Effect of the Magnetic Field on the Dense Matter EoS

Veronica Dexheimer

Main Reference:
M.Strickland, VD, D.Menezes Phys.Rev.D 2012

KENT STATE
UNIVERSITY

1) Motivation

- neutron star central densities: a few times ρ_o
- core composition: mainly neutrons, but also protons, leptons, hyperons and/or quarks
- magnetic fields: so far up to 10^{15} G on surface and 10^{16} G inside
- anomalous magnetic moment of baryons: due to quark composition and present even for neutrons
- temperature: about 0 MeV for old neutron stars, up to 30 MeV for proto-neutron stars, up to 80 MeV for neutron star mergers

Relativistic fermi gas under strong magnetic fields with AMM corrections at finite temperature !

2) Relativistic fermi gas under strong magnetic fields with AAA corrections at finite temperature

a) Modified Dirac Lagrangian density for fermions with spin ½ and charge q (assuming Lorentz-Heaviside natural units $\hbar=c=\mu_o=1$)

strength of fermion coupling to $T^{\mu\nu}$

$$\mathcal{L} = \underline{\bar{\psi}}(\underline{i\partial\!\!\!/} - \underline{qA\!\!\!/} - \underline{m} + \underline{\frac{1}{2}\kappa\sigma^{\mu\nu}F_{\mu\nu}})\underline{\psi} - \underline{\frac{1}{4}F^{\mu\nu}F_{\mu\nu}},$$

kinetic term fermions

interaction term fermions/ electromag.

mass term fermions

AMM term

kinetic term electromag.

Dirac matrix

vector potential in Landau gauge that gives B=∇xA in z-direction

with $a\!\!\!/ = \gamma^\mu a_\mu$, $A^\mu = B\,(0, -y, 0, 0)$, $\sigma^{\mu\nu} = i[\gamma^\mu, \gamma^\nu]/2$
and $F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu$

b) Modified Dirac equation of motion for fermions

Using the Euler-Lagrange equation $\frac{\partial \mathcal{L}}{\partial \bar{\psi}} - \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \bar{\psi})} \right) = 0$

on $\mathcal{L} = \bar{\psi}(i\partial\!\!\!/ - qA\!\!\!/ - m + \frac{1}{2}\kappa\sigma^{\mu\nu}F_{\mu\nu})\psi \; - \frac{1}{4}F^{\mu\nu}F_{\mu\nu},$

we obta $(i\partial\!\!\!/ - qA\!\!\!/ - m + \frac{1}{2}\kappa\sigma^{\mu\nu}F_{\mu\nu})\psi = 0$

and the adjoint $\bar{\psi}(i\partial\!\!\!/ - qA\!\!\!/ - m + \frac{1}{2}\kappa\sigma^{\mu\nu}F_{\mu\nu}) - i\partial_\mu\bar{\psi}\gamma^\mu = 0$

$$i\partial_\mu\bar{\psi}\gamma^\mu + \bar{\psi}(qA\!\!\!/ + m - \tfrac{1}{2}\kappa\sigma^{\mu\nu}F_{\mu\nu}) = 0$$

Using $A^\mu = B\,(0, -y, 0, 0)$ and $\sigma^{\mu\nu} \; = \; i[\gamma^\mu, \; \gamma^\nu]/2$

in $F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu$, we get $F^{\mu\nu} = B(\delta^{\mu x}\delta^{\nu y} - \delta^{\nu x}\delta^{\mu y})$ and

$$\frac{1}{2}\kappa\sigma^{\mu\nu}F_{\mu\nu} \underset{\uparrow}{=} i\kappa B\gamma^x\gamma^y = \kappa B \begin{pmatrix} \sigma_3 & 0 \\ 0 & \sigma_3 \end{pmatrix} \equiv \kappa B \mathcal{S}_3$$

$1/4\ ikB(\gamma^x\gamma^y - \gamma^y\gamma^x - \gamma^y\gamma^x + \gamma^x\gamma^y)$

And the modified Dirac equation of motion

$$(i\partial\!\!\!/ - qA\!\!\!/ - m + \tfrac{1}{2}\kappa\sigma^{\mu\nu}F_{\mu\nu})\psi = 0$$

with separated temporal and spacial components

$$\partial_\mu \equiv \frac{\partial}{\partial x^\mu} = \left(\frac{\partial}{\partial t}, \boldsymbol{\nabla}\right), \quad A_\mu = (0, -\mathbf{A})$$

becomes in Hamiltonian form

$$(\boldsymbol{\alpha}\cdot\boldsymbol{\pi} + \underset{\overset{\uparrow}{\beta}}{\gamma^0} m - \kappa B\gamma^0\mathcal{S}_3)\Psi = E\Psi$$

with $\boldsymbol{\alpha} \equiv \gamma^0\boldsymbol{\gamma}$ and $\boldsymbol{\pi} \equiv -i\boldsymbol{\nabla} - q\mathbf{A}$

c) Energy-momentum tensor

$\eta_{\mu\nu} = \mathrm{diag}(1,-1,-1,-1)$

$$T^{\mu\nu} = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \psi)}\frac{\partial \psi}{\partial x_\nu} + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \bar{\psi})}\frac{\partial \bar{\psi}}{\partial x_\nu} + \frac{\partial \mathcal{L}}{\partial(\partial_\mu A_\alpha)}\frac{\partial A_\alpha}{\partial x_\nu} - \eta^{\mu\nu}\mathcal{L}$$

with $\mathcal{L} = \bar{\psi}(i\not{\partial} - \xcancel{q\not{A} -} m + \frac{1}{2}\kappa\sigma^{\mu\nu}F_{\mu\nu})\psi - \frac{1}{4}F^{\mu\nu}F_{\mu\nu},$

gives $T^{\mu\nu} = i\bar{\psi}\gamma^\mu\partial^\nu\psi - F^{\mu\alpha}F^\nu_\alpha + \kappa\sigma^{\mu\alpha}F^\nu_\alpha \ \psi + \eta^{\mu\nu}\frac{1}{4}F^{\alpha\beta}F_{\alpha\beta}$

where we used $X^\lambda = \eta^{\lambda\mu}X_\mu$, $F^{ij} = -F^{ji}$,
added the term $F^{\mu\nu}\partial_\alpha A^\nu$ (allowed by Noether's theorem) to remain gauge invariant and used the equation of motion for fermions

d) Electromagnetic energy-momentum tensor

$$T^{\mu\nu} = i\bar{\psi}\gamma^{\mu}\partial^{\nu}\psi - F^{\mu\alpha}F^{\nu}_{\alpha} + \kappa\sigma^{\mu\alpha}F^{\nu}_{\alpha}\ \ \psi + \eta^{\mu\nu}\frac{1}{4}F^{\alpha\beta}F_{\alpha\beta}$$

