Relativistic quantum mechanics

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Outline

- Relativity in Quantum Mechanics
- Poincaré covariant theories
- Lorentz covariant theories
- Euclidean covariant theories
- Dynamics

Relativity in quantum mechanics

- **Quantum observables:**

$$P = |\langle\psi|\phi\rangle|^2 \qquad \langle\psi|A|\psi\rangle \qquad \mathbf{Tr}(\rho A)$$

- **Inertial coordinate systems:**

$$X \underbrace{\longrightarrow}_{(\Lambda,a)} X'$$

- **Relativistic invariance of quantum observables:**

$$P' = P \qquad \langle\psi'|A'|\psi'\rangle = \langle\psi|A|\psi\rangle \qquad \mathbf{Tr}(\rho' A') = \mathbf{Tr}(\rho A)$$

- **Wigner's theorem:** $U(\Lambda, a)$

$$|\psi'\rangle = U(\Lambda, a)|\psi\rangle \qquad A' = U(\Lambda, a) A U^\dagger(\Lambda, a)$$

$$\rho' = U(\Lambda, a)\rho U^\dagger(\Lambda, a)$$

The Lorentz group and $SL(2,\mathbb{C})$

2×2 matrix representation of coordinates

$$X = \sigma_\mu x^\mu = \begin{pmatrix} x^0 + x^3 & x^1 - ix^2 \\ x^1 + ix^2 & x^0 - x^3 \end{pmatrix} = X^\dagger$$

$$\det(X) = (x^0)^2 - \mathbf{x}^2$$

Poincaré transformations

$$X \to X' = \Lambda X \Lambda^\dagger + A \qquad \det(\Lambda) = 1 \qquad A = A^\dagger$$

$$\Lambda^\mu{}_\nu = \frac{1}{2}\mathrm{tr}(\sigma_\mu \Lambda \sigma_\nu \Lambda^\dagger) \qquad a^\mu = \frac{1}{2}\mathrm{tr}(\sigma_\mu A)$$

Polar decomposition - Boosts

$$\Lambda = \underbrace{((\Lambda\Lambda)^{\dagger})^{1/2}}_{P}\underbrace{((\Lambda\Lambda)^{\dagger})^{-1/2}\Lambda}_{U}$$

Rotationless (canonical) boost ($\boldsymbol{\rho}$ = rapidity)

$$P = e^{\frac{1}{2}\boldsymbol{\rho}\cdot\boldsymbol{\sigma}} = \Lambda_c(p)$$

Rotation (generalized Melosh rotation)

$$U = e^{\frac{i}{2}\boldsymbol{\theta}\cdot\boldsymbol{\sigma}} = R(p)$$

$$\Lambda(p) = \Lambda_c(p)R(p) \qquad R(p) = \Lambda_c^{-1}(p)\Lambda(p)$$

Relativistic dynamics

$$U(\Lambda, a) : \mathcal{H} \to \mathcal{H}$$

Relativistic analog of diagonailzing Hamiltonian

$$WU(\Lambda, a)W^{\dagger} = \int_{\oplus_{m,s,d}} U_{m,s}(\Lambda, a) d\mu(m, s, d)$$

$m =$ **mass** $\qquad s =$ **spin** $\qquad d =$ **degeneracy**

- $U_{m,s}(\Lambda, a)$ **fixed by group theory, basis choice and representation of** $\mathcal{H}$

Poincaré covariant RQM - basis choice:

$U(\Lambda, a) \to P^{\mu}, J^{\mu\nu} \to$ **commuting observables**

$$\{|(m, s; d)b\rangle\}$$

Invariants:

$$M^2 = -P^{\mu}P_{\mu} \qquad M^2 S^2 = W^{\mu}W_{\mu} \qquad W^{\mu} = \frac{1}{2}\epsilon^{\alpha\beta\gamma\mu}P_{\alpha}J_{\beta\gamma}$$

Momentum observables:

$$\mathbf{P} \qquad \mathbf{V} = \mathbf{P}/M \qquad P^{+} = P^{0} + \hat{\mathbf{n}} \cdot \mathbf{P}, \mathbf{P}_{\perp}$$

Spin observables (depend on choice of $\Lambda_x(p)$):

$$\mathbf{s}^{i}_{x} = \Lambda^{-1}_{x}(P)^{i}{}_{\mu}\Lambda^{-1}_{x}(P)^{i}{}_{\nu}J^{\mu\nu}$$

