Global numerical simulations of giant glitches in full general relativity

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in collaboration with

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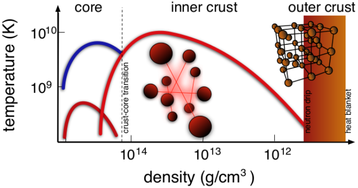
Sourie, Oertel & Novak, *PRD*, 2016; Sourie, Chamel, Novak & Oertel, *MNRAS*, 2017

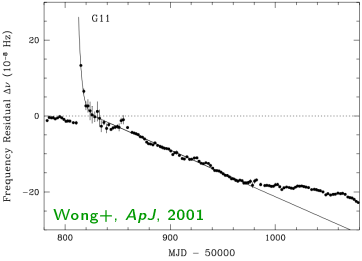
Superfluidity in neutron stars

Theoretical predictions:

$$T \lesssim T_c^{\max} \sim 10^8 - 10^{10}\ \mathrm{K}$$

⇝ *superfluid neutrons* in the core & in the inner crust of NSs.





Wong+, *ApJ*, 2001

Observational evidence:

- *Long relaxation time scales in pulsar glitches,*
- Fast cooling in Cassiopeia A,
- QPOs from SGRs, ...

This work

Consequence of superfluidity:

several **dynamically distinct** fluids inside NSs,
coupled through both *dissipative* and *non-dissipative* effects.

Questions:

What is the impact of **general relativity** on
the *non-dissipative couplings* between the fluids ?

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What is the impact of **general relativity** on
the *non-dissipative couplings* between the fluids ?

Are **general-relativistic effects** important on
the *global dynamics* of *giant pulsars glitches* ?

1. Introduction

2. Equilibrium configurations of superfluid NSs
   - Model assumptions
   - Fluid couplings

3. Applications to the dynamics of giant glitches
   - Transfer of angular momentum
   - Impact of GR on the dynamics of pulsar glitches

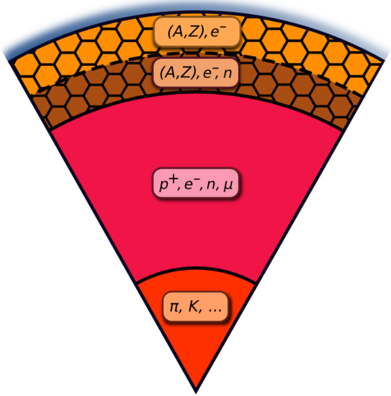
4. Conclusion

Assumptions & ingredients

Prix et al., *PRD*, 2005 & Sourie et al., *PRD*, 2016

Equilibrium configurations:

- $T = 0$ and no magnetic field,
- dissipative effects are neglected,
- **uniform** composition: p, e^-, n
  ⇝ the crust is **not** included,
- asymptotically flat, *stationary*, *axisymmetric* & *circular* metric,
- **rigid-body** rotation: Ω_n, Ω_p
  ⇝ **global** model.



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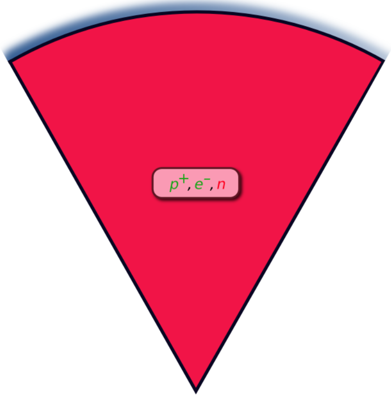

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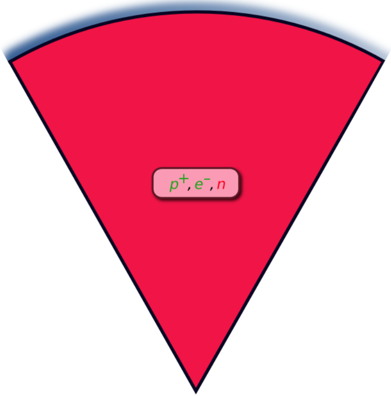
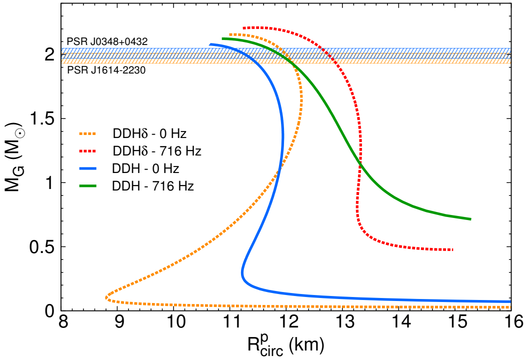


The neutron star is thus described by **two perfect fluids**:

⇢ a neutron superfluid and a fluid of charged particles

Equations of state

- Polytropic EoSs,
- *Density-dependent RMF models* (DDH & DDHδ).



$$\mathcal{E}\left(n_n, n_p, \Delta^2\right)$$

Fluid angular momenta

⇝ Komar angular momentum (*axisymmetry*):

$$J_{\mathrm{K}} = J_{\mathrm{n}} + J_{\mathrm{p}}$$

see Langlois, Sedrakian & Carter, *MNRAS*, 1998.

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Moments of inertia:

$$\mathrm{d}J_X = I_{XX}\ \mathrm{d}\Omega_X + I_{XY}\ \mathrm{d}\Omega_Y \qquad X, Y \in \{\mathrm{n}, \mathrm{p}\}$$

$$\hat{I}_X = I_{XX} + I_{XY} \qquad \hat{I} = \hat{I}_{\mathrm{n}} + \hat{I}_{\mathrm{p}}$$

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⇢ I_{XY} contains any possible non-dissipative couplings between the fluids.