In Landau gauge $A^{\mu} = B\,(0, -y, 0, 0)$ so

$$F^{\mu\alpha}F^{\nu}_{\alpha} = (B\delta^{\mu}_{x}\delta^{\alpha}_{y} - B\delta^{\mu}_{y}\delta^{\alpha}_{x})(B\delta^{\nu}_{y}\delta^{x}_{\alpha} - B\delta^{\nu}_{x}\delta^{y}_{\alpha}) = -B^{2}(\delta^{\mu}_{x}\delta^{\nu}_{x} + \delta^{\mu}_{y}\delta^{\nu}_{y})$$

$$F^{\alpha\beta}F_{\alpha\beta} = (B\delta^{\alpha}_{y}\delta^{\beta}_{x} - B\delta^{\beta}_{y}\delta^{\alpha}_{x})(B\delta^{y}_{\alpha}\delta^{x}_{\beta} - B\delta^{y}_{\beta}\delta^{x}_{\alpha}) = 2B^{2}$$

giving diagonal terms $T^{\mu\nu}_{elec} = B^{2}(\delta^{\mu}_{x}\delta^{\nu}_{x} + \delta^{\mu}_{y}\delta^{\nu}_{y})) + \eta^{\mu\nu}\frac{1}{4}2B^{2}$

in 0-coordinate: $T^{00}_{elec} = B^{2}(0+0) + \frac{1}{2}B^{2} = \frac{1}{2}B^{2}$

in x-coordinate: $T^{xx}_{elec} = B^{2}(1+0) - \frac{1}{2}B^{2} = \frac{1}{2}B^{2}$

in y-coordinate: $T^{yy}_{elec} = B^{2}(0+1) - \frac{1}{2}B^{2} = \frac{1}{2}B^{2}$

in z-coordinate: $T^{zz}_{elec} = B^{2}(0+0) - \frac{1}{2}B^{2} = -\frac{1}{2}B^{2}$

$\}$ Asymmetric !

e) Fermion energy-momentum tensor

$$T^{\mu\nu} = i\bar{\psi}\gamma^{\mu}\partial^{\nu}\psi - F^{\mu\alpha}F^{\nu}_{\alpha} + \bar{\psi}\,\kappa\sigma^{\mu\alpha}F^{\nu}_{\alpha}\;\;\psi + \eta^{\mu\nu}\frac{1}{4}F^{\alpha\beta}F_{\alpha\beta}$$

giving diagonal terms

$$\left.\begin{array}{l} \text{in 0-coordinate: } \mathcal{T}^{00}_{fermi} = \bar{\psi}\,(i\gamma^{0}\;\partial^{0})\,\psi \\ \text{in x-coordinate: } \mathcal{T}^{xx}_{fermi} = \bar{\psi}\,(i\gamma^{x}\;\partial^{x} - \kappa B\sigma^{xy})\,\psi \\ \text{in y-coordinate: } \mathcal{T}^{yy}_{fermi} = \bar{\psi}\,(i\gamma^{y}\;\partial^{y} - \kappa B\sigma^{xy})\,\psi \\ \text{in z-coordinate: } \mathcal{T}^{zz}_{fermi} = \bar{\psi}\,(i\gamma^{z}\;\partial^{z})\,\psi \end{array}\right\}$$

Different Contributions !

where we disregarded the added term $\partial_{\alpha}A^{\nu}$ and used

$$\sigma^{\mu\alpha}\partial^{\nu}A_{\alpha} = B\sigma^{\mu\alpha}\delta^{\nu}_{y}\delta^{x}_{\alpha} = -B\sigma^{xy}$$

f) Solution of Modified Dirac equation for charged fermions

We are going to assume a static solution $\Psi(\mathbf{x}) = e^{ik_x x} e^{ik_z z} u_l^{(s)}(\underset{\uparrow}{y})$

due to choice of Landau gauge

with

$$u_l^{(s)}(y) = \begin{pmatrix} c_1 \phi_\nu(y) \\ c_2 \phi_{\nu-1}(y) \\ c_3 \phi_\nu(y) \\ c_4 \phi_{\nu-1}(y) \end{pmatrix}$$

where l=0, 1, 2, ...refers to quantum orbital numbers, s refers to the spin alignment s=±1, the constants c_i depend on the spin alignment and the function $\phi_n(\xi) = N_n e^{-\xi^2/2} H_n(\xi)$, which contains an Hermite polynomial, the new variable $\xi = \sqrt{|q|B}\left(y + \frac{k_x}{qB}\right)$ and the normalization constant $N_n = (qB)^{1/4}(\sqrt{\pi}2^n n!)^{-1/2}$ that insures $\int_{-\infty}^{\infty} dy\, \phi_n^2(y) = 1$

I) Landau levels $\nu = l + \frac{1}{2} - \frac{s}{2}\frac{q}{|q|}$ (new quantum number)

For positive charge and spin up: $\nu = l = 0, 1, 2, 3, ...$

For positive charge and spin down: $\nu = l + 1 = 1, 2, 3, ...$

For negative charge and spin up: $\nu = l = 0, 1, 2, 3, ...$

For negative charge and spin down: $\nu = l + 1 = 1, 2, 3, ...$

So, only positive charges with spin up and negative charges with spin down can possess the zeroth Landau level

All fermions possess levels larger than zero

Back to f)

Inserting $\Psi(\mathbf{x})$ in the modified Dirac equation and simplifying,

we obtain

$$\begin{pmatrix} m-\kappa B & 0 & k_z & k_\nu \\ 0 & m+\kappa B & k_\nu & -k_z \\ k_z & k_\nu & -m+\kappa B & 0 \\ k_\nu & -k_z & 0 & -m-\kappa B \end{pmatrix} \underbrace{\begin{pmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{pmatrix}}_{\chi^{(s)}} = E \underbrace{\begin{pmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{pmatrix}}_{\chi^{(s)}}$$

With $k_\nu \equiv \sqrt{2|q|B\nu}$

The energy eigenvalues $E_s = \pm\sqrt{k_z^2 + (\lambda - s\kappa B)^2}$

fermions

anti-fermions

$\lambda \equiv \sqrt{m^2 + k_\nu^2}$

are obtained from the determinant of the matrix

The positive energy eigenvectors are

$$\chi^{(s)} = \frac{1}{\sqrt{2\lambda\alpha_s\beta_s}} \begin{pmatrix} s\alpha_s\beta_s \\ -k_z k_\nu \\ s\beta_s k_z \\ \alpha_s k_\nu \end{pmatrix}$$

with $\alpha_s \equiv E_s - \kappa B + s\lambda$

$$\int_{-\infty}^{\infty} dy\, u_n^{(r)\dagger}(\mathbf{x})\, u_m^{(s)}(\mathbf{x}) = 2E_s \delta^{rs} \delta_{nm}$$

and $\beta_s \equiv \lambda + sm$

so we can rewrite the spinors

$$u_l^{(s)} = \frac{1}{\sqrt{2\lambda\alpha_s\beta_s}} \begin{pmatrix} s\alpha_s\beta_s\varphi_\nu(y) \\ -k_z k_\nu \varphi_{\nu-1}(y) \\ s\beta_s k_z \varphi_\nu(y) \\ \alpha_s k_\nu \varphi_{\nu-1}(y) \end{pmatrix}$$

and define the quantum state for positive energy states

$$\psi(x) = \sum_{s=\pm 1} \sum_{l,\mathbf{k}} \frac{1}{L} \frac{1}{\sqrt{2E}} b_s(\mathbf{k}) u_l^{(s)}(\mathbf{k}) e^{-i\kappa_\mu x^\mu}$$

fermion annihilation operator

$$\psi^\dagger(x) = \sum_{s=\pm 1} \sum_{l,\mathbf{k}} \frac{1}{L} \frac{1}{\sqrt{2E}} b_s^\dagger(\mathbf{k}) \bar{u}_l^{(s)}(\mathbf{k}) e^{i\kappa_\mu x^\mu}$$

fermion creation operator

with $\kappa = (E_k, k_x, 0, k_z)$ and $\{b_s(\mathbf{k}), b_{s'}^\dagger(\mathbf{k}')\} = (2\pi)\delta_{ss'}\delta_{kk'}$

where $V = L^3$ and in the thermodynamical limit

$$\frac{1}{L}\sum_{l,\mathbf{k}} \to \frac{|q|B}{(2\pi)^2}\sum_{l}\int_{-\infty}^{\infty} dk_z$$