P **operator**

Hilbert space representations: $\mathcal{H} = L^2(m, s, d, b)$

- $\{(m, s; d)\mathbf{p}, \hat{s}_c \cdot \hat{\mathbf{n}}\}$ **(Instant form)**
- $\{(m, s; d)\mathbf{v}, \hat{s}_c \cdot \hat{\mathbf{n}}\}$ **(Point form)**
- $\{(m, s; d)\mathbf{p}, \hat{s}_h \cdot \hat{\mathbf{z}}\}$ **(Jacob-Wick helicity)**
- $\{(m, s; d)p^+, \mathbf{p}_\perp, \hat{s}_f \cdot \hat{\mathbf{n}}\}$ **(Front form)**
- $\vdots$

Spectrum of b in irreducible subspaces fixed by group theory.

Different basis choices are related by unitary transformations.

Irreducible representations

$$U(\Lambda, a)|(m,s,d)b\rangle = \sum_{b'} \int db' |(m,s,d)b'\rangle \mathcal{D}^{m,s}_{b'b}(\Lambda, a)$$

$$\mathcal{D}^{m,s}_{b'b}(\Lambda, a) = \langle (m,s,d)b|U(\Lambda, a)|(m,s,d)b'\rangle$$

$$\sum \int db' \mathcal{D}^{m,s}_{bb''}(\Lambda_2, a_2)\mathcal{D}^{m,s}_{b''b'}(\Lambda_1, a_1) = \mathcal{D}^{m,s}_{bb'}(\Lambda_2\Lambda_1, \Lambda_2 a_1 + a_2)$$

Structure of $\mathcal{D}^{m,s}_{b'b}(\Lambda, a)$

Depends on basis choice

$$\delta(\mathbf{b}' - \mathbf{\Lambda}(b))|\frac{\partial(\mathbf{\Lambda}(b))}{\partial(\mathbf{b})}|^{\frac{1}{2}} e^{i(\Lambda p)\cdot a} D^s_{\mu\nu}[\Lambda_x^{-1}(\Lambda p)\Lambda\Lambda_x(p)]$$

Example $b = \{\mathbf{p}, \mathbf{s}_c\}$ - (Instant form)

$$\delta(\mathbf{p}' - \mathbf{\Lambda}(p))|\frac{\omega_m(p'))}{\omega_m(p)}|^{1/2} e^{i(\Lambda p)\cdot a} D^s_{\mu\nu}[\Lambda_c^{-1}(\Lambda p)\Lambda\Lambda_c(p)]$$

Dynamics non-trivial

$$[K^i, P^j] = i\delta_{ij} H = i\delta_{ij}(H_0 + V)$$

Kinematic subgroups $\mathcal{K}$

$$(\Lambda, a) \in \mathcal{K} \Leftrightarrow \mathcal{D}^{m,s}_{b'b}(\Lambda, a) \qquad \textbf{independent of } m$$

$\mathcal{K}$ depends on basis

$b = \{\mathbf{p}, \hat{s}_c \cdot \hat{\mathbf{n}}\}$ $\quad \mathcal{K} =$ **3 dimensional Euclidean group**

$b = \{\mathbf{v}, \hat{s}_c \cdot \hat{\mathbf{n}}\}$ $\quad \mathcal{K} =$ **Lorentz group**

$b = \{p^+, \mathbf{p}_\perp, \hat{s}_f \cdot \hat{\mathbf{n}}\}$ $\quad \mathcal{K} =$ **subgroup preserving** $x^+ = 0$

Lorentz covariant relativistic quantum mechanics (Hilbert space representations with kernels)

$$D^s_{\mu\nu}[R] = \langle s, \mu | e^{i\frac{\boldsymbol{\theta}}{2}\cdot\boldsymbol{\sigma}} | s, \nu \rangle \qquad \textbf{entire function of } \boldsymbol{\theta}$$

$$\sum_\alpha D^s_{\mu\alpha}[R_2] D^s_{\alpha\nu}[R_1] - D^s_{\mu\nu}[R_2 R_1] = 0$$

$$\langle s, \mu | s_1, \mu_1, s_2, \mu_2 \rangle D^{s_1}_{\mu_1\nu_1}[R] D^{s_2}_{\mu_2\nu_2}[R] \langle s_1, \nu_1, s_2, \nu_2 | s, \nu \rangle - D^s_{\mu\nu}[R] = 0$$

$$D^{s_1}_{\mu_1,\nu_1}[R] D^{s_2}_{\mu_2,\nu_2}[R] - \sum_{s\mu\nu} \langle s_1, \mu_1, s_2, \mu_2 | s, \mu \rangle D^s_{\mu\nu}[R] \langle s, \nu | s_1, \nu_1, s_2, \nu_2 \rangle = 0$$

