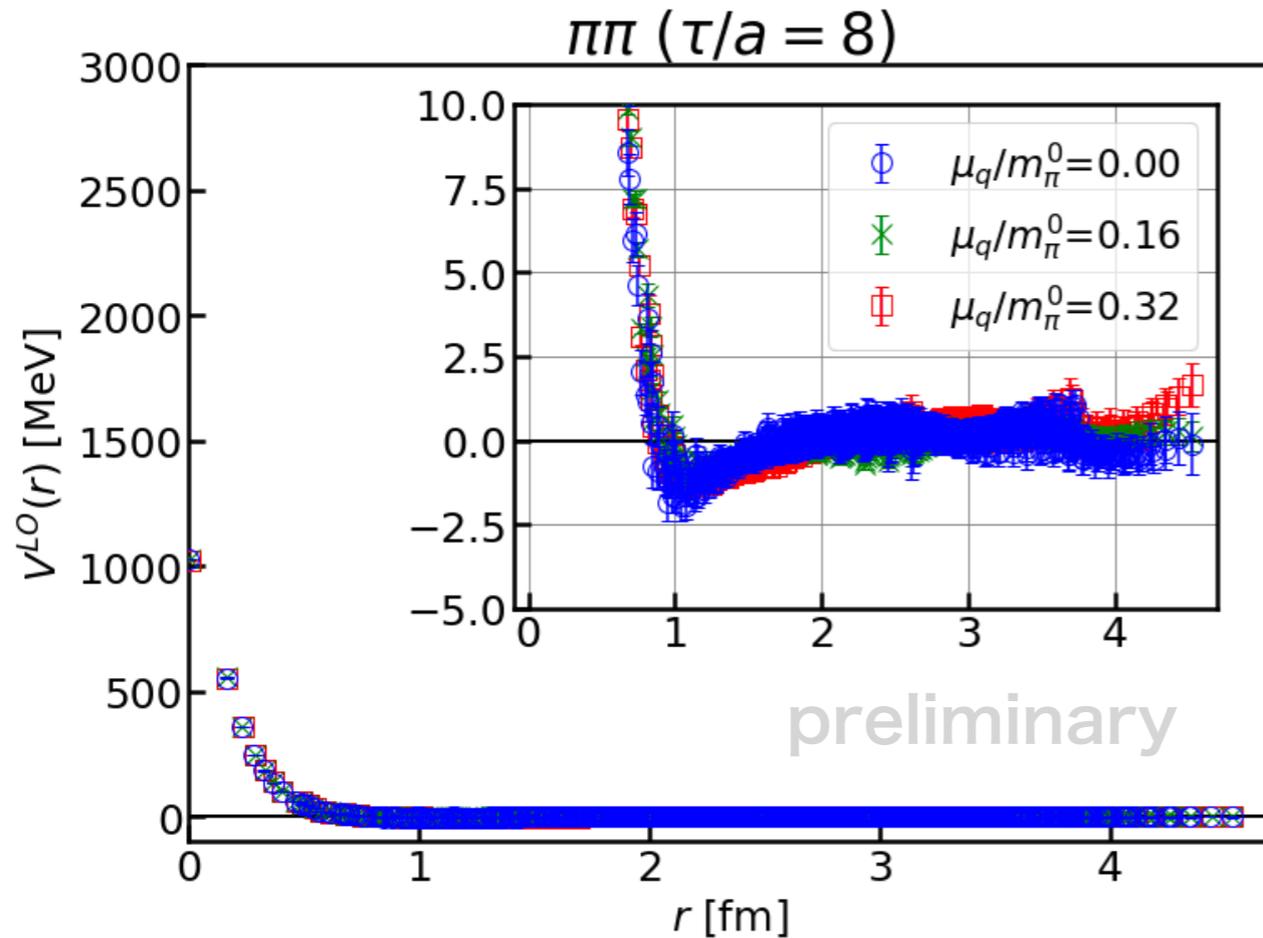

- both have no μ_q-dependence

← consistent with our formulation

$\pi\pi$ and DD LO potentials

$$V^{LO}(\mathbf{r}) = \frac{\frac{1}{2\tilde{m}}\left(\nabla^2 - m\frac{\partial}{\partial\tau} + \frac{1}{4}\frac{\partial^2}{\partial\tau^2}\right)R(\mu;\mathbf{r},\tau)}{R(\mu;\mathbf{r},\tau)}$$