II) Number density

For fermions, the number density is defined as

$$n = \langle N \rangle = \frac{1}{L} \sum_{\pm s=1} \sum_{\mathbf{k}} \langle b_s^\dagger(\mathbf{k}) b_s(\mathbf{k}) \rangle \quad \text{with} \quad \frac{1}{L} \sum_{l,\mathbf{k}} \to \frac{|q|B}{(2\pi)^2} \sum_{l} \int_{-\infty}^{\infty} dk_z$$

fermion creation/annihilation operator

Using $\langle b_s^\dagger(\mathbf{k}) b_s(\mathbf{k}) \rangle = \dfrac{1}{e^{\beta(E \ \ -\mu)} + 1} = f_+(E \ \ , T, \mu)$

and $\{b_r(\mathbf{p}), b_s^\dagger(\mathbf{k})\} = (2\pi)\delta_{rs}\delta_{nm}\delta(p_z - k_z)$

we obtain $$n \ = \langle N \rangle = \frac{|q|B}{2\pi} \sum_{s=\pm1} \sum_{l} \int_{-\infty}^{\infty} \frac{dk_z}{2\pi} f_+(E \ \ , T, \mu)$$

For anti-fermions we simply replace the distribution function f_+

by $f_-(E, T, \mu) = \dfrac{1}{e^{\beta(E+\mu)} + 1}$

III) Number density at zero temperature

The fermion distribution is replaced by a Heaviside theta function

$$n = \frac{|q|B}{(2\pi)^2} \sum_{s=\pm 1} \sum_{l=0}^{\nu \leq \nu_{\max}} \int_{-\infty}^{\infty} dk_z \, \Theta(\mu - E)$$

$$= \frac{|q|B}{2\pi^2} \sum_{s=\pm 1} \sum_{l=0}^{\nu \leq \nu_{\max}} k_{z,F}(\nu)$$

where the integral goes until the Fermi level and there is a maximum Landau level defined as

$$\nu_{\max} = \left\lfloor \frac{(\mu + s\kappa B)^2 - m^2}{2|q|B} \right\rfloor$$

in order to have real energy states when the momentum is zero

$$E = \sqrt{k_z^2 + ((m^2 + 2\nu|q|B)^{1/2} - s\kappa B)^2}$$

IV) Energy density

For fermions, the energy density is defined as

$\epsilon \equiv \langle \mathcal{T}^{00} \rangle = \langle \mathcal{H} \rangle = \langle i\psi^\dagger \partial_t \psi \rangle$ with $\int_{-\infty}^{\infty} dy\, u_n^{(r)\dagger}(\mathbf{x})\, u_m^{(s)}(\mathbf{x}) = 2E_s \delta^{rs} \delta_{nm}$

and $$\psi(x) = \sum_{s=\pm 1} \sum_{l,\mathbf{k}} \frac{1}{L} \frac{1}{\sqrt{2E}} b_s(\mathbf{k}) u_l^{(s)}(\mathbf{k}) e^{-i\kappa_\mu x^\mu}$$

$$\kappa = (E\ , k_x, 0, k_z)$$

$$\bar{\psi} = \psi^\dagger \gamma_0$$

$$\psi^\dagger(x) = \sum_{s=1,2} \sum_{l,\mathbf{k}} \frac{1}{L} \frac{1}{\sqrt{2E}} b_s^\dagger(\mathbf{k}) \bar{u}_l^{(s)}(\mathbf{k}) e^{i\kappa_\mu x^\mu}$$

same index s

Giving $$\epsilon \equiv \langle H \rangle = \frac{|q|B}{2\pi} \sum_{s=\pm 1} \sum_{l} \int_{-\infty}^{\infty} \frac{dk_z}{2\pi} E \langle b_s^\dagger(\mathbf{k}) b_s(\mathbf{k}) \rangle$$

$$= \frac{|q|B}{2\pi} \sum_{s=\pm 1} \sum_{l} \int_{-\infty}^{\infty} \frac{dk_z}{2\pi} E\ f_+(E\ , T, \mu)$$

For anti-fermions we simply replace the distribution function

f_+ by f_-

V) Energy density at zero temperature

The er distribution is replaced by a Heaviside theta function

$$\epsilon = \frac{|q|B}{2\pi^2} \sum_{s=\pm 1} \sum_{l=0}^{\nu \le \nu_{\max}} \int_0^{k_{z,F}} dk_z \sqrt{k_z^2 + \bar{m}^2(\nu)}$$

$$= \frac{|q|B}{4\pi^2} \sum_{s=\pm 1} \sum_{l=0}^{\nu \le \nu_{\max}} \left[\mu\, k_{z,F}(\nu) + \bar{m}^2(\nu) \log\left(\frac{\mu + k_{z,F}(\nu)}{\bar{m}(\nu)} \right) \right]$$

with $\bar{m} \equiv \sqrt{m^2 + 2|q|B\nu} - s\kappa B$ being the modified mass from

$$E = \sqrt{k_z^2 + ((m^2 + 2\nu|q|B)^{1/2} - s\kappa B)^2}$$

and $\nu_{\max} = \left\lfloor \frac{(\mu + s\kappa B)^2 - m^2}{2|q|B} \right\rfloor$

VI) Pressure parallel to the field

For fermions, it is defined as

$P_{\|} = \langle \mathcal{T}^{zz} \rangle = \langle \bar{\psi} \gamma^z \partial^z \psi \rangle$ with

$$\psi(x) = \sum_{s=\pm 1} \sum_{l,\mathbf{k}} \frac{1}{L} \frac{1}{\sqrt{2E}} b_s(\mathbf{k}) u_l^{(s)}(\mathbf{k}) e^{-i\kappa_\mu x^\mu}$$

$$\kappa = (E\ \ , k_x, 0, k_z)$$

$$\psi^\dagger(x) = \sum_{s=1,2} \sum_{l,\mathbf{k}} \frac{1}{L} \frac{1}{\sqrt{2E}} b_s^\dagger(\mathbf{k}) \bar{u}_l^{(s)}(\mathbf{k}) e^{i\kappa_\mu x^\mu}$$

$$\bar{\psi} = \psi^\dagger \gamma_0$$

giving $P_{\|} = -\frac{1}{2} \frac{|q|B}{2\pi} \sum_{s=\pm 1} \sum_{l} \int_{-\infty}^{\infty} \frac{dk_z}{2\pi} \frac{k^z}{E} \langle b_s^\dagger(\mathbf{k}) b_s(\mathbf{k}) \rangle \int_{-\infty}^{\infty} dy \left[u^{(s)\dagger}(\mathbf{k}) \gamma^0 \gamma^z u^{(s)}(\mathbf{k}) \right]$

with $\int_{-\infty}^{\infty} dy\, u^{(s)\dagger}(\mathbf{k}) \gamma^0 \gamma^z u^{(s)}(\mathbf{k}) = -2k^z$ so

$$P_{\|} = \frac{|q|B}{2\pi} \sum_{s=\pm 1} \sum_{l} \int_{-\infty}^{\infty} \frac{dk_z}{2\pi} \frac{k^z k^z}{E} f_+(E\ \ , T, \mu)$$