Valid for complex angles by analytic continuation

$$U(\Lambda, a)|(m,s,d)\mathbf{p},\mu\rangle =$$

$$\sum_{\nu} |(m,s,d)\boldsymbol{\Lambda} p,\nu\rangle e^{i\Lambda p\cdot a} D^{s}_{\nu\mu}[\Lambda_x^{-1}(\Lambda p)\Lambda\Lambda_x(p)]\sqrt{\frac{\omega_m(\boldsymbol{\Lambda} p)}{\omega_m(\mathbf{p})}}$$

Decompose Wigner rotation

$$\sum_{\nu\alpha\beta} |(m,s,d)\boldsymbol{\Lambda} p,\nu\rangle e^{i\Lambda p\cdot a} D^{s}_{\nu\alpha}[\Lambda_x^{-1}(\Lambda p)] D^{s}_{\alpha\beta}[\Lambda] D^{s}_{\beta\mu}[\Lambda_x(p)]\sqrt{\frac{\omega_m(\Lambda p)}{\omega_m(\mathbf{p})}}$$

Multiply on right by $\sqrt{\omega_m(\mathbf{p})} D^{s}_{\alpha\mu}[\Lambda_x^{-1}(p)]$

$$\Downarrow$$

$$U(\Lambda, a)\underbrace{|(m,s,d)\mathbf{p},\alpha\rangle\sqrt{\omega_m(\mathbf{p})} D^{s}_{\alpha\mu}[\Lambda_x^{-1}(p)]}_{|(m,s)p,\mu,d\rangle_{cov}} =$$

$$\sum_{\nu}\underbrace{\sum_{\alpha} |(m,s,d)\boldsymbol{\Lambda} p,\alpha\rangle\sqrt{\omega_m(\boldsymbol{\Lambda} p)} D^{s}_{\alpha\mu}[\Lambda_x^{-1}(\Lambda p)]}_{|(m,s)\Lambda p,\nu,d\rangle_{cov}}\, e^{i\Lambda p\cdot a} D^{s}_{\nu\mu}[\Lambda]$$

Lorentz covariant representations of the Poincaré group

$$U(\Lambda, a)|(m,s)p,\mu,d\rangle_{cov} = |(m,s)\Lambda p,\nu,d\rangle_{cov} e^{i\Lambda p\cdot a} D^s_{\nu\mu}[\Lambda]$$

Hilbert space inner product

non-trivial dynamical kernel

$$\langle\psi|\phi\rangle = \sum \int \langle\psi|(m,s)\mathbf{p},\mu\rangle d\mathbf{p} \langle(m,s)\mathbf{p},\mu|\phi\rangle =$$

$$\sum \int \langle\psi|(m,s)\mathbf{p},\mu\rangle_{cov} \frac{d\mathbf{p}}{\omega_m(\mathbf{p})} D^s_{\mu\nu}[\Lambda_x(p)\Lambda_x^\dagger(p)]_{cov} \langle(m,s)\mathbf{p},\nu|\phi\rangle$$

$$\sum \int \langle\psi|(m,s)\mathbf{p},\mu\rangle_{cov} 2d^4p\delta(p^2+m^2)\theta(p^0) D^s_{\mu\nu}[\sigma\cdot p]_{cov} \langle(m,s)\mathbf{p},\nu|\phi\rangle$$

$$\Lambda_x(p)\Lambda_x^\dagger(p) = \Lambda_c(p)R_m(p)R_m^\dagger(p)\Lambda_c^\dagger(p) = \Lambda_c^2(p) = e^{\boldsymbol{\rho}\cdot\boldsymbol{\sigma}} = p\cdot\sigma$$