- $\pi\pi$ and DD have the same potential at $\mu_q = 0$

  - global symmetry in two-flavor QC$_2$D

    $m = 0,\ \mu = 0$ $m \neq 0,\ \mu = 0$ 5 pseudo-NG

    $$SU(4) \longrightarrow Sp(4) \sim SO(3)$$ bosons

    $\pi^{\pm},\ \pi^0,\ D,\ \bar{D}$

- both potentials are independent of μ_q consistent with
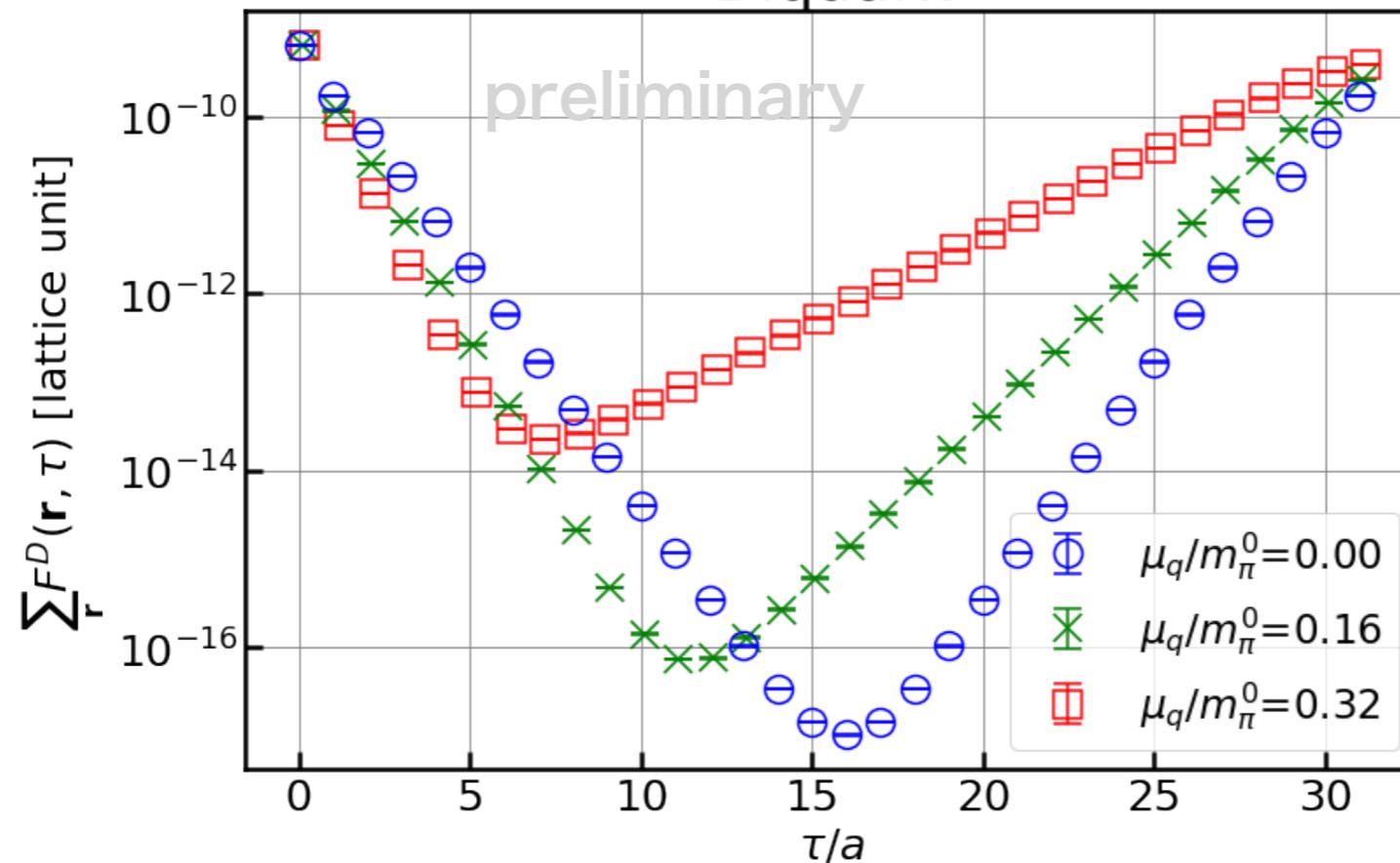
  analytical prediction

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Suppression of the periodicity in $F^D(\mu_q, \mathbf{r}, aN_\tau - \tau)$

- To obtain the reliable potential, we have to take long τ region to suppress the inelastic contribution

- $F^\pi(\mu_q; \mathbf{r}, \tau)$: at most $\tau/a \sim 8$

- $F^D(\mu_q, \mathbf{r}, aN_\tau - \tau)$: $N_\tau - \tau/a \sim 17$ with large μ_q

$$\sum_{\mathbf{r}} F(\mu_q; \mathbf{r}, \tau)$$

Diquark

DD LO potential at longer timeslice for $\mu_q \neq 0$

- Note: $V_{\pi\pi}^{LO}(r) = V_{DD}^{LO}(r)$: μ_q-independent

DD ($\mu_q/m_\pi^0 = 0.32$)

preliminary

$\pi\pi$ LO potential with heavy pion mass in 3-color QCD

(Akahoshi et al. (HAL QCD), PTEP 2019, no.8, 083B02 (2019))

- attractive pocket disappears at long τ

- <u>repulsive force remains</u> at long τ

same as 3-color $\pi\pi$ potential

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Phase shifts of DD at longer timeslice for $\mu_q \neq 0$