Angular momentum of fluid X

In the *slow-rotation approximation* and to *first order in the lag* $\delta\Omega = \Omega_{\rm n} - \Omega_{\rm p}$, we get:

$$\begin{aligned} J_X \ &\simeq \int_{\Sigma_t} n_X \mu^X \frac{B}{N} \left(\Omega_X - \omega\right) r^2 \sin^2\theta \ \mathrm{d}^3 V \\ &+ \int_{\Sigma_t} n_X \mu^X \varepsilon_X \frac{B}{N} \left(\Omega_Y - \Omega_X\right) r^2 \sin^2\theta \ \mathrm{d}^3 V \end{aligned}$$

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\end{aligned}
$$

- entrainment effect

  due to the strong interactions between nucleons *in the core*:

  $$\boxed{p^\alpha_X = \mathcal{K}^{XX} n_X u^\alpha_X + \mathcal{K}^{XY} n_Y u^\alpha_Y}$$

  Andreev & Bashkin, *SJETP*, 1976

- relativistic frame-dragging effect

  associated with the rotation of the two fluids, $\Omega_{\rm n}$ and $\Omega_{\rm p}$:

  $$\boxed{g_{t\varphi} \neq 0}$$

  Carter, *Annals of Physics*, 1975

Fluid couplings

Angular momentum of fluid X:

$$\begin{aligned} J_X &= \int n_X \mu^X \tfrac{B}{N} r^2 \sin^2\theta \, \mathrm{d}^3V \times \Omega_X \\ &+ \int \varepsilon_X \, n_X \mu^X \tfrac{B}{N} r^2 \sin^2\theta \, \mathrm{d}^3V \times (\Omega_Y - \Omega_X) \\ &- \int \omega \, n_X \mu^X \tfrac{B}{N} r^2 \sin^2\theta \, \mathrm{d}^3V \end{aligned}$$

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- I_X = "moment of inertia" of fluid X $\dashrightarrow$ $\int_{\Sigma_t} \rho_X r^2 \sin^2\theta \, \mathrm{d}^3 V$

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- $\tilde{\varepsilon}_X$ = entrainment parameter averaged over the star.

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- $\tilde{\varepsilon}_X$ = entrainment parameter averaged over the star.
- ε^{LT}_{YX} & ε^{LT}_{XX} = contribution of fluids Y and X on Lense-Thirring effects on X.

Entrainment VS frame-dragging

In the general-relativistic framework, one thus gets:

$$J_X = I_X \left(1 - \varepsilon_{XX}^{LT} - \tilde{\varepsilon}_X\right) \Omega_X + I_X \left(\tilde{\varepsilon}_X - \varepsilon_{YX}^{LT}\right) \Omega_Y$$

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- In the **Newtonian limit** (see, e.g., Sidery+, *MNRAS*, 2010):

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- In **GR**: *additional* coupling through **frame-dragging** effects.

--→ *already* pointed out by Carter, *Annals of Physics*, 1975

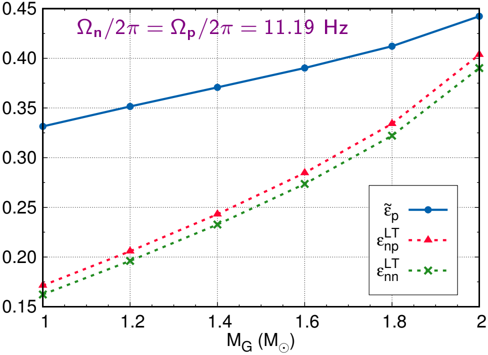
Entrainment VS frame-dragging

Total coupling coefficients:

$$\hat{\varepsilon}_X = I_{XY} / \left(I_{XX} + I_{XY} \right)$$

In the slow-rotation approximation:

$$\boxed{\hat{\varepsilon}_{\rm p} = \frac{\tilde{\varepsilon}_{\rm p} - \varepsilon^{LT}_{\rm n\,p}}{1 - \varepsilon^{LT}_{\rm p\,p} - \varepsilon^{LT}_{\rm n\,p}}}$$



Remarks:

- in Newt. gravity: $\hat{\varepsilon}_X = \tilde{\varepsilon}_X$
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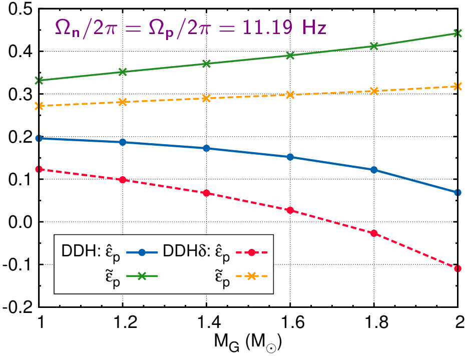

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Vortex-mediated glitch theory

Anderson & Itoh, *Nature*, 1975

Two-fluid model

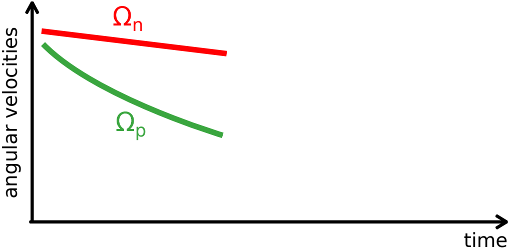
Baym et al., *Nature*, 1969

- Charged particles:

  $\Omega_p = \Omega \leftrightarrow$ pulsar

- Superfluid neutrons:

  $\Omega_n \gtrsim \Omega_p$



Key assumption: the vortices can **pin** to the crust and/or to flux tubes.