For anti-fermions we simply replace the distribution function f_+ by f_-

VII) Pressure parallel to the field at zero temperature

The fermion distribution is replaced by a Heaviside theta function

$$P_{\parallel} = \frac{|q|B}{2\pi^2} \sum_{s=\pm 1} \sum_{l=0}^{\nu \leq \nu_{\max}} \int_0^{k_{z,F}} dk_z \, \frac{k_z^2}{\sqrt{k_z^2 + \bar{m}^2(\nu)}}$$

$$= \frac{|q|B}{4\pi^2} \sum_{s=\pm 1} \sum_{l=0}^{\nu \leq \nu_{\max}} \left[\mu \, k_{z,F}(\nu) - \bar{m}^2(\nu) \log\left(\frac{\mu + k_{z,F}(\nu)}{\bar{m}(\nu)} \right) \right]$$

with $\bar{m} \equiv \sqrt{m^2 + 2|q|B\nu} - s\kappa B$ and $\nu_{\max} = \left\lfloor \frac{(\mu + s\kappa B)^2 - m^2}{2|q|B} \right\rfloor$

VIII) Thermodynamic consistency at T=0

We can calculate $\epsilon + P_{||} = \mu n$?

using

$$\epsilon = \frac{|q|B}{4\pi^2} \sum_{s=\pm 1} \sum_{l=0}^{\nu \leq \nu_{\max}} \left[\mu\, k_{z,F}(\nu) + \bar{m}^2(\nu) \log\left(\frac{\mu + k_{z,F}(\nu)}{\bar{m}(\nu)} \right) \right]$$

$$P_{||} = \frac{|q|B}{4\pi^2} \sum_{s=\pm 1} \sum_{l=0}^{\nu \leq \nu_{\max}} \left[\mu\, k_{z,F}(\nu) - \bar{m}^2(\nu) \log\left(\frac{\mu + k_{z,F}(\nu)}{\bar{m}(\nu)} \right) \right]$$

which results in

$$\mu \times \left(\frac{|q|B}{2\pi^2} \sum_{s=\pm 1} \sum_{l=0}^{\nu \leq \nu_{\max}} k_{z,F}(\nu) \right) = \mu n$$ ✓

so $\Omega = \epsilon - \mu n = -P_{||}$

VIX) Pressure perpendicular to the field

For fermions, it is defined as

$P_\perp \equiv \langle \mathcal{T}^{yy}\rangle = \langle \mathcal{T}^{xx}\rangle = \langle\bar{\psi}\,(i\gamma^y\,\partial^y - \kappa B\sigma^{xy})\,\psi\rangle$ with

$$\psi(x) = \sum_{s=\pm 1}\sum_{l,\mathbf{k}}\frac{1}{L}\frac{1}{\sqrt{2E}}b_s(\mathbf{k})u_l^{(s)}(\mathbf{k})e^{-i\kappa_\mu x^\mu}$$

$$\kappa = (E\ , k_x, 0, k_z)$$

$$\psi^\dagger(x) = \sum_{s=1,2}\sum_{l,\mathbf{k}}\frac{1}{L}\frac{1}{\sqrt{2E}}b_s^\dagger(\mathbf{k})\bar{u}_l^{(s)}(\mathbf{k})e^{i\kappa_\mu x^\mu}$$

$$\bar{\psi} = \psi^\dagger\gamma_0$$

giving $P_\perp = \frac{|q|B}{2\pi}\sum_{s=\pm 1}\sum_{l}\int_{-\infty}^{\infty}\frac{dk_z}{2\pi}\frac{1}{E}\langle b_s^\dagger(\mathbf{k})b_s(\mathbf{k})\rangle$

$$\times\left\{ i\int_{-\infty}^{\infty} dy\ u^{(s)\dagger}(\mathbf{k})\gamma^0\gamma^y\partial^y u^{(s)}(\mathbf{k}) - \kappa B\int_{-\infty}^{\infty} dy\, u^{(s)\dagger}(\mathbf{k})\gamma^0\sigma^{xy}u^{(s)}(\mathbf{k})\right\}$$

with $\int_{-\infty}^{\infty} dy\, u^{(s)\dagger}(\mathbf{k})\gamma^0\sigma^{xy}u^{(s)}(\mathbf{k}) = 2s(\lambda - s\kappa B)$

$$P_{\perp} = \frac{|q|B^2}{2\pi^2} \sum_{s=\pm 1} \sum_{l} \int_{-\infty}^{\infty} dk_z \frac{1}{E} f_{+}(E \ \ , T, \mu) \left[\frac{|q|\nu \bar{m}(\nu)}{\sqrt{m^2 + 2\nu|q|B}} - s\kappa \bar{m}(\nu) \right]$$

with $\bar{m} \equiv \sqrt{m^2 + 2|q|B\nu} - s\kappa B$ being the modified mass from

$$E = \sqrt{k_z^2 + ((m^2 + 2\nu|q|B)^{1/2} - s\kappa B)^2}$$

For anti-fermions we simply replace the distribution function f_+ by f_-

X) Pressure perpendicular to the field at zero temperature

The fermion distribution is replaced by a Heaviside theta function

$$P_\perp = \frac{|q|B^2}{2\pi^2}\sum_{s=\pm1}\sum_{l=0}^{\nu\le\nu_{\max}}\left[\frac{|q|\nu\bar{m}(\nu)}{\sqrt{m^2+2\nu|q|B}} - s\kappa\bar{m}(\nu)\right]\int_0^{k_{z,F}} dk_z\,\frac{1}{\sqrt{k_z^2+\bar{m}^2(\nu)}}$$

$$= \frac{|q|B^2}{2\pi^2}\sum_{s=\pm1}\sum_{l=0}^{\nu\le\nu_{\max}}\left[\frac{|q|\nu\bar{m}(\nu)}{\sqrt{m^2+2\nu|q|B}} - s\kappa\bar{m}(\nu)\right]\log\left(\frac{\mu+k_{z,F}(\nu)}{\bar{m}(\nu)}\right)$$

with $\bar{m} \equiv \sqrt{m^2+2|q|B\nu} - s\kappa B$ and $\nu_{\max} = \left\lfloor \frac{(\mu+s\kappa B)^2-m^2}{2|q|B} \right\rfloor$