- Note: $\delta_{\pi\pi}(p) = \delta_{DD}(p)$: μ_q-independent

- scattering length, effective range

$$p \cot \delta(p) = -\frac{1}{a_0} + \frac{r_{\text{eff}}}{2} p^2 + \cdots$$
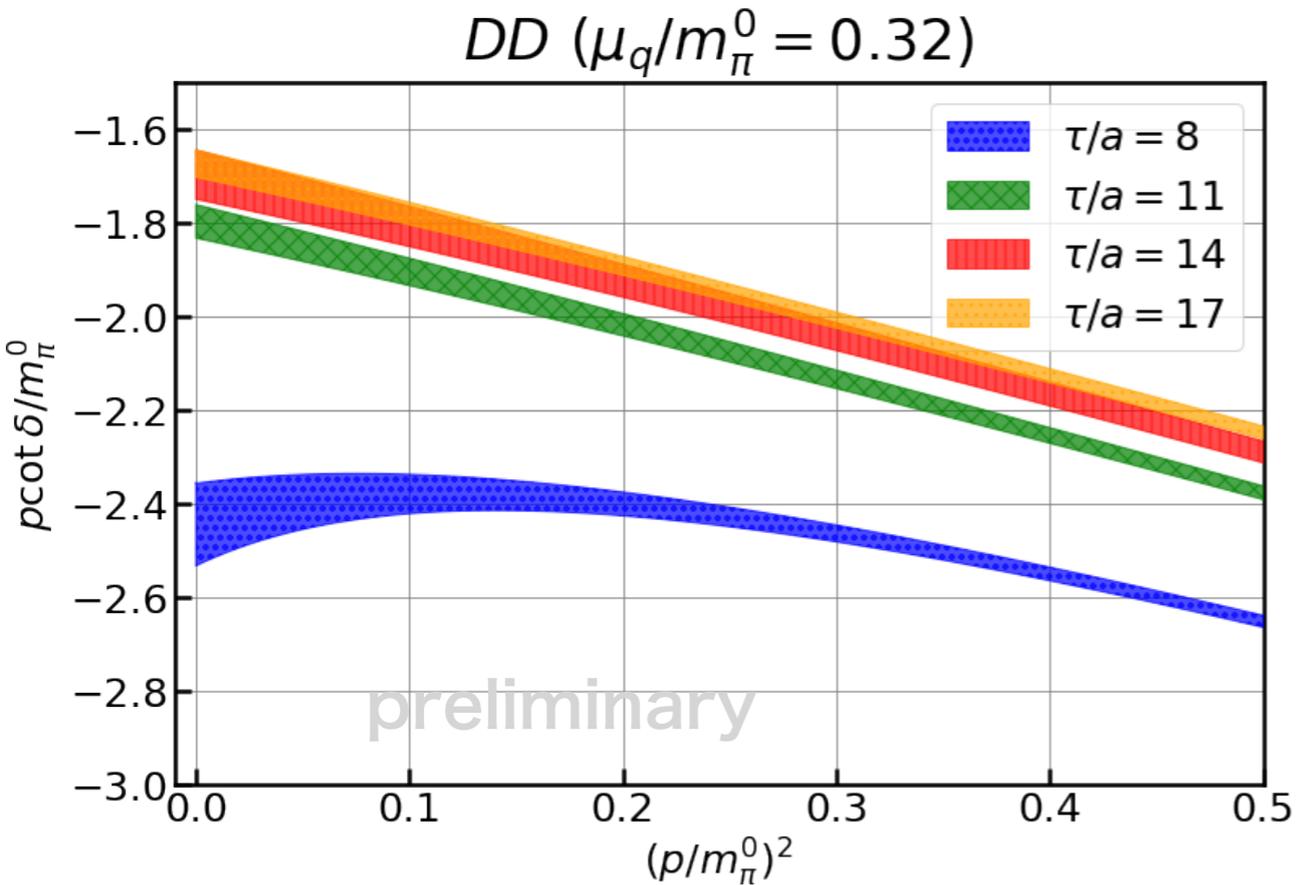

DD ($\mu_q/m_\pi^0 = 0.32$)

preliminary

t	8	11	14	17
a_0	0.108(4)	0.146(2)	0.155(5)	0.157(2)
r_{eff}	0.685(404)	-0.535(18)	-0.571(58)	-0.525(17)

- depend on τ for $\tau/a \leq 14$

  ➡ inelastic contribution remains until $\tau/a = 14$

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Conclusion

- we formulate the HAL QCD method at small but nonzero μ and see that the potential is independent of μ

- our numerical results of the demonstration in QC_2D are consistent with our formulation

- we see the potential at long τ using the suppression of the periodicity in DD 4-point correlation function at small μ_q

Future work

- Application of the small-μ technique to μ_I

- our formulation is valid only in the hadronic phase
  ➡ what would happen after phase transition?

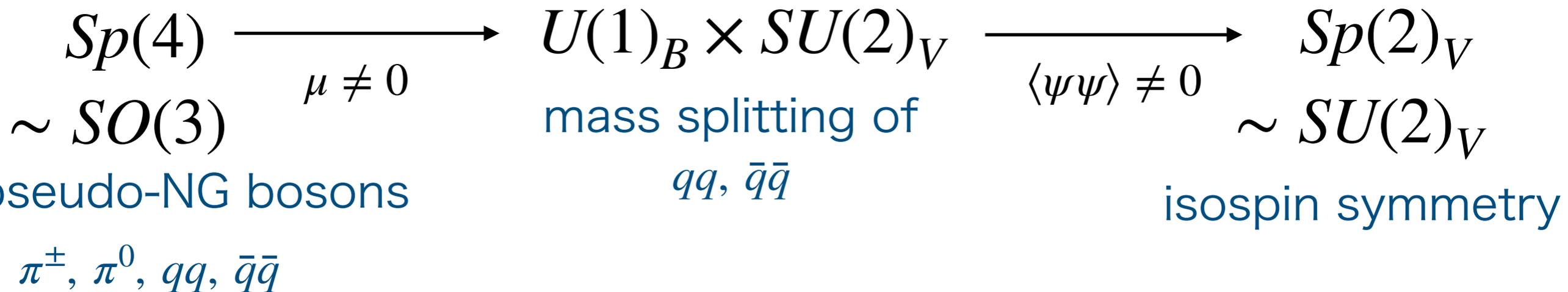# Backups

Global symmetry in 2-color QCD $(N_f = 2, j = 0)$

$m = 0, \mu = 0$

$SU(4)$ enhanced symmetry because of
pseudo-reality of $SU(N_c = 2)$

$m \neq 0,$

$\langle \bar{\psi}\psi \rangle \neq 0$

$Sp(4)$ $\xrightarrow{\mu \neq 0}$ $U(1)_B \times SU(2)_V$ $\xrightarrow{\langle \psi\psi \rangle \neq 0}$ $Sp(2)_V$

$\sim SO(3)$ $\qquad$ mass splitting of $\qquad$ $\sim SU(2)_V$

5 pseudo-NG bosons $\qquad$ $qq, \bar{q}\bar{q}$ $\qquad$ isospin symmetry

$\pi^{\pm}, \pi^0, qq, \bar{q}\bar{q}$

- in superfluid phase, the states are labeled by
  spin, **parity** and **isospin** (no baryon number)

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- dispersion relation of a single hadron at $\mu \neq 0$

$$\blacktriangleright \quad E(\mathbf{p}, \mu) = \sqrt{\mathbf{p}^2 + m^2} - \mu n_O$$