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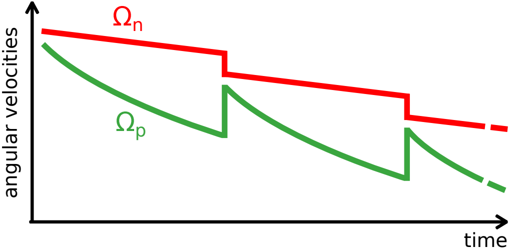

Once a critical lag $\Omega_n - \Omega_p$ is reached:

some vortices get **unpinned** and are allowed to move **radially**

⇢ angular momentum transfer between the fluids

Angular momentum transfer

Langlois et al., *MNRAS*, 1998 & Sidery et al., *MNRAS*, 2010

$\Omega_n - \Omega_p = \delta\Omega_0 \Rightarrow$ the dynamics is governed by **mutual friction forces**

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Assuming *straight vortices*, the mutual friction moment considered reads

$$\Gamma_{\text{int}} = -\int \frac{\mathcal{R}}{1+\mathcal{R}^2} \Gamma_n n_n \varpi_n \chi_\perp^2 \, \mathrm{d}\Sigma \times (\Omega_n - \Omega_p) = -2\bar{\mathcal{B}}\hat{I}_n \Omega_n \zeta \times \delta\Omega$$

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superfluid vorticity

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superfluid vorticity

mean mutual friction parameter

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superfluid vorticity

mean mutual friction parameter

⇝ the geometry of the vortex array and the interactions between superfluid vortices and superconducting flux tubes are **poorly known**.

Spin-up time scale

Evolution equations:

$$\begin{cases} \dot{j}_{\rm n} &= +\,\Gamma_{\rm int}, \\ \dot{j}_{\rm p} &= -\,\Gamma_{\rm int}. \end{cases} \quad \dashrightarrow \quad \frac{\delta\dot{\Omega}}{\delta\Omega} = -\frac{\hat{I}\hat{I}_{\rm n}}{I_{\rm nn}I_{\rm pp} - I_{\rm np}^2} \times 2\bar{\mathcal{B}}\zeta\Omega_{\rm n}$$

- Theoretical rise time:

$$\rightsquigarrow \quad \delta\Omega(t) = \delta\Omega_0 \times e^{-\frac{t}{\tau_{\rm r}}}$$

$$\boxed{\tau_{\rm r} = \frac{\hat{I}_{\rm p}}{\hat{I}} \times \frac{1 - \hat{\varepsilon}_{\rm p} - \hat{\varepsilon}_{\rm n}}{2\zeta\bar{\mathcal{B}}\Omega_{\rm n}}}$$

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- Numerical modelling:

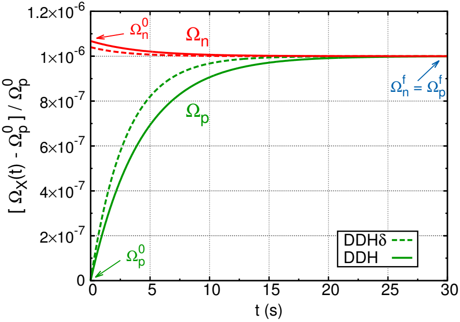
hyp.: hydrodynamical time ~ 0.1 ms $\ll$ rise time (dissipation)

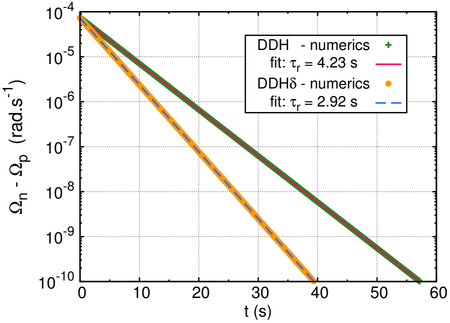
$\dashrightarrow$ Computation of $\Omega_\mathrm{n}(t)$ & $\Omega_\mathrm{p}(t)$ profiles from $\Omega_{\mathrm{n},0} > \Omega_{\mathrm{p},0}$ using a quasi-stationary sequence of equilibrium configurations.

Time evolution

$$\Delta\Omega/\Omega = 10^{-6},\ \Omega_{\mathrm{n}}^{f} = \Omega_{\mathrm{p}}^{f} = 2\pi \times 11.19\ \mathrm{Hz},$$

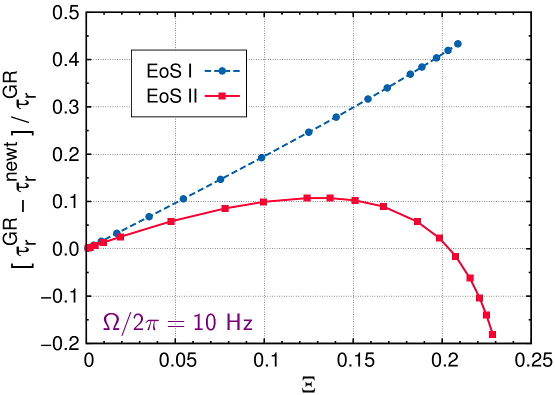
$$M_{\mathrm{G}} = 1.4\ \mathrm{M}_{\odot}\ \&\ \bar{\mathcal{B}} = 10^{-4}$$





⇢ the spin-up time scale can be very precisely estimated from *stationary configurations* only.

Influence of general relativity on τ_r



- polytropic EoSs
- **compactness** parameter:

$$\boxed{\Xi = \frac{G\,M_G}{R_{c,eq}\,c^2}}$$

NB: for NSs, $\Xi \simeq 0.2$

- these relative differences also depend on Ω

--→ **GR** can have a **large impact** on the dynamics of pulsar glitches!

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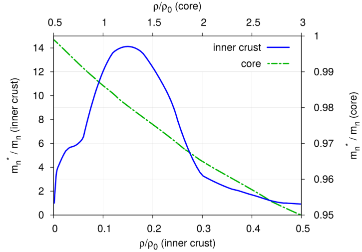
Conclusion & perspectives

- *Additional coupling* through relativistic frame-dragging effects,
- *Relativistic corrections* on the spin-up time: $\sim 50\%$ (core),

  ↪ should be included in a quantitative model of glitches.