Comments

$$(R^\dagger)^{-1} = R \qquad (\Lambda^\dagger)^{-1} \neq \Lambda$$

Leads to two inequivalent Hilbert space kernels:

$$K_r^{m,s}(p,\mu,\nu) := 2\delta(p^2+m^2)\theta(p^0)D^s_{\mu\nu}[\sigma\cdot p]$$

$$K_l^{m,s}(p,\mu,\nu) := 2\delta(p^2+m^2)\theta(p^0)D^s_{\mu\nu}[\sigma\cdot \Pi p]$$

$\Pi =$ **space reflection**

The kernels are familiar - $K_{l/r}^{m,s}(p,\mu\nu) =$ **two-point Wightman function for a spin-**s **right or left handed free field.**

Summary: Poincaré covariant reps. $\Rightarrow$ Lorentz covariant reps.

- **Start with irreducible representations of the Poincaré group in an L^2 Hilbert space of functions of eigenvalues commuting functions of the Poincaré generators.**
- **Use analyticity of the Wigner rotations to factor the Wigner rotations to construct Lorentz $(SL(2,\mathbb{C}))$ covariant representation of the Poincaré group.**
- **Hilbert spaces have kernels - space reflection does not commute with the kernels - the direct sum of right- and left-handed kernels can be used to get a Lorentz covariant representation of space reflection.**
- **Kernels are independent of original basis choice.**
- **Kernels are familiar Wightman distributions.**

Relation of Poincaré covariant wave functions to Lorentz covariant wave functions - spin choice "x" enters in boost:

$$\boxed{{}_x\langle (m,s,d)\mathbf{p},\mu|\psi\rangle = \sum_{\nu} \frac{D^s_{\mu\nu}[\Lambda_x^{\dagger}(p)]}{\sqrt{\omega_m(\mathbf{p})}}\, {}_{cov,r}\langle (m,s,d)p,\nu|\psi\rangle_{|_{p^0=\omega_m(\mathbf{p})}}}$$

$$\boxed{{}_x\langle (m,s,d)\mathbf{p},\mu|\psi\rangle = \sum_{\nu} \frac{D^s_{\mu\nu}[\Lambda_x^{-1}(p)]}{\sqrt{\omega_m(\mathbf{p})}}\, {}_{cov,l}\langle (m,s,d)p,\nu|\psi\rangle_{|_{p^0=\omega_m(\mathbf{p})}}}$$

Euclidean covariant representations of the Poincaré group

Some definitions:

$$\sigma_{e\mu} = (i\sigma_0, \boldsymbol{\sigma})$$

$$p_e^\mu = (-ip^0, \mathbf{p})$$

$$P_e := p_e^\mu \sigma_{e\mu} = \begin{pmatrix} ip_e^0 + p^3 & p^1 - ip^2 \\ p^1 + ip^2 & ip_e^0 - p^3 \end{pmatrix}$$

$$\det(P_e) = -(p_e^0)^2 - \mathbf{p}^2$$

$$P := p^\mu \sigma_\mu = \begin{pmatrix} p^0 + p^3 & p^1 - ip^2 \\ p^1 + ip^2 & p^0 - p^3 \end{pmatrix}$$

$$\det(P) = (p^0)^2 - \mathbf{p}^2$$

Complex orthogonal group $\sim$ complex Lorentz group

$$P'_e = AP_eB^t \qquad \det(A) = \det(B) = 1 \qquad \textbf{orthogonal}$$

$$P' = APB^t \qquad \det(A) = \det(B) = 1 \qquad \textbf{Lorentz}$$

$$O_{\mu,\nu} = \frac{1}{2}\mathrm{Tr}(\sigma^{\dagger}_{e\mu} A \sigma_{e\nu} B^T) \qquad \Lambda_{\mu,\nu} = \frac{1}{2}\mathrm{Tr}(\sigma_{\mu} A \sigma_{\nu} B^T)$$

- $B = A^* =$ **real Lorentz transformations**

- A, B **unitary** $=$ **real orthogonal transformation**

- $A, B \in SL(\mathbb{C}) =$ **complex orthogonal transformation; complex Lorentz transformation.**

- **Real Lorentz group $\sim$ complex subgroup of complex orthogonal group**

- **Real orthogonal group $\sim$ complex subgroup of complex Lorentz group**

- **Euclidean and Poincaré generators related by related by**

$$H_m = iH_e \qquad \mathbf{K}_c \cdot \hat{\mathbf{n}} = -i\mathbf{J}_{e\hat{\mathbf{n}},x_e^0}$$