- μ-**independent scheme**: $m_{\text{eff}} = m$ [Muroya, Nakamura, Nonaka 2003]

$\longrightarrow$ μ-independence of the hadron spectrum

- μ-**dependent scheme**: $m_{\text{eff}} = E(\mathbf{p} = \mathbf{0}, \mu) = m - \mu n_O$

$\longrightarrow$ trivial shift of the   [Hands, Sitch, Skullerud 2008]

effective mass   [Wilhelm et al., 2019]

appear as a pole of the   [K.M., Suenaga, Iida, E. Itou 2023]

2-point correlation functions

Review of the HAL QCD method at $\mu = 0$ (For simplicity, $m_1 = m_2 = m$)

- key quantity: equal-time Nambu-Bethe-Salpeter (NBS) wave function

$$\Psi^W(\mathbf{r}) = \sum_{\mathbf{x}} \langle 0|\hat{O}(\mathbf{r}+\mathbf{x}, 0)\hat{O}(\mathbf{x}, 0)|HH; W\rangle$$

2-body hadron state
with energy W

hadron operators at
one spacetime point

$(W = 2\sqrt{p^2 + m^2})$

$$\Psi_l^W(r) \underset{r\to\infty}{\propto} \frac{\sin(pr - \frac{l}{2}\pi + \delta^l(p))}{pr}e^{i\delta^l(p)} \quad \& \quad (p^2 + \nabla^2)\Psi^W(\mathbf{r}) \underset{r\to\infty}{\to} 0$$

phase shift

[Lin, Martinelli, Sachrajda, Testa, 2001]

- HAL QCD method: obtain the interaction potential defined by

[Ishii, Aoki, Hatsuda 2007]
[Ishii et al. 2011]

$$\int d^3r'\, U(\mathbf{r}, \mathbf{r}')\Psi^W(\mathbf{r}') = \frac{1}{2\tilde{m}}(p^2 + \nabla^2)\Psi^W(\mathbf{r})$$

($\tilde{m}$: reduced mass)

from the R-correlator (defined in the next slide)

$$R(\mathbf{r}, \tau) \propto \sum_n \tilde{A}_n \Psi^{W_n}(\mathbf{r})e^{-(W_n - 2m)\tau}$$

➡ we can extract the phase shift $\delta^l(p)$ by
solving the Schrödinger equation with $U(\mathbf{r}, \mathbf{r}')$

HAL QCD method at $\mu \neq 0$ (1/2)

$$(W = \sqrt{k^2 + m_1^2} + \sqrt{k^2 + m_2^2})$$

- R-correlator

$$R(\mu; \mathbf{r}, \tau) = \frac{F(\mu; \mathbf{r}, \tau)}{C(\mu; \tau)C(\mu; \tau)}$$

2-point correlation function with $\mathbf{p} = 0$

- $F(\mu; \mathbf{r}, \tau)$: four-point correlation function at $\mu \neq 0$ and $T = 0$

$$F(\mu; \mathbf{r}, \tau) = \lim_{T \to 0} \frac{1}{Z} \mathrm{Tr}[e^{-\frac{1}{T}(\hat{H} - \mu\hat{N})} \sum_{\mathbf{x}} \hat{O}(\mathbf{r} + \mathbf{x}, \tau)\hat{O}(\mathbf{x}, \tau)\ \hat{O}^\dagger(0)\hat{O}^\dagger(0)]$$

$$= \sum_{\mathbf{x}} \langle 0|\hat{O}(\mathbf{r} + \mathbf{x}, \tau)\hat{O}(\mathbf{x}, \tau)\hat{O}^\dagger(0)\hat{O}^\dagger(0)|0\rangle$$

- in the same way as for the two-point correlation functions,

$$F(\mu; \mathbf{r}, \tau) = e^{2\mu n_O \tau}\langle 0|\hat{O}(\mathbf{r}, 0)\hat{O}(\mathbf{r}, 0)e^{-\tau\hat{H}}\ \hat{O}^\dagger(0)\hat{O}^\dagger(0)|0\rangle$$

$$\mathbf{1} = \sum_n |n\rangle\langle n|$$

$$= \sum_n A_n \Psi^{W_n}(\mathbf{r})e^{-(W_n - 2\mu n_O)\tau} + \cdots$$

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HAL QCD method at $\mu \neq 0$ (2/2)

$$(\Delta W_n = W_n - 2m)$$

- thus

$$R(\mu; \mathbf{r}, \tau) \simeq \frac{\sum_n A_n \Psi^{W_n}(\mathbf{r}) e^{-(W_n - 2\mu n_O)\tau}}{C e^{-(m - \mu n_O)\tau} \; C e^{-(m - \mu n_O)\tau}} = \sum_n \tilde{A}_n \Psi^{W_n}(\mathbf{r}) e^{-\Delta W_n \tau}$$

independent of μ

- each term satisfies the Schrödinger equation

$$\int d^3 r' \; U(\mathbf{r}, \mathbf{r}') \tilde{A}_n \Psi^{W_n}(\mathbf{r}') \; e^{-\Delta W_n \tau} = \frac{1}{2\tilde{m}}(p_n^2 + \nabla^2) \tilde{A}_n \Psi^{W_n}(\mathbf{r}) \; e^{-\Delta W_n \tau}$$

$$\approx V^{LO}(r)\delta^{(3)}(\mathbf{r} - \mathbf{r}')$$
(leading-order (LO) approximation)

$$= \Delta W_n + \frac{1}{8\mu}\Delta W_n^2 = -\frac{\partial}{\partial t} + \frac{1}{8\mu}\frac{\partial^2}{\partial t^2}$$