Future work:

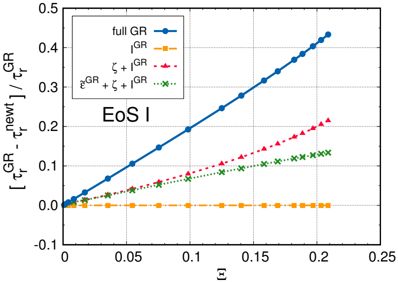
- Build a **local** model in which only a small part of the superfluid is decoupled from the rest of the star (differential rotation),
- Take the **crust** into account!



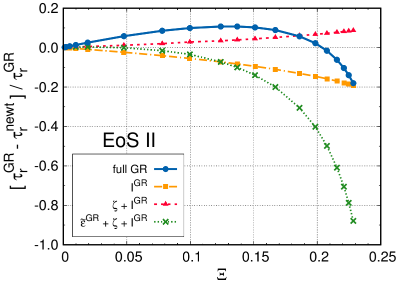
Antonelli & Pizzochero, 2017

Thank you!

Influence of general relativity on τ_r



EoS I



EoS II

Spacetime metric

Bonazzola, Gourgoulhon, Salgado & Marck, *A&A*, 1993

Rotating neutron stars, at **equilibrium**, described by $(\mathcal{E}, \boldsymbol{g})$:

- asymptotically flat: $\boldsymbol{g} \to \boldsymbol{\eta}$ at spatial infinity ($r \to +\infty$),
- stationary & axisymmetric: $\frac{\partial g_{\alpha\beta}}{\partial t} = \frac{\partial g_{\alpha\beta}}{\partial \varphi} = 0$,
- circular: perfect fluids ⇒ *purely circular* motion around the rotation axis with Ω_n, Ω_p (+ **rigid rotation**).

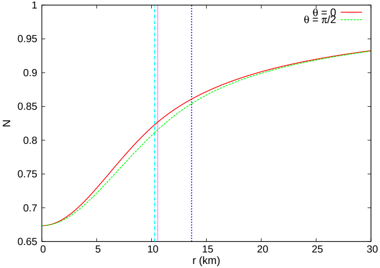
Spacetime metric in quasi-isotropic coordinates:

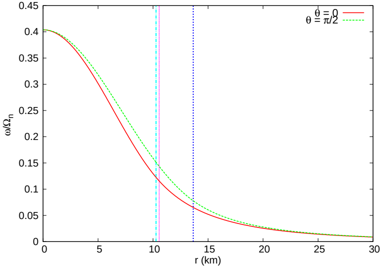
$$g_{\alpha\beta}\,\mathrm{d}x^\alpha\,\mathrm{d}x^\beta = -N^2\,\mathrm{d}t^2 + A^2(\mathrm{d}r^2 + r^2\,\mathrm{d}\theta^2) + B^2 r^2 \sin^2\theta(\mathrm{d}\varphi - \omega\,\mathrm{d}t)^2$$

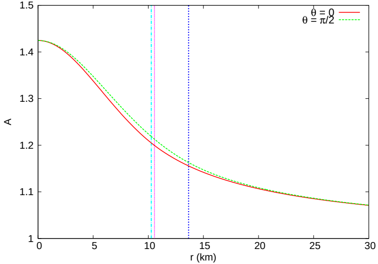
At spatial infinity

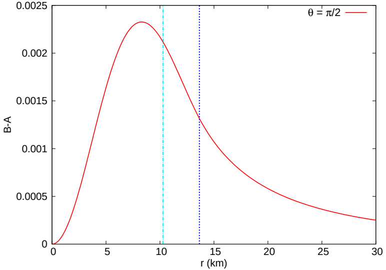
$$N, A, B \to 1 \quad \& \quad \omega \to 0$$

Metric potentials









Relativistic two-fluid hydrodynamics

Carter, "Covariant theory of conductivity in ideal fluid or solid media", 1989 & Carter & Langlois, *Nuc. Phys. B*, 1998

System = two **perfect** fluids:

- superfluid neutrons $\rightarrow \vec{\boldsymbol{n}}_{\mathrm{n}} = n_{\mathrm{n}} \vec{\boldsymbol{u}}_{\mathrm{n}}$,
- protons & electrons $\rightarrow \vec{\boldsymbol{n}}_{\mathrm{p}} = n_{\mathrm{p}} \vec{\boldsymbol{u}}_{\mathrm{p}}$.

Energy-momentum tensor

$$T_{\alpha\beta} = n_{\mathrm{n}\alpha} p^{\mathrm{n}}_{\beta} + n_{\mathrm{p}\alpha} p^{\mathrm{p}}_{\beta} + \Psi g_{\alpha\beta}$$

$\hookrightarrow$ **conjugate momenta**

Entrainment matrix:

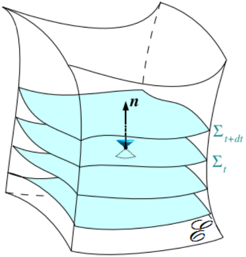
$$\begin{cases} p^{\mathrm{n}}_{\alpha} & = \mathcal{K}^{\mathrm{nn}} n^{\mathrm{n}}_{\alpha} + \mathcal{K}^{\mathrm{np}} n^{\mathrm{p}}_{\alpha} \\ p^{\mathrm{p}}_{\alpha} & = \mathcal{K}^{\mathrm{pn}} n^{\mathrm{n}}_{\alpha} + \mathcal{K}^{\mathrm{pp}} n^{\mathrm{p}}_{\alpha} \end{cases}$$

$\dashrightarrow$ entrainment effect

Equation of state

$$\mathcal{E}(n_{\mathrm{n}}, n_{\mathrm{p}}, \Delta^2)$$

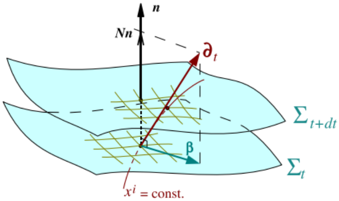
3+1 formalism



Foliation of the spacetime $(\mathcal{E}, \boldsymbol{g})$ by $(\Sigma_t)_{t\in\mathbb{R}}$, with unit normal $\vec{\boldsymbol{n}}$

Eulerian observer $\mathcal{O}_n$: 4-velocity $= \vec{\boldsymbol{n}}$

- lapse function N : $\vec{\boldsymbol{n}} = -N\vec{\boldsymbol{\nabla}}t$,
- shift vector $\vec{\boldsymbol{\beta}}$: $\vec{\boldsymbol{\partial}}_t = N\vec{\boldsymbol{n}} + \vec{\boldsymbol{\beta}}$.