- **Euclidean and Poincaré generators cannot both be self-adjoint on the same representation of the Hilbert space.**

Define the irreducible Euclidean covariant kernels

$$K^{m,s}_{e,r/l}(x_e - y_e;\mu,\nu) := \int d^4 p_e K^{m,s}_{e,r/l}(p_e,\mu,\nu) e^{ip_e\cdot(x_e-y_e)}$$

$$K^{m,s}_{e,r}(p_e,\mu,\nu) := \frac{1}{\pi}\frac{D^s_{\mu\nu}[p_e\cdot\sigma_e]}{p_e^2+m^2}$$

$$K^{m,s}_{e,l}(p_e,\mu,\nu) := \frac{1}{\pi}\frac{D^s_{\mu\nu}[\Pi p_e\cdot\sigma_e]}{p_e^2+m^2}$$

- **These kernels transform covariantly under spin s representations of the 4-d orthogonal transformations.**

- **Define $\theta(x_e^0, \mathbf{x}) = (-x_e^0, \mathbf{x}) =$ Euclidean time reflection.**

- **Replacing $K^{m,s}_{e,r/l}(x_e - y_e; \mu, \nu)$ by $K^{m,s}_{e,r/l}(\theta x_e - y_e; \mu, \nu)$ makes the H_m and $\mathbf{K}_c \cdot \hat{\mathbf{n}}$ Hermitian with respect to the quadratic form with this kernel.**

- **The resulting sesquilinear form has negative norm vectors on the space of Euclidean test functions (x_e^0 even and odd functions cannot both have positive norm).**

- **The sesquilinear form can be made positive by projecting on a suitable subspace.**

  (O.S.) Subspace = Schwartz functions in Euclidean space-time variables with support for positive relative Euclidean times.

- **The projection of** $K^{m,s}_{e,r/l}(\theta x_e - y_e; \mu, \nu)$ **on spinor functions with positive Euclidean time support is non-negative. It can be completed to construct a new Hilbert space representation,** $\mathcal{H}$

- **The Euclidean kernels,** $K^{m,s}_{e,r/l}(x_e - y_e; \mu, \nu)$**, are reflection positive:**

$$\Pi_+ \Theta K \Pi_+ \geq 0 \qquad = \qquad \textbf{reflection positivity}$$

On $\mathcal{H}$ the Poincaré generators ($s=0$ case) are self-adjoint:

$$H\Psi(x_e) = \frac{\partial}{\partial x_e^0}\Psi(x_e)$$

$$\mathbf{P}\Psi(x_e) = -i\frac{\partial}{\partial \mathbf{x}_e}\Psi(x_e)$$

$$\mathbf{J}\Psi(x_e) = -i\mathbf{x}\times\boldsymbol{\nabla}_x\Psi(x_e)$$

$$K^j\Psi(x_e) = (x^j\frac{\partial}{\partial x_e^0} - x_e^0\frac{\partial}{\partial x^j})\Psi(x_e).$$

- **In this representation the wave functions involve Euclidean times, there is no analytic continuation!**

- **Generators have the same form in dynamical theories - only the Euclidean Green functions change.**

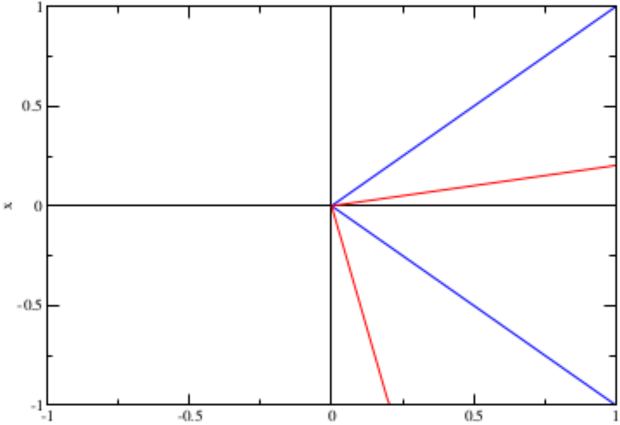
These generators are self-adjoint on $\mathcal{H}$.