$$\times \sum_n$$

$$\Longrightarrow \quad V^{LO}(\mathbf{r}) = \frac{\frac{1}{2\tilde{m}}\left(\nabla^2 - m\frac{\partial}{\partial \tau} + \frac{1}{4}\frac{\partial^2}{\partial \tau^2}\right) R(\mu; \mathbf{r}, \tau)}{R(\mu; \mathbf{r}, \tau)}$$

possible μ-dependence only here

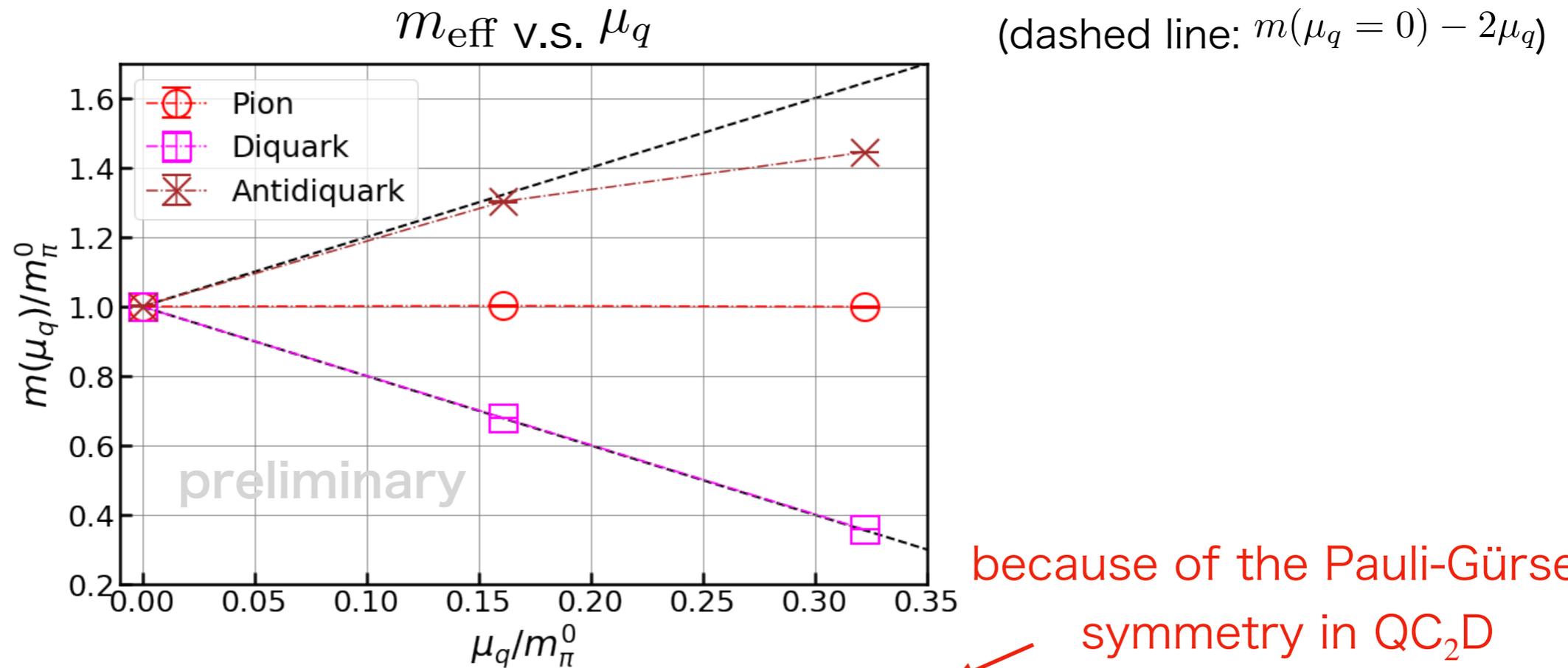- Note: $\tilde{m}$ can be set by hand in the HAL QCD method; **varying** $\tilde{m}$ just alters the definition of $U(\mathbf{r}, \mathbf{r}')$ **without changing** $\Psi^W(\mathbf{r})$