3+1 metric:

$$g_{\alpha\beta}\,\mathrm{d}x^{\alpha}\,\mathrm{d}x^{\beta} = -N^2\,\mathrm{d}t^2 + \gamma_{ij}\left(\mathrm{d}x^i + \beta^i\,\mathrm{d}t\right)\left(\mathrm{d}x^j + \beta^j\,\mathrm{d}t\right)$$

Numerical procedure

Paramètres d'entrée :

- une EOS
- H_c^n, H_c^p
- Ω_n, Ω_p

$i = 0$

Initialisation :

- $N = A = B = 1$ et $\omega = 0, \forall\,(r, \theta)$
- $U_n = U_p = 0$
- $H_0^i(r, \theta) = H_c^i\left(1 - \frac{r^2}{R^2}\right)$

Convergence threshold

$$|H_{k+1}^i(r, \theta) - H_k^i(r, \theta)| < \epsilon$$

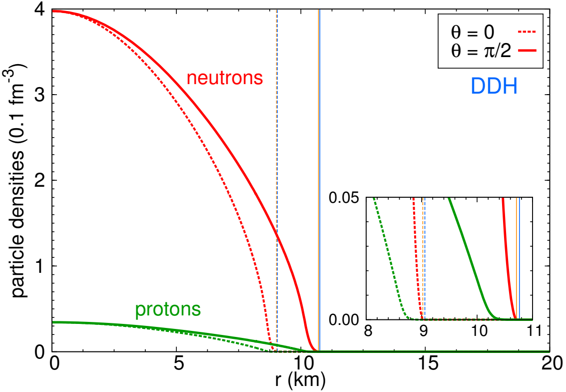
At each iteration

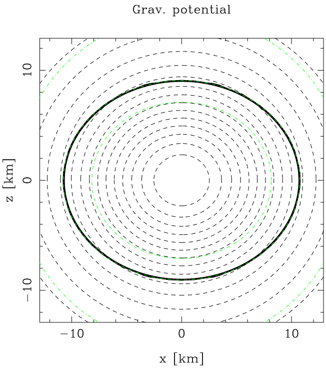
For given values of $(\mu^{\mathrm{n}}, \mu^{\mathrm{p}}, \Delta^2)$, we compute:

1. Ψ, n_{n}, n_{p} and α from the EoS
2. The source terms E, p_φ, ${S^i}_i$,
3. Einstein Equations are solved,
4. Kinetic terms U_{i} et Γ_{i},
5. Computation of H_{k+1}^i.

Density profiles

$$M_G = 1.4\ M_\odot,\ \Omega_n/2\pi = \Omega_p/2\pi = 716\ \mathrm{Hz}$$

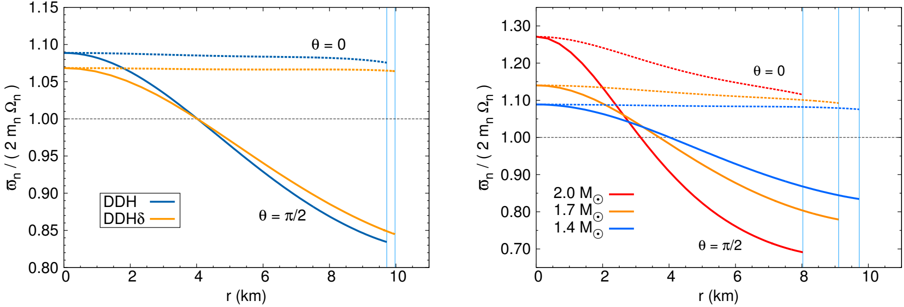




Vorticity

Superfluid vorticity

$$w_{\mu\nu} = \nabla_\mu p^{\mathrm{n}}_\nu - \nabla_\nu p^{\mathrm{n}}_\mu \dashrightarrow \varpi_{\mathrm{n}} = \sqrt{\frac{w_{\mu\nu} w^{\mu\nu}}{2}}$$



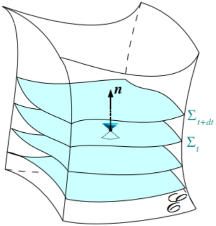
$$\Omega^{\mathrm{n}}/2\pi = \Omega^{\mathrm{p}}/2\pi = 716 \text{ Hz}$$

Angular momenta

Axisymmetry $\leftrightarrow$ $\vec{\chi}$

Komar definition:

$$J_{\mathrm{K}} = - \int_{\Sigma_t} \underbrace{\boldsymbol{T}(\vec{\boldsymbol{n}}, \vec{\chi})}_{-p_\varphi} \ \mathrm{d}^3 V$$



Eulerian observer $\vec{\boldsymbol{n}}$ (3+1)