XI) Magnetization

It is defined as $M \equiv -\partial\Omega/\partial B = \partial P_{\|}/\partial B$

For fermions, it becomes

$$M = \frac{P_{\|}}{B} + \frac{|q|B}{2\pi^2}\sum_{s=\pm 1}\sum_{l}\int_{-\infty}^{\infty} dk_z \frac{1}{E} f_+(E \ ,T,\mu)$$
$$\times \ \bar{m}(\nu)\left[s\kappa - \frac{|q|\nu}{\sqrt{m^2+2\nu|q|B}}\right]$$

with $\bar{m} \equiv \sqrt{m^2+2|q|B\nu} - s\kappa B$

which can be written as $M = \frac{P_{\|}}{B} - \frac{P_{\perp}}{B}$

For anti-fermions we simply replace the distribution function f_+ by f_- and we can generalize $P_{\perp,\pm} = P_{\|,\pm} - M_{\pm}B$

XII) Magnetization at zero temperature

The fermion distribution is replaced by a Heaviside theta function

$$M = \frac{\partial P_{\parallel}}{\partial B} = \frac{P_{\parallel}}{B} + \frac{|q|B}{2\pi^2} \sum_{s=\pm 1} \sum_{l=0}^{\nu \leq \nu_{\max}}$$

$$\times \left[s\kappa \bar{m}(\nu) - \frac{|q|\nu \bar{m}(\nu)}{\sqrt{m^2 + 2\nu |q| B}} \right] \log \left(\frac{\mu + k_{z,F}(\nu)}{\bar{m}(\nu)} \right)$$

with $\bar{m} \equiv \sqrt{m^2 + 2|q|B\nu} - s\kappa B$

and again

$$M = \frac{P_{\parallel}}{B} - \frac{P_{\perp}}{B}$$

g) Solution of Modified Dirac equation for uncharged fermions

In this case the momenta of fermions in the direction perpendicular to the field are not quantized so we are going to assume a static solution $\Psi(\mathbf{x}) = e^{i\mathbf{k}\cdot\mathbf{x}}u$ for the modified Dirac equation with $u = (c_1 \; c_2 \; c_3 \; c_4)^{\mathrm{T}}$

Inserting $\Psi(\mathbf{x})$ in the modified Dirac equation and simplifying, we obtain

$$\begin{pmatrix} m-\kappa B & 0 & k_z & k_- \\ 0 & m+\kappa B & k_+ & -k_z \\ k_z & k_- & -m+\kappa B & 0 \\ k_+ & -k_z & 0 & -m-\kappa B \end{pmatrix} u = Eu$$

with $k_\pm \equiv k_x \pm ik_y$

The same energy eigenvalues $E_s = \pm\sqrt{k_z^2 + (\lambda - s\kappa B)^2}$

fermions

anti-fermions

are obtained from the determinant of the matrix but now

$\lambda \equiv \sqrt{m^2 + k_\perp^2}$ with $k_\perp^2 = k_x^2 + k_y^2$

We can rewrite the spinor

$$u^{(s)} = \frac{1}{\sqrt{2\lambda\alpha_s\beta_s}} \begin{pmatrix} s\alpha_s\beta_s \\ -k_z k_+ \\ s\beta_s k_z \\ \alpha_s k_+ \end{pmatrix}$$

$$u^{(r)\dagger} u^{(s)} = 2E_s\delta^{rs}$$

with, as before,

$\alpha_s \equiv E_s - \kappa B + s\lambda$ and $\beta_s \equiv \lambda + sm$

and we can define the quantum state for positive energy states

$$\psi(x) = \sum_{s=\pm1} \frac{1}{(2\pi)^3} \int d^3k \frac{1}{\sqrt{2E}} b_s(\mathbf{k}) u^{(s)}(\mathbf{k}) e^{-ik_\mu x^\mu}$$

fermion annihilation operator

not trivial!

I) Change of variables

$$k_x = \sqrt{\lambda^2 - m^2}\cos\phi\,, \qquad \lambda \equiv \sqrt{m^2 + k_\perp^2}$$

$$k_y = \sqrt{\lambda^2 - m^2}\sin\phi\,,$$

$$k_z = \sqrt{E^2 - (\lambda - s\kappa B)^2}$$

with the Jacobian for the transformation

$$d^3k = \frac{E\lambda}{\sqrt{E^2 - (\lambda - s\kappa B)^2}}\, dE\, d\lambda\, d\phi$$

and new limits

$$\lambda \geq m$$

$$\lambda \leq E + s\kappa B$$

$$E \geq m - s\kappa B$$

II) Number density

For fermions, the number density is becomes

$$n = \frac{1}{2\pi^2}\sum_{s=\pm 1}\int_{m-s\kappa B}^{\infty} dE\, E\, f_+(E,T,\mu)\int_{m}^{E+s\kappa B} d\lambda \frac{\lambda}{\sqrt{E^2-(\lambda-s\kappa B)^2}}$$

$$= \frac{1}{2\pi^2}\sum_{s=\pm 1}\int_{m-s\kappa B}^{\infty} dE\, E\, f_+(E,T,\mu)\left[\hat{k} + s\kappa B\left(\arctan\left(\frac{s\kappa B - m}{\hat{k}}\right)+\frac{\pi}{2}\right)\right]$$

with $\hat{k} \equiv \sqrt{E^2-(m-s\kappa B)^2}$

For anti-fermions we replace the distribution function f_+ by f_-

At zero temperature

$$n = \frac{1}{4\pi^2}\sum_{s=\pm 1}\left[\frac{k_F}{3}\left(2k_F^2 - 3s\kappa B\hat{m}\right) - s\kappa B\mu^2\left(\arctan\left(\frac{\hat{m}}{k_F}\right)-\frac{\pi}{2}\right)\right]$$

with $\hat{m} = m - s\kappa B$ and $k_F = \sqrt{\mu^2-\hat{m}^2}$

III) Energy density

For fermions, the energy density becomes

$$\epsilon = \frac{1}{2\pi^2}\sum_{s=\pm1}\int_{m-s\kappa B}^{\infty} dE\, E^2\, f_+(E,T,\mu)\int_{m}^{E+s\kappa B} d\lambda \frac{\lambda}{\sqrt{E^2-(\lambda-s\kappa B)^2}}$$

$$= \frac{1}{2\pi^2}\sum_{s=\pm1}\int_{m-s\kappa B}^{\infty} dE\, E^2\, f_+(E,T,\mu)\left[\hat{k} + s\kappa B\left(\arctan\left(\frac{s\kappa B - m}{\hat{k}}\right)+\frac{\pi}{2}\right)\right]$$

For anti-fermions we replace the distribution function f_+ by f_-

At zero temperature

$$\epsilon = \frac{1}{48\pi^2}\sum_{s=\pm1}\Bigg[k_F\mu(6\mu^2-3\hat{m}^2-4s\kappa B\hat{m}) - 8s\kappa B\mu^3\left(\arctan\left(\frac{\hat{m}}{k_F}\right)-\frac{\pi}{2}\right)$$
$$-\hat{m}^3(3\hat{m}+4s\kappa B)\log\left(\frac{k_F+\mu}{\hat{m}}\right)\Bigg]$$

with $\hat{k} \equiv \sqrt{E^2-(m-s\kappa B)^2}$, $\hat{m} = m - s\kappa B$, $k_F = \sqrt{\mu^2-\hat{m}^2}$