➡️ **the scattering phase shifts are independent of μ**

Effective masses of pion and diquark

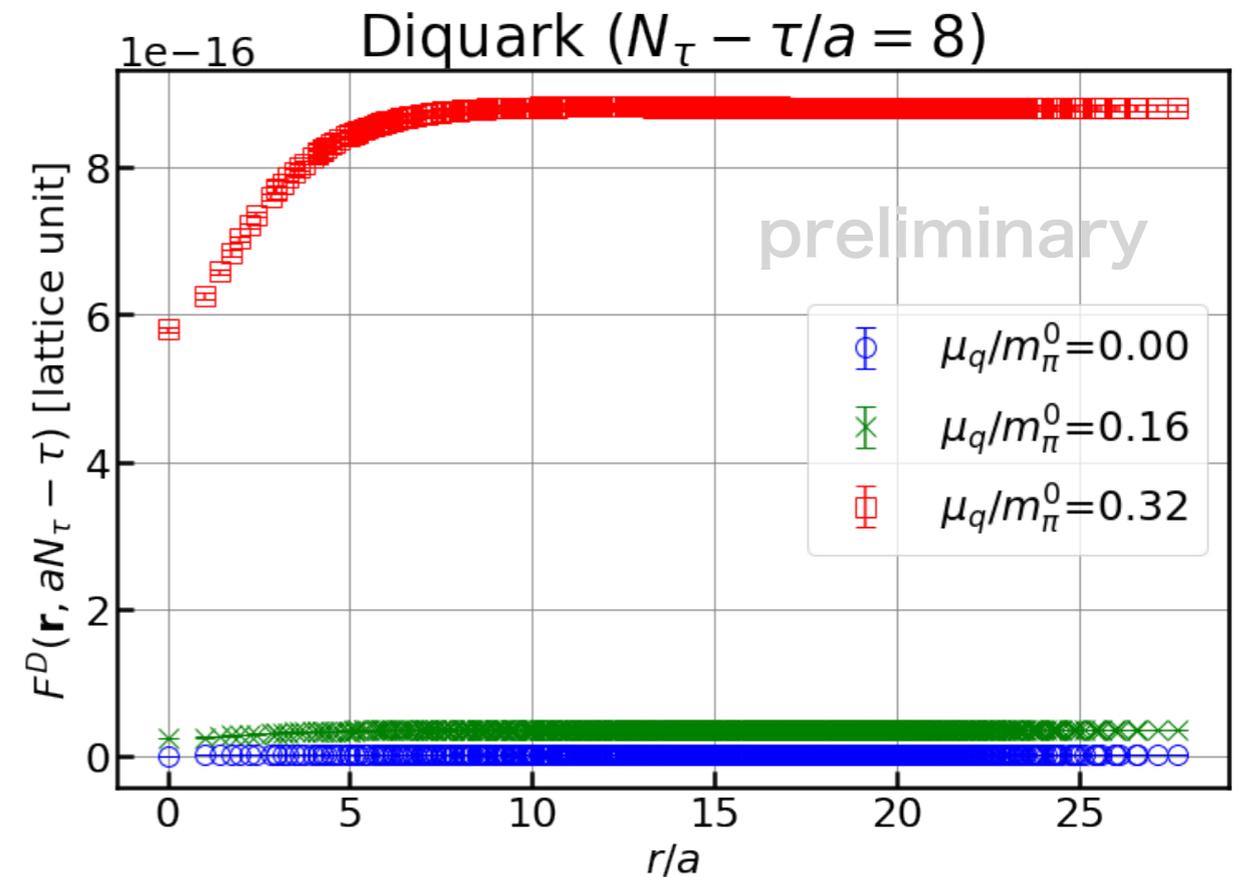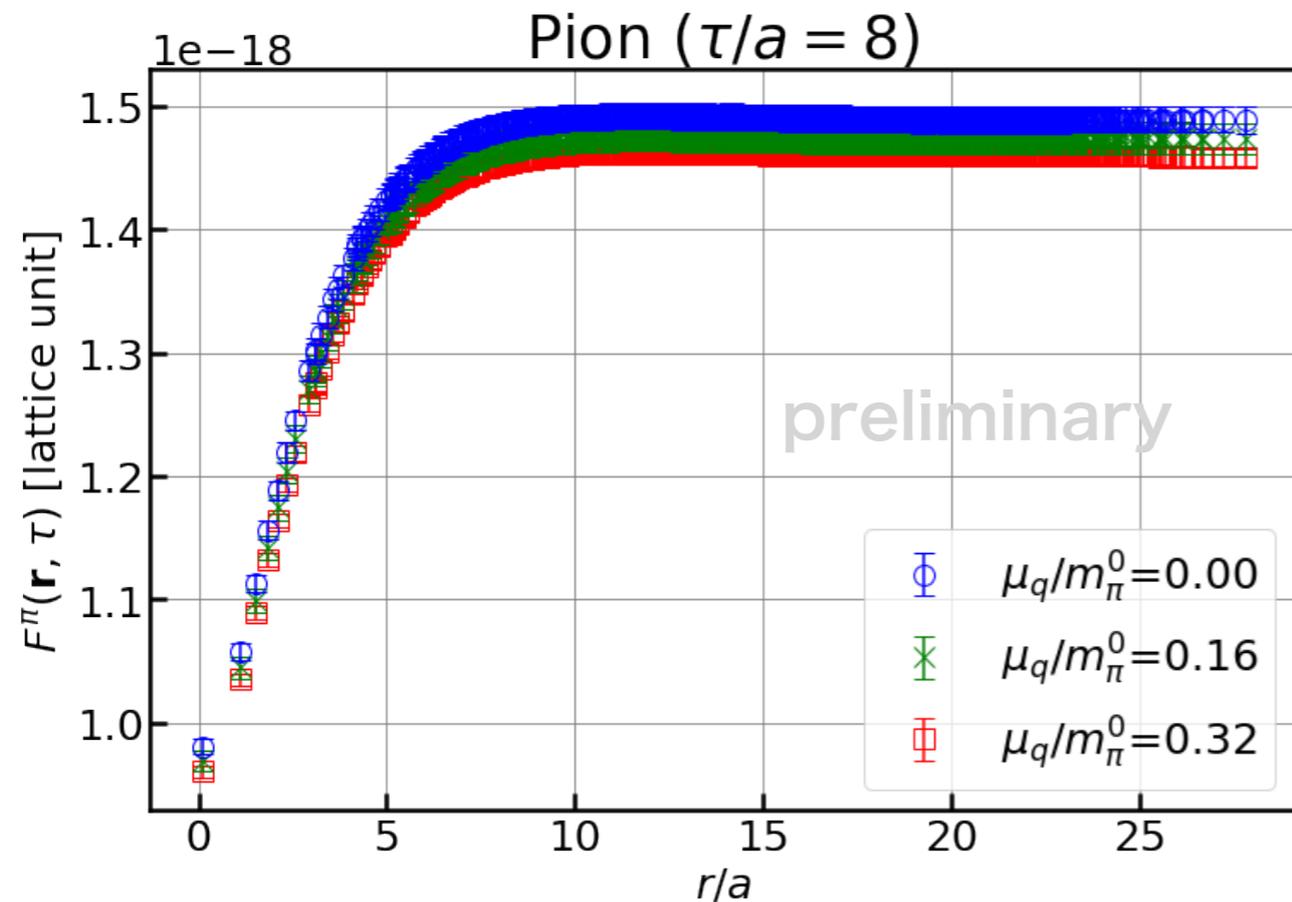- we choose μ-dependent shceme $m_{\text{eff}} = m - \mu n_O$