- **$\mathbf{P}, \mathbf{J}$ generate one parameter unitary groups.**

Positive Euclidean time translations and rotations in Euclidean space-time planes are used to construct the Hamiltonian and Lorentz boost generators

- **H generates a contractive Hermitian semigroup under x_e^0 translations.**
- **$\mathbf{K}$ generates a local symmetric semigroup under rotations in x_e^0, x^i planes.**
- **All have self-adjoint generators by Stone's theorem and extensions.**

Domains for local symmetric semigroups



Hilbert space inner product on $\mathcal{H}$:

$$\langle f|g\rangle := (f, \Theta K g) =$$

$$\frac{1}{\pi}\int f^*(x_e,\mu)\frac{D^s_{\mu\nu}[p_e\cdot\sigma_e]}{p_e^2+m^2}e^{ip_e\cdot(\theta x_e - y_e)}g(y_e,\nu)d^4p_e d^4x_e d^4y_e$$

Integrate over p_e^0 - use the support condition to close the contour in the lower half plane

$$\langle f|g\rangle =$$

$$\int d^4x_e d^4y_e f^*(x_e,\mu)\frac{e^{-\omega_m(\mathbf{p})(x_e^0+y_e^0)+i\mathbf{p}\cdot(\mathbf{x}-\mathbf{y})}}{(2\pi)^3}\frac{d\mathbf{p}}{\omega_m(\mathbf{p})}D^s_{\mu\nu}[p\cdot\sigma]g(y_e,\nu)$$

Relation to Poincaré covariant and Lorentz covariant representations

$$\int f^*(x_e,\mu)K^{m,s}_{e,r}(\theta x - y;\mu\nu)g(y_e,\nu)d^4x_e d^4y_e =$$

$$\int \underbrace{\left(\int d^4x_e f^*(x_e,\mu)e^{-\omega_m(\mathbf{p})x_e^0}\frac{e^{i\mathbf{p}\cdot\mathbf{x}}}{(2\pi)^{3/2}}\right)}_{\langle\psi|(m,s)p,\mu\rangle_{cov}}$$

$$\frac{d\mathbf{p}}{\omega_m(\mathbf{p})}D^s_{\mu\nu}[p\cdot\sigma]$$

$$\underbrace{\left(\int d^4y_e g(y_e,\nu)e^{-\omega_m(\mathbf{p})y_e^0}\frac{e^{-i\mathbf{p}\cdot\mathbf{y}}}{(2\pi)^{3/2}}\right)}_{{}_{cov}\langle(m,s)p,\nu|\phi\rangle}$$

The right and left handed wave functions in Lorentz covariant, Euclidean covariant and Poincaré covariant representations are related by:

$$
{}_{cov,r}\langle (m,s)p,\nu|\phi\rangle|_{p^0=\omega_m(\mathbf{p})} =
$$

$$
\int d^4y_e g(y_e,\nu) e^{-\omega_m(\mathbf{p})y_e^0} \frac{e^{-i\mathbf{p}\cdot\mathbf{y}}}{(2\pi)^{3/2}} =
$$

$$
\sqrt{\omega_m(\mathbf{p})} D^s_{\mu\nu}[\Lambda_x^{-1}(p)]_x\langle (m,s)p,\nu|\phi\rangle
$$

$$
{}_{cov,l}\langle (m,s)p,\nu|\phi\rangle|_{p^0=\omega_m(\mathbf{p})} =
$$

$$
\int d^4y_e g(y_e,\nu) e^{-\omega_m(\mathbf{p})y_e^0} \frac{e^{-i\mathbf{p}\cdot\mathbf{y}}}{(2\pi)^{3/2}} =
$$

$$
\sqrt{\omega_m(\mathbf{p})} D^s_{\mu\nu}[\Lambda_x(p)]_x\langle (m,s)p,\nu|\phi\rangle
$$

$$ {}_{x}\langle (m,s)p,\nu|\phi\rangle, \qquad {}_{cov}\langle (m,s)p,\nu|\phi\rangle, \qquad \text{and} \qquad g(y_e,\nu) $$

- **All represent the same physical state in different Hilbert space representations.**

- **These relations hold for free particles of irreducible basis states of an interacting theory.**

Euclidean representation - properties

- **Calculations of physical inner products can be performed directly in the Euclidean representation without analytic continuation.**