IV) Pressure parallel to the field

For fermions, the parallel pressure becomes

$$P_{\|} = \frac{1}{2\pi^2} \sum_{s=\pm 1} \int_{m-s\kappa B}^{\infty} dE\, f_+(E,T,\mu) \int_{m}^{E+s\kappa B} d\lambda\, \lambda \sqrt{E^2 - (\lambda^2 - s\kappa B)^2}$$

$$= \frac{1}{24\pi^2} \sum_{s=\pm 1} \int_{m-s\kappa B}^{\infty} dE\, f_+(E,T,\mu) \Big\{ 2\hat{k}(s\kappa B - m)(2m + s\kappa B)$$

$$+E^2 \left[4\hat{k} + 6s\kappa B \left(\arctan\left(\frac{s\kappa B - m}{\hat{k}} \right) + \frac{\pi}{2} \right) \right] \Big\}$$

For anti-fermions we replace the distribution function f_+ by f_-

At zero temperature $P_{\|} = \frac{1}{48\pi^2} \sum_{s=\pm 1} \Big[k_F \mu (2\mu^2 - 5\hat{m}^2 - 8s\kappa B\hat{m})$

$$-4s\kappa B\mu^3 \left(\arctan\left(\frac{\hat{m}}{k_F} \right) - \frac{\pi}{2} \right) + \hat{m}^3 (3\hat{m} + 4s\kappa B) \log\left(\frac{k_F + \mu}{\hat{m}} \right) \Big]$$

with $\hat{k} \equiv \sqrt{E^2 - (m - s\kappa B)^2}, \hat{m} = m - s\kappa B,\ k_F = \sqrt{\mu^2 - \hat{m}^2}$

V) Magnetization

For fermions, it becomes

$$M = \frac{\kappa}{4\pi^2} \sum_{s=\pm 1} s \int_{m-s\kappa B}^{\infty} dE\, f_+(E,T,\mu)$$

$$\times \left[\hat{k}(s\kappa B + m) + E^2 \left(\arctan\left(\frac{s\kappa B - m}{\hat{k}} \right) + \frac{\pi}{2} \right) \right]$$

For anti-fermions we replace the distribution function f_+ by f_-

At zero temperature $M = \frac{\kappa}{12\pi^2} \sum_{s=\pm 1} s \left[\mu k_F (3s\kappa B + \hat{m}) \right.$

$$\left. -\mu^3 \left(\arctan\left(\frac{\hat{m}}{k_F} \right) - \frac{\pi}{2} \right) - \hat{m}^2 (3s\kappa B + 2\hat{m}) \log\left(\frac{k_F + \mu}{\hat{m}} \right) \right]$$

with $\hat{k} \equiv \sqrt{E^2 - (m - s\kappa B)^2}$, $\hat{m} = m - s\kappa B$, $k_F = \sqrt{\mu^2 - \hat{m}^2}$

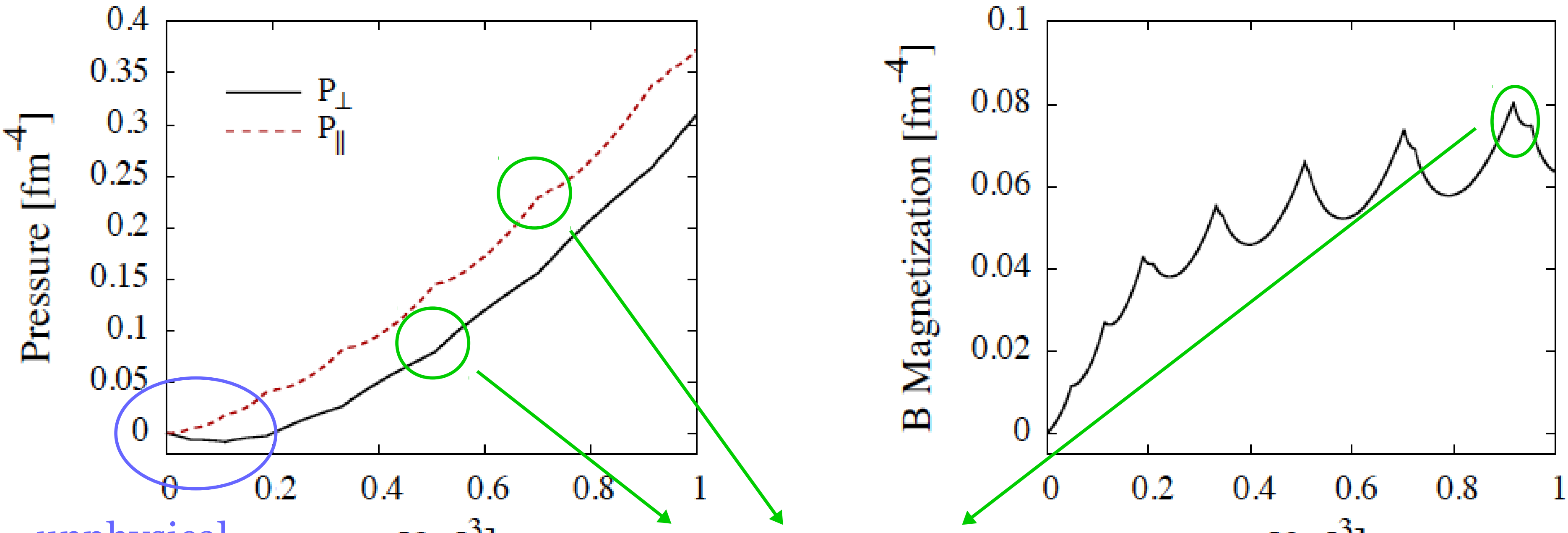
We can verify that the magnetization vanish in the absence of AMM

We can also still generalize

$$P_{\perp,\pm} = P_{\|,\pm} - M_{\pm}B$$

3) Numerical Results

a) Free Fermi gas with spin ½ and positive charge at zero temperature including AMM



unphysical region

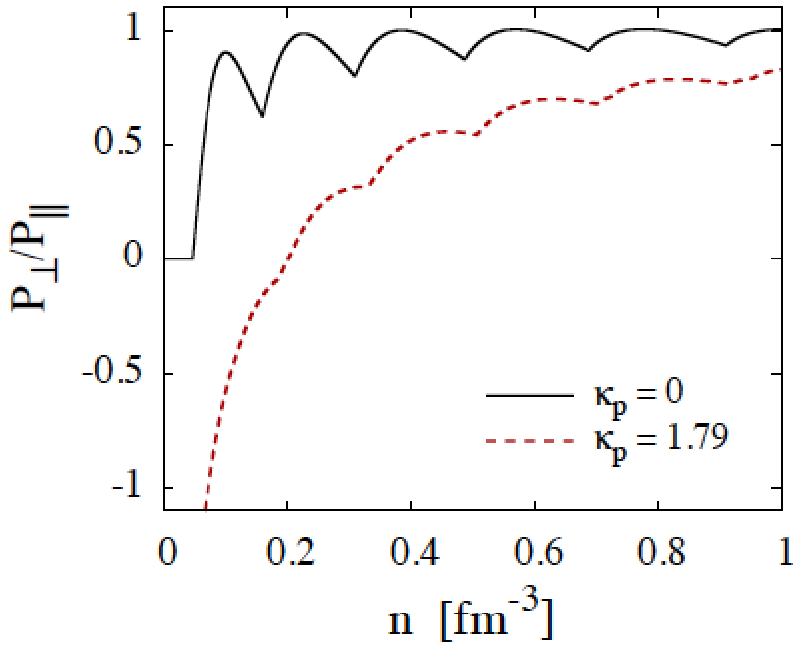
threshold crossing to populate new Landau level