  ➡ fit function: $C(\tau) \sim A e^{-m_{\text{eff}}\tau}$

m_{eff} v.s. μ_q

(dashed line: $m(\mu_q = 0) - 2\mu_q$)

because of the Pauli-Gürsey symmetry in QC$_2$D

- the three hadrons are degenerated at $\mu_q = 0$

- pion, diquark: consistent with $m_{\text{eff}} = m - \mu_q n_q$

- antidiquark: <u>slightly deviates from</u> $m_{\text{eff}} = m + 2\mu_q$

due to the periodicity on finite lattices

Spatial dependence of four-point correlation functions

- see spatial dependence of $F(\mu_q, \mathbf{r}, \tau)$ at fixed timeslice

- pion ($\pi\pi$): no drastic change for different μ_q

- diquark (DD): the scale increases with μ_q, keeping its shape

➡ μ_q-dependence appears only in the overall factor $e^{2\mu n_O \tau}$?
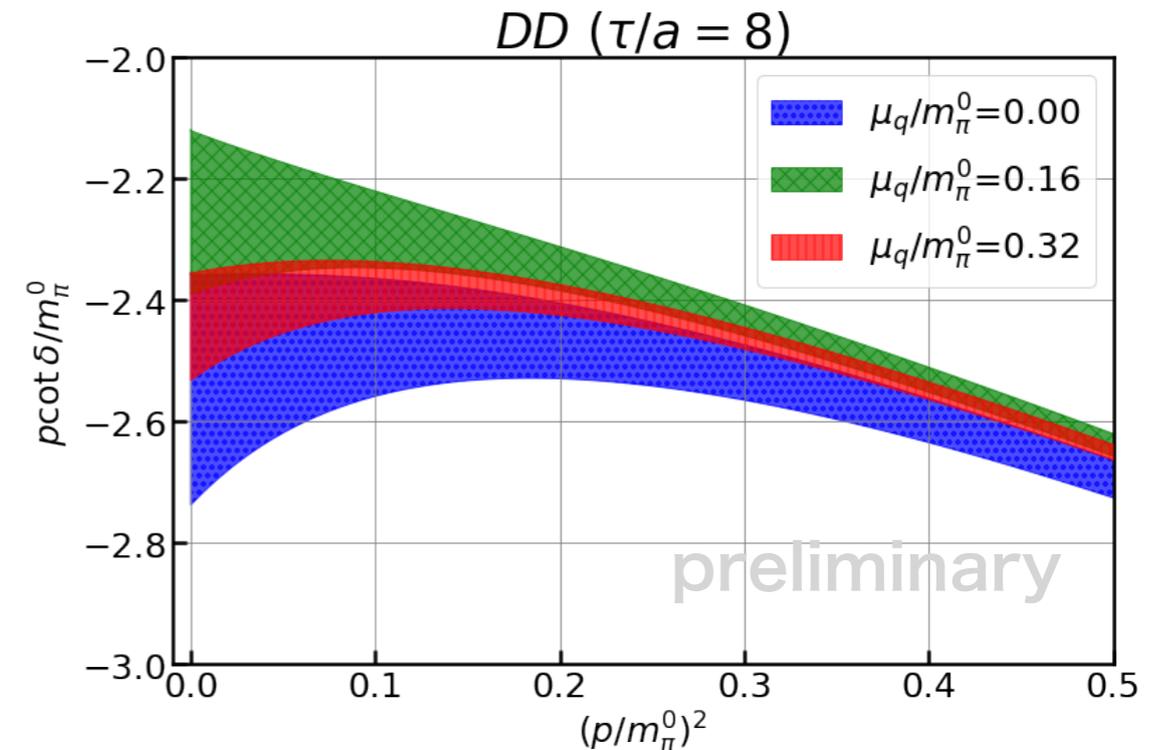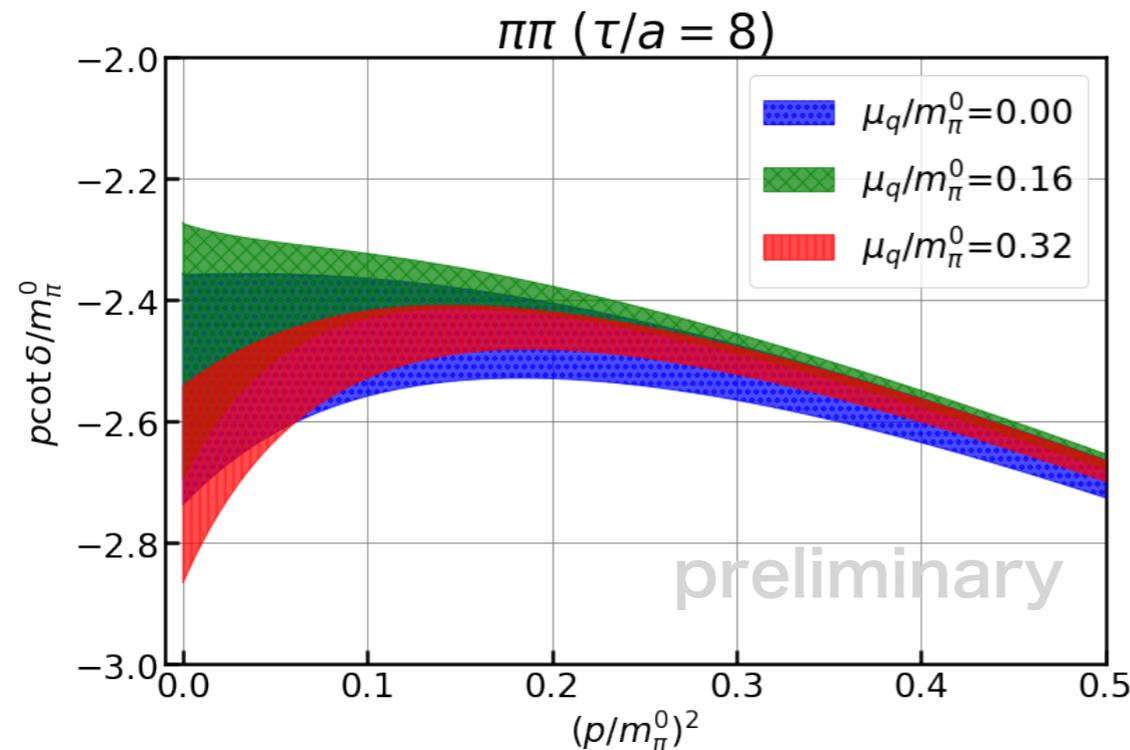
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$p \cot \delta(p)$ of $\pi\pi$ and DD scatterings