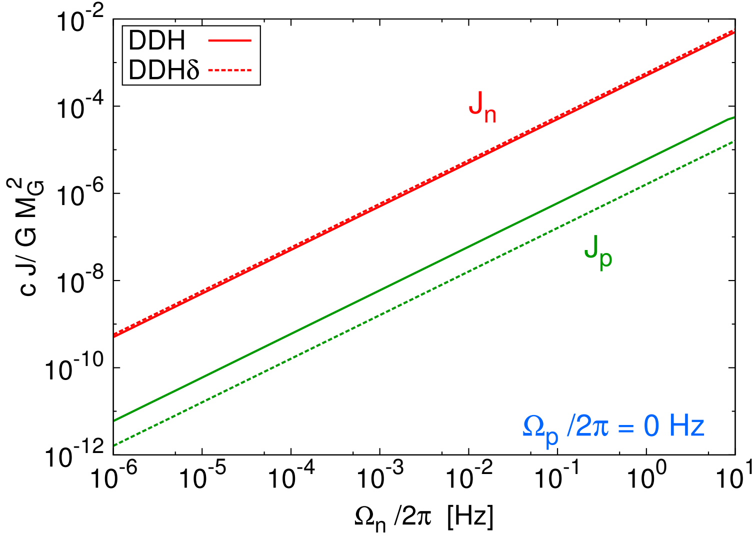
Angular momentum of each fluid

Langlois, Sedrakian & Carter, *MNRAS*, 1998

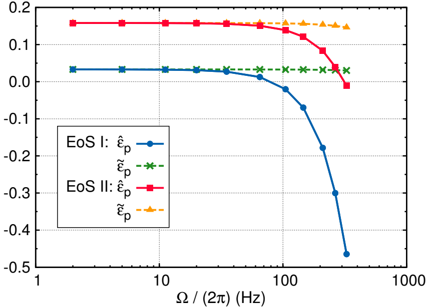
$$p_\varphi = \underbrace{\Gamma_{\mathrm{n}} n_{\mathrm{n}} p^{\mathrm{n}}_\varphi}_{j^{\mathrm{n}}_\varphi} + \underbrace{\Gamma_{\mathrm{p}} n_{\mathrm{p}} p^{\mathrm{p}}_\varphi}_{j^{\mathrm{p}}_\varphi}$$

$$J_X = \int_{\Sigma_t} j^X_\varphi \ A^2 B r^2 \sin\theta \, \mathrm{d}r \, \mathrm{d}\theta \, \mathrm{d}\varphi$$

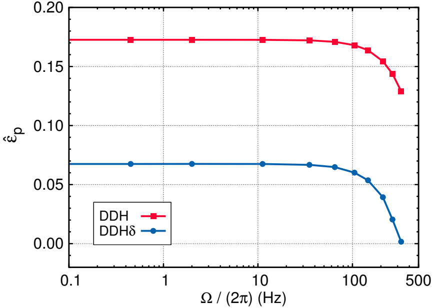
Fluid couplings



Influence of Ω on the couplings



Newtonian gravity



general relativity

Where does the vortex unpinning take place?

Glitches have been generally thought to originate from the crust, because:

- the core superfluid was expected to be strongly coupled to the crust Alpar et al., *ApJ*, 1984
- the analysis of glitch data suggested that the superfluid represents a few percent of the total angular momentum of the star Link et al., *PRL*, 1999

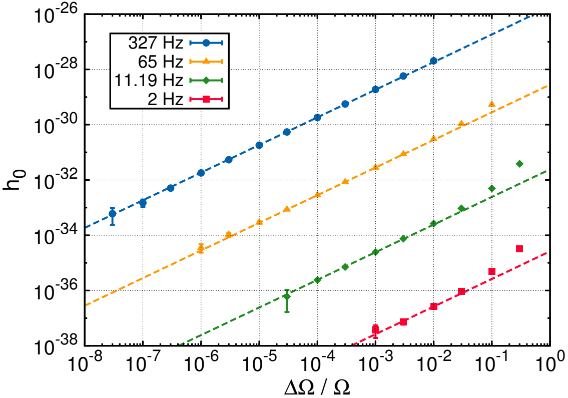
However, this scenario has been recently **challenged**:

- considering entrainment effects, the crust does not carry enough angular momentum Andersson et al., *PRL*, 2012 & Chamel, *PRL*, 2013
- a huge glitch has been observed in PSR 2334+61 Alpar, *AIP Conf.Proc.*, 2011
- the core superfluid could be decoupled from the rest of the star, if vortices are pinned to flux tubes Gügercinoglu & Alpar, *ApJ*, 2014

The core superfluid plays a more important role than previously thought.

Gravitational wave amplitude

$$h_+(t) = -\frac{3}{2}\sin^2 i\,\frac{G}{Dc^4}\,\ddot{Q} = h_0 \sin^2 i \; e^{-\frac{t}{\tau_r}}$$



- $D = 1$ kpc,
- $\bar{\mathcal{B}} = 10^{-3}$,
- $M_G = 1.4\ \mathrm{M}_\odot$,
- DDH EoS.

$$h_0 \simeq 1.0 \times 10^{-37}\left(\frac{D}{1\ \mathrm{kpc}}\right)^{-1}\left(\frac{\bar{\mathcal{B}}}{10^{-3}}\right)^{2}\left(\frac{\Omega}{10^{2}\ \mathrm{rad.s}^{-1}}\right)^{4}\left(\frac{\Delta\Omega/\Omega}{10^{-6}}\right)$$

Entrainment effects

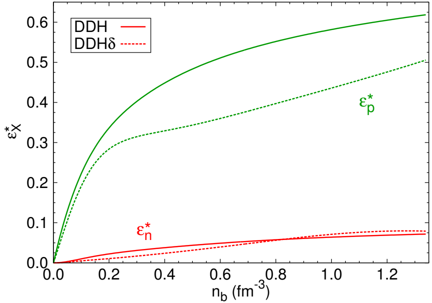
Dynamical effective mass:

$$\boxed{{}^3\vec{\boldsymbol{p}}_X = m_X^* \; {}^3\vec{\boldsymbol{u}}_X}$$

→ in the *rest frame* of the second fluid.