- **Reflection positivity gives both a positive norm and the spectral condition $H \geq 0$.**

- **The representation of the Poincaré generators are the same for dynamical or non-interacting theories. The dynamics enters in the Euclidean kernel.**

- **For gauge theories the above considerations only apply to gauge invariant states.**

- **Because of the time reflection in the kernel, delta functions are normalizable states on $\mathcal{H}$!**

Dynamics - models

General considerations

- The general dynamical problem is to decompose the unitary representation of the Poincaré group into a direct integral of irreducible representations.
- The first step is to construct the dynamical unitary representation of the Poincaré group.
- The second step is to decompose $U(\Lambda, A)$ into a direct integral of representations labeled by mass, spin and degeneracy parameters.
- It is natural to start with a model of non-interacting particles. Interactions are then added that should the preserve positivity of the Hilbert space norm, the unitarity of the representation of Poincaré group, the spectral condition, cluster properties. and should allow scattering.

Poincaré covariant dynamics ($N = 2$)
(generalized Bakamjian-Thomas construction)

- **Use Clebsch-Gordan coefficients of the Poincaré group**

$$\langle (m_1, s_1) b_1, (m_2, s_2) | (m_{12}, s_{12}) b_{12}, d_{12} \rangle$$

to decompose the two-body Hilbert space into a direct intergal of irreducible representations in the same basis (b**).**

- **Add interactions to the mass that (1) commute with** b**, (2) are independent of** b **(3) commute with the free two-body spin and (4) preserve the spectral condition.**
- **Diagonalize the interacting mass in the free irreducible representation.**
- **Simultaneous eigenstates of interactng mass, free spin and free vector variables (**b**) transform irreducibly, defining a relativistic two-body dynamics**

Realistic model interactions = ($N = 2$)

$$|(m_{12}, s_{12})b_{12}, d_{12}\rangle \to |(k, j)\mathbf{p}, \mu; l, s\rangle$$

$$\langle (k', j')\mathbf{p}', \mu'; l', s'|V_{nn}|(k, j)\mathbf{p}, \mu; l, s\rangle =$$

$$\delta(\mathbf{p}' - \mathbf{p})\delta_{j'j}\delta_{\mu'\mu}\langle k', l', s'\|V_{nn}^{j}\|k, l, s\rangle$$

$$M = \sqrt{m_1^2 + 2m_r(k^2/2m_r + V_{nn})} + \sqrt{m_2^2 + 2m_r(k^2/2m_r + V_{nn})}$$

- **Eigenstates of H_{nr} are eigenstates of M.**
- **Relativistic S-matrix fits same 2-body CM data as NR 2-body S-matrix (invariance principle)!**

$$S_{exp} = S_{nr} = \lim_{t\to\infty} e^{iH_0 t} e^{2iH_{nr}} e^{iH_0 t} = \lim_{t\to\infty} e^{iM_0 t} e^{2iMt} e^{iM_0 t} = S_r$$

- **Construction violates cluster properties for $N \geq 3$.**

Poincaré covariant dynamics (Sokolov construction) ($2 \to 3$ - cluster properties - fixed up to 3BF)

- **N=3 Take tensor products of N=2 with N=1. Use unitary scattering equivalence to transform to representations with free three-body spin.**
- **Add the transformed 2+1 interactions to the free three-body mass. Use Poincaré group Racah coefficients to put is a common basis.**
- **Multiply the result by a symmetrized product of the inverse of the 2+1 unitary scattering equivalences.**
- **This construction gives a dynamical unitary representation of the Poincaré group satisfying cluster properties, spectral condition, and fits experimental two-body data.**

Sokolov construction ($N = 3$ example)

$$H =$$

$$e^{\sum \ln A^{\dagger}_{ij,k}} \left(\sum (A_{ij,k} H_{ij} \otimes H_k A^{\dagger}_{ij,k} - 2H_1 \otimes H_2 \otimes H_3 + V_{123} \right) e^{\sum \ln A_{ij,k}}$$

- **Method preserves Poincaré invariance, cluster properties and spectral condition (for suitable interactions).**
- **Preserves kinematic subgroup (for suitable interactions).**
- $A_{ij,k}$ **generates frame-dependent many interactions.**
- **Resulting spin is dynamical.**
- **Recursive construction is messy, result depends on choice of** $A_{ij,k}$**, connection with QFT not direct.**