- solve (radial part of) Schrödinger equation ($l = 0$)

$$-\frac{1}{2\tilde{m}}\left(\frac{1}{r}\frac{d^2}{dr^2}r - \frac{l(l+1)}{r^2}\right)\psi_l^W(r) + V^{LO}(r)\psi_l^W(r) = \frac{p^2}{2\tilde{m}}\psi_l^W(r)$$

➡ $\psi_{l=0}^W(r) \underset{r\to\infty}{\propto} \frac{\sin(pr + \delta(p))}{pr}e^{i\delta(p)}$

- $p \cot \delta(p)$ for $\pi\pi$ and DD scatterings

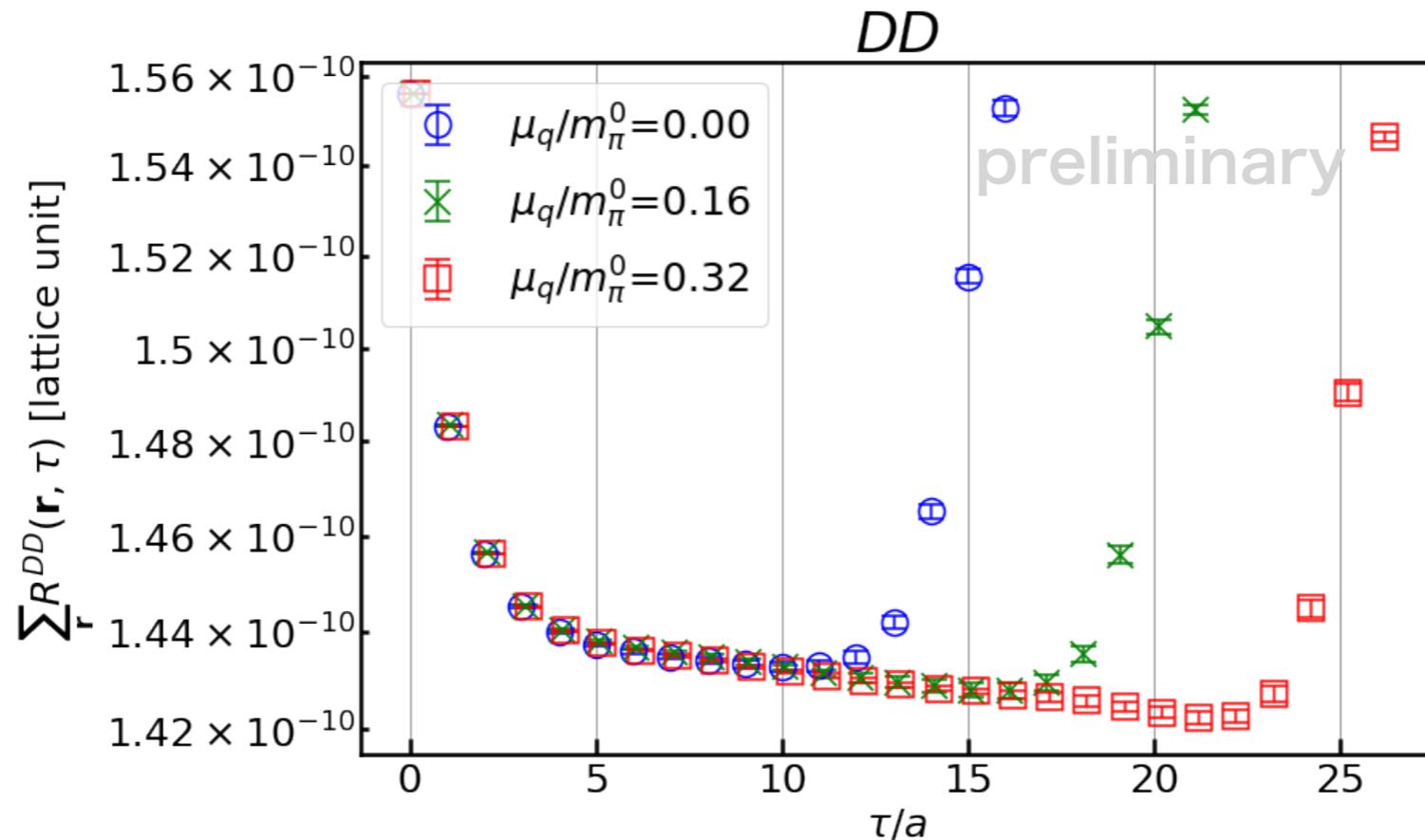- <u>results for different μ_q agree with each other</u> for both $\pi\pi$ and DD

↖ consistent with our conclusion

(Silver Blaze phenomenon)

Temporal dependence of DD R-correlators

- see temporal dependence of $\sum_{\mathbf{r}} R^{DD}(\mu_q; \mathbf{r}, \tau)$

- depend on μ_q at larger τ due to the finite-T effect

- the finite-T effect (periodicity) is more suppressed for larger μ_q

  occur for positive baryon number $n_O > 0$ in general