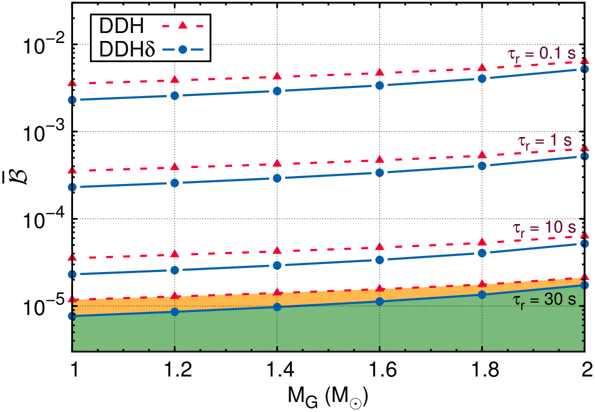
Zero-velocity frame:

$$m_X^* = \overset{\text{special relativity}}{\boxed{\mu^X}} \times \left(1 - \underset{\textbf{entrainment}}{\boxed{\varepsilon_X^*}}\right)$$



The Vela pulsar

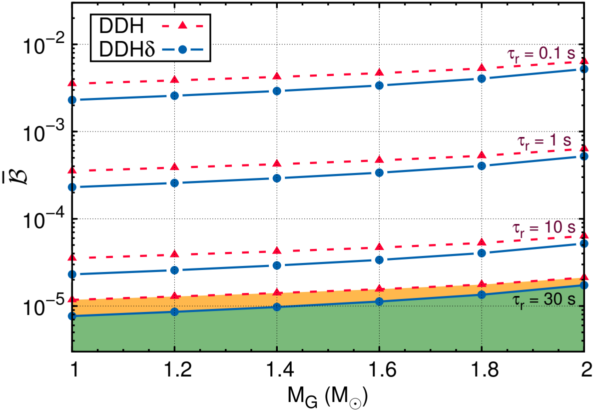
$$\Delta\Omega/\Omega = 10^{-6},\ \Omega_{\rm n}^f = \Omega_{\rm p}^f = 2\pi \times 11.19\ \text{Hz}$$



- $\bar{\mathcal{B}} \nearrow \Longrightarrow \tau_{\rm r} \searrow$

The Vela pulsar

$$\Delta\Omega/\Omega = 10^{-6},\ \Omega_n^f = \Omega_p^f = 2\pi \times 11.19\ \text{Hz}$$



- $\bar{\mathcal{B}} \nearrow \implies \tau_r \searrow$
- *Constraint on* $\bar{\mathcal{B}}$:

  $\tau_r < 30$ s $\Rightarrow \bar{\mathcal{B}} > 10^{-5}$

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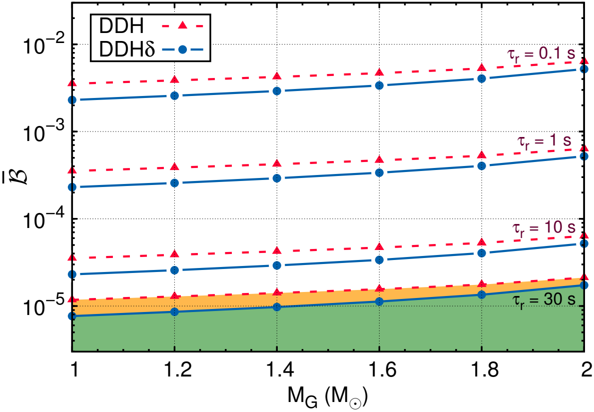

- $\bar{\mathcal{B}} \nearrow \implies \tau_{\rm r} \searrow$
- *Constraint on* $\bar{\mathcal{B}}$:

  $\tau_{\rm r} < 30\ \text{s} \Rightarrow \bar{\mathcal{B}} > 10^{-5}$
- $\bar{\mathcal{B}} < 0.5 \rightsquigarrow \tau_{\rm r} > 0.6\ \text{ms}$

↪ the glitch event is a **quasi-stationary** process

In Newtonian gravity

In the Newtonian limit ($\mu^X \simeq m_X$, $B = N = 1$, $\omega = 0$), we get:

$$J_X = \int_{\Sigma_t} \rho_X \left(1 - \varepsilon_X\right) \Omega_X r^2 \sin^2\theta \ \mathrm{d}^3 V + \int_{\Sigma_t} \rho_X \varepsilon_X \left(\Omega_Y - \Omega_X\right) r^2 \sin^2\theta \ \mathrm{d}^3 V$$

Defining the moment of inertia I_X and the mean entrainment parameter $\tilde{\varepsilon}_X$ as

$$I_X \equiv \int_{\Sigma_t} \rho_X r^2 \sin^2\theta \, \mathrm{d}^3 V \qquad I_X \tilde{\varepsilon}_X \equiv \int_{\Sigma_t} \rho_X r^2 \sin^2\theta \, \varepsilon_X \ \mathrm{d}^3 V$$

$$J_X = I_X \left(1 - \tilde{\varepsilon}_X\right) \Omega_X + I_X \tilde{\varepsilon}_X \Omega_Y$$

$$\rightsquigarrow \quad I_{XY} = \frac{\partial J_X}{\partial \Omega_Y} = I_X \tilde{\varepsilon}_X$$

see, e.g., Sidery, Passamonti & Andersson, *MNRAS*, 2010.

In General Relativity

Let's go back to

$$\begin{aligned} J_X &\simeq \int_{\Sigma_t} n_X \mu^X \tfrac{B}{N} \left(\Omega_X - \omega\right) r^2 \sin^2\theta \;\, \mathrm{d}^3 V \\ &+ \int_{\Sigma_t} n_X \mu^X \varepsilon_X \tfrac{B}{N} \left(\Omega_Y - \Omega_X\right) r^2 \sin^2\theta \;\, \mathrm{d}^3 V \end{aligned}$$

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where $i_X \equiv n_X \mu^X \frac{B}{N} r^2 \sin^2\theta$ ($\rightarrow$ in **Newt. grav.**, $i_X = \rho_x r^2 \sin^2\theta$).