Lorentz covariant constraint dynamics (method I)

Add interactions to Wightman functions

- **Treat** $C_i := p_i^2 + m_i^2 = 0, \quad p_i^0 > 0$ **as first class constraints.**

$$K = \prod_i \delta(C_i)$$

- **Add Lorentz covariant interactions** $C_i \to C_i' = C_i + \sum_j V_{ij}$ **so** C_i' **are compatible constraints (first class)**

$$[C_i', C_j'] = f_{ijk} C_k' \qquad \Rightarrow \qquad K' = \Pi \delta(C_i')$$

- **Two-body interactions satisfying first class condition:**

$$[V_{ij}, p_i^2 - p_j^2] = 0$$

- K' **are model Wightman functions - normally imposed by solving coupled Dirac or Klein Gordon equations. The first class condition is an integrability condition.**
- **First class constraints satisfying cluster properties for** $N > 2$ **not known.**

Lorentz covaraint Schwinger-Dyson equations (method II)

- **Solutions are matrix elements of time-ordered products of fields.**
- **Key requirement - $\exists$ complete sets of positive energy intermediate states (plus vacuum) bewteen all operators and states.**
- **This is where the direct integral of irreducible representations appears.**
- **Using the integral representation of the Heaviside function with the direct integral leads to poles associated with the energies of the intermediate states:**

$$\theta(t) = \frac{1}{2\pi i} \int \frac{e^{ist}}{s - i\epsilon}$$

- **Residues of the (double) poles have the general strufture**

  $$\langle 0|T(\Pi\phi_i(x_i))|(m,s)b\rangle\langle(m,s)b|O|(m',s')b'\rangle\langle(m',s')b'|T(\Pi\phi_j(x_j))|0$$

- **The matrix elements of any operator** $\langle(m,s)b|O|(m',s')b'\rangle$ **in irreducible eigenstates can be extracted using a normalization condition to eliminate the "wave functions"** $\langle(m,s)b|T(\Pi\phi_j(x_j))|0\rangle$**.**

- **Equations are non-linear - but the solutions can be expressed as moments of a path integral.**

- **The connection with the general requirements of relativistic quantum theory is not direct. This makes it difficult to determine what properties are preserved under truncation.**

- **The method has a natural connection with the** S **matrix. Bogoliubov's S-matrix axioms may be easier to check.**

Euclidean covaraint dynamics

- Direct connection with Lagrangian field theory. The Euclidean kernels (before time reflection) are moments of a Euclidean path integral.

- They satisfy non-linear Euclidean Schwinger-Dyson equations.

- The connection with a relativistic quantum field theory is more straightforward in the Euclidean case - reflection positivity, Euclidean covariance and cluster properties are the requirements for a relativistic quantum theory.

- For gauge theories reflection positivity is only needed for gauge invariant intermediate states.

Things to think about

- The Euclidean representation has a **Minkowski interpretation** provided we use the Euclidean time reflection operator and Euclidean states with positive relative Euclidean time support.
- **Analytic continuation is not needed**, and there are explicit representations of all 10 Poincaré generators in the Euclidean representaion. These **generators are self adjoint**.
- Four dimensional **delta functions are square integrable** in the Euclidean representation - this may facilitate numerical calculations.
- Locality logically independent of the other Euclidean Axioms.

- **The requirements of reflection positivity need to be better understood - for example what are sufficient conditions on Euclidean Bethe-Salpeter kernels for G to be reflection positive**

$$G^{-1} = \prod G_i{}^{-1} - \sum K_{ij} - \sum K_{ijk}$$

- **Conditions on K_{ij}, K_{ijk} so G is reflection positive?**

- **By giving up locality it is possible to have different N-point Green functions (i.e. $G(1:3), G(2:2), G(3:1)$) corresponding to different numbers of initial and final particles). This makes the reflection positivity constraint easier. In a local theory these are related by analytic continuation.**
- **Haag-Ruelle scattering can be directly formulated in the Euclidean representation.**
- **Light front generators are given by complex differential operators, $p^+ := \frac{\partial}{\partial x_e^0} - i\frac{\partial}{\partial x^3}$**

  these should be self adjoint on $\mathcal{H}$.

Thanks to the organizers!