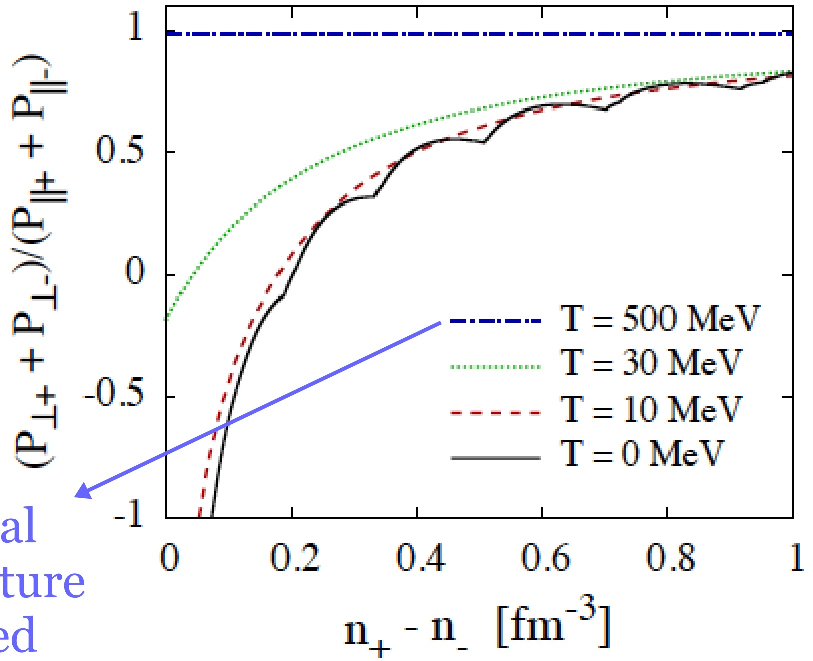
$$\nu = l + \frac{1}{2} - \frac{s}{2}\frac{q}{|q|}$$

with mass m=m_p=0.939 GeV,

charge q=+e, $\kappa=\kappa_p\mu_N$=0.288633/GeV and B=5x10^{18} G

b) Free Fermi gas with spin ½ and positive charge at zero and finite temperature

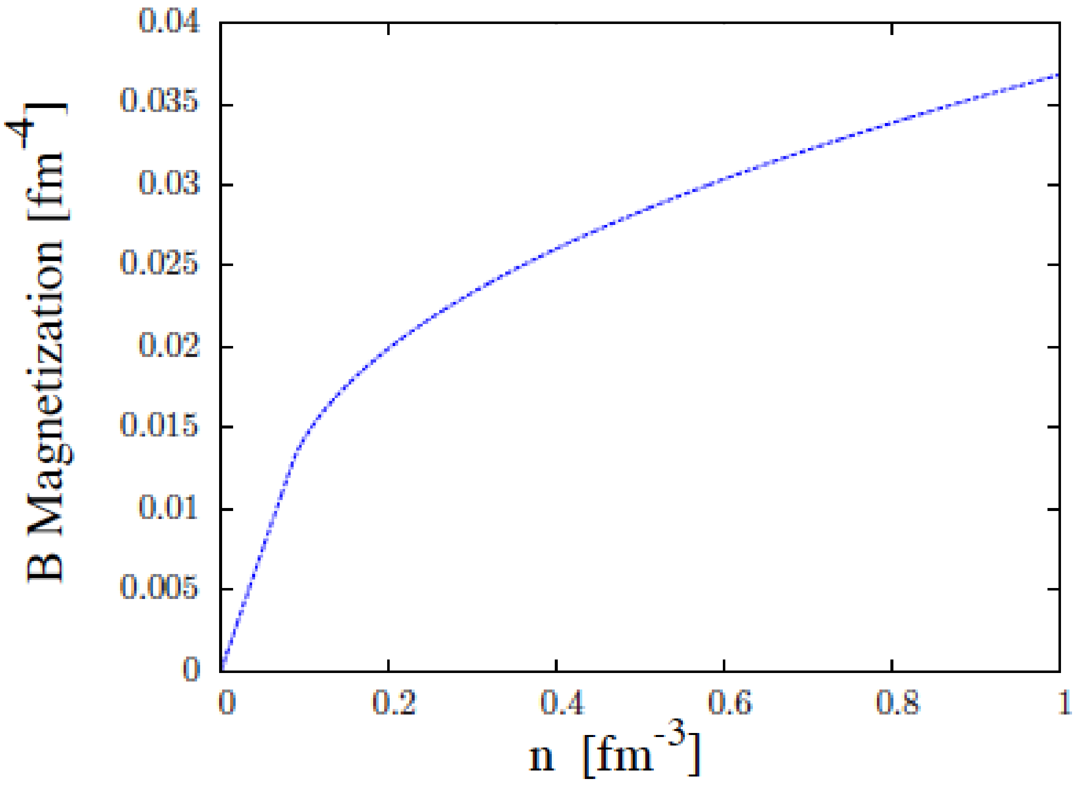




~ initial temperature reached in HIC's in CERN

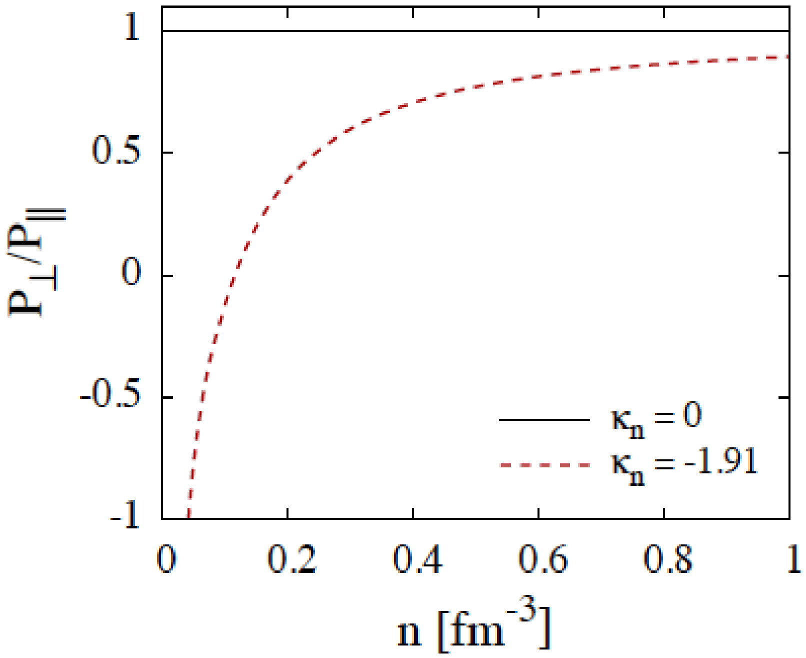
The pressure ratio is always less than 1, it is farther from 1 with AMM but gets closer to 1 with the increase of temperature

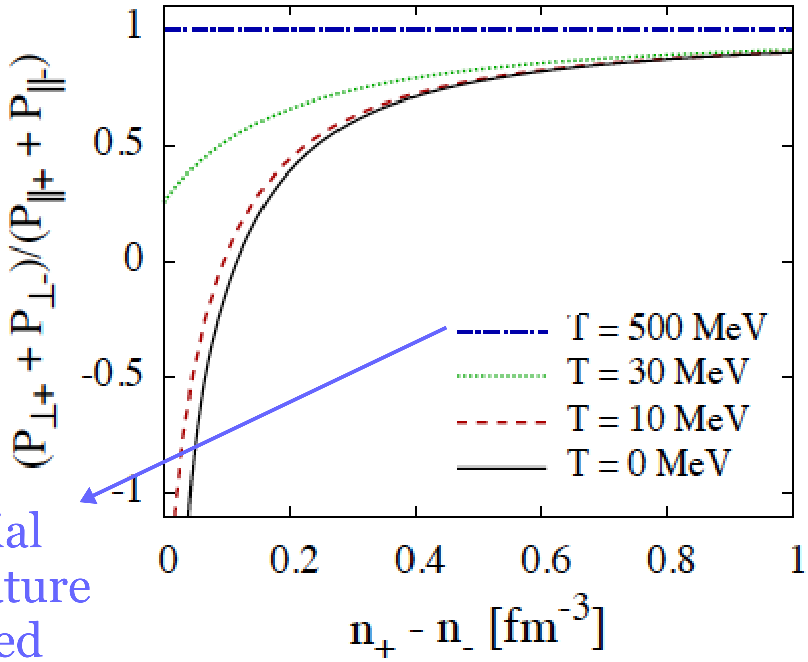
c) Free Fermi gas with spin ½ and no charge at zero temperature including AMM



with mass $m=m_n=0.939$ GeV, charge $q=0$, $\kappa=\kappa_n\mu_N=$ -0.307983/GeV and $B=5\mathrm{x}10^{18}$ G

d) Free Fermi gas with spin ½ and no charge at zero and finite temperature

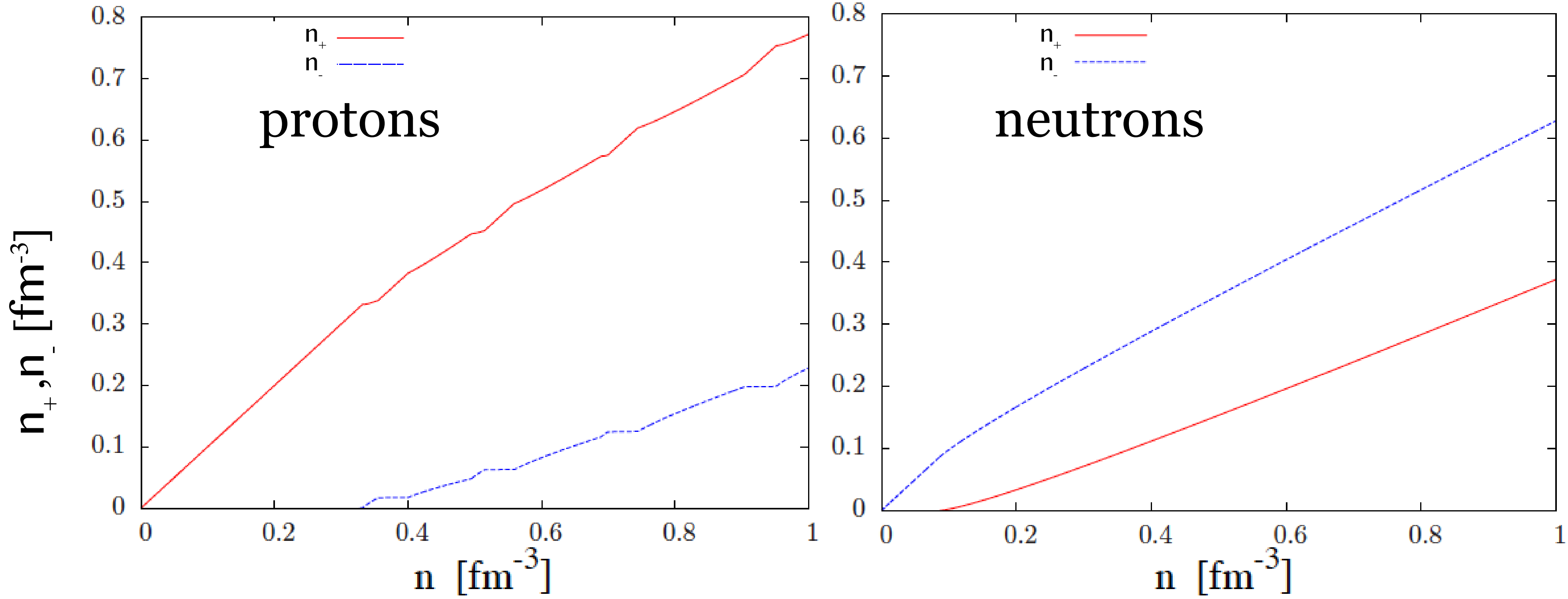




~ initial temperature reached in HIC's in CERN

The pressure ratio is always less than 1 with AMM but gets closer to 1 with the increase of temperature

e) Free Fermi gas with spin ½ at zero temperature including AMM



Spin "+" protons minimize the energy (κ>0), while spin "-" neutrons (κ<0) minimize the energy

$$E_s = \pm\sqrt{k_z^2 + (\lambda - s\kappa B)^2}$$

VD, S.Schramm Astrophys.J. 2008

VD, S.Schramm Phys.Rev.C 2010

4) Realistic Model

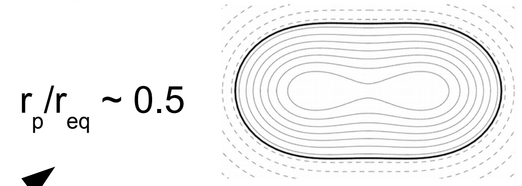
M.Hempel, VD, S.Schramm, I.Iosilevskiy Phys Rev.C 2013

- non-Linear Realization of the SU(3) Sigma Model
- effective quantum relativistic model → mean field
- describes hadrons and quarks interacting via meson exchange (σ,δ,ζ,ω,ρ,ϕ)
- constructed from symmetry relations → allow it to be chirally invariant → masses from interaction with medium
- includes hadrons and quarks as degrees of freedom
- 1st order phase transitions or crossovers between phases
- calibrated to nuclear constraints at low densities, agrees with lattice QCD at high temperature and PQCD at high density
- includes magnetic field and AMM effects

a) Inclusion of gravity