In General Relativity

Let's go back to

$$J_X \simeq \int_{\Sigma_t} i_X \left(\Omega_X - \omega\right) \, \mathrm{d}^3 V + \int_{\Sigma_t} i_X \varepsilon_X \left(\Omega_Y - \Omega_X\right) \, \mathrm{d}^3 V$$

where $i_X \equiv n_X \mu^X \frac{B}{N} r^2 \sin^2 \theta$ (→ in **Newt. grav.**, $i_X = \rho_x r^2 \sin^2 \theta$).

The "moment of inertia" I_X and the mean entrainment parameter $\tilde{\varepsilon}_X$ are now given by

$$I_X \equiv \int_{\Sigma_t} i_X \, \mathrm{d}^3 V \qquad\qquad I_X \tilde{\varepsilon}_X \equiv \int_{\Sigma_t} i_X \varepsilon_X \, \mathrm{d}^3 V$$

In General Relativity

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The additional term associated with **frame-dragging effect** can be expressed as

$$\int_{\Sigma_t} i_X \; \omega \; \mathrm{d}^3V \equiv I_X \left(\varepsilon^{LT}_{X\to X} \Omega_X + \varepsilon^{LT}_{Y\to X} \Omega_Y\right)$$

In General Relativity

Let's go back to

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$$J_X = I_X \left(1 - \varepsilon^{LT}_{X\to X} - \tilde{\varepsilon}_X\right)\Omega_X + I_X \left(\tilde{\varepsilon}_X - \varepsilon^{LT}_{Y\to X}\right)\Omega_Y$$

Frame-dragging contribution

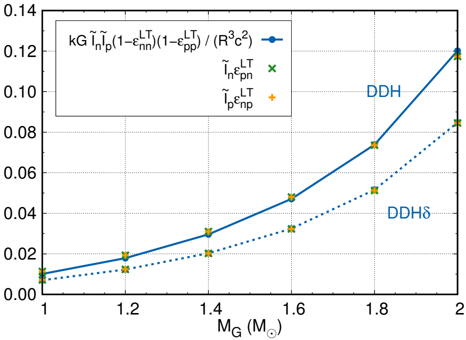
$$\rightsquigarrow \; I_{XY} = \frac{\partial J_X}{\partial \Omega_Y} = I_X \left(\tilde{\varepsilon}_X - \varepsilon^{LT}_{Y \to X} \right)$$

⇢ additional coupling arising from **frame-dragging effects**.

Already pointed out by B. Carter in 1975. By dimensional considerations:

$$\begin{aligned} I_{\mathrm{n}} \varepsilon^{LT}_{\mathrm{p\,n}} &= I_{\mathrm{p}} \varepsilon^{LT}_{\mathrm{n\,p}} \\ &\simeq \kappa G I_{\mathrm{n}} I_{\mathrm{p}} / (R^3 c^2) \\ &\times \left(1 - \varepsilon^{LT}_{\mathrm{n\,n}}\right)\left(1 - \varepsilon^{LT}_{\mathrm{p\,p}}\right) \end{aligned}$$

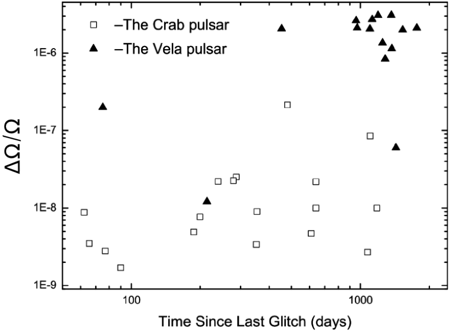
Carter, *Annals of Physics*, 1975



$\kappa_{num} \simeq 3.8$

Distinct glitching behaviors

Wang et al., *Ap&SS*, 2012



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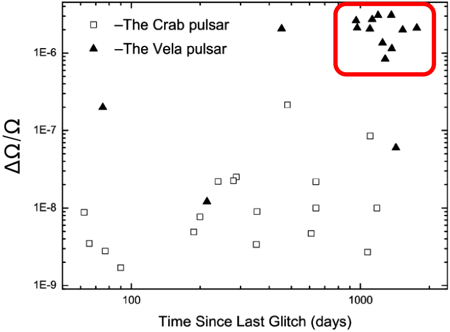
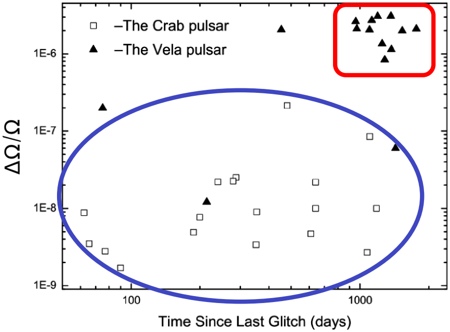


quasi-periodic giant glitches with a very narrow spread in size

Distinct glitching behaviors

Wang et al., *Ap&SS*, 2012

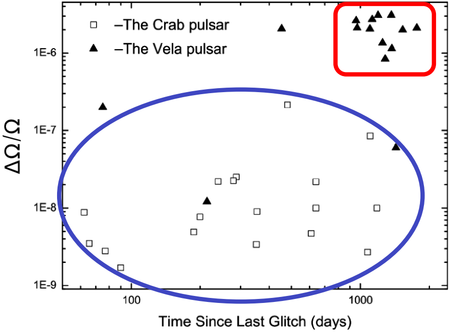


quasi-periodic giant glitches with a very narrow spread in size

glitches of various sizes at random intervals of time

Distinct glitching behaviors

Wang et al., *Ap&SS*, 2012



quasi-periodic giant glitches with a very narrow spread in size

glitches of various sizes at random intervals of time

Different models of glitches Haskell & Melatos, *IJMPD*, 2015

- Rearrangement of the moment of inertia --→ crustquakes,
- Angular momentum transfer between *two* fluids --→ **superfluidity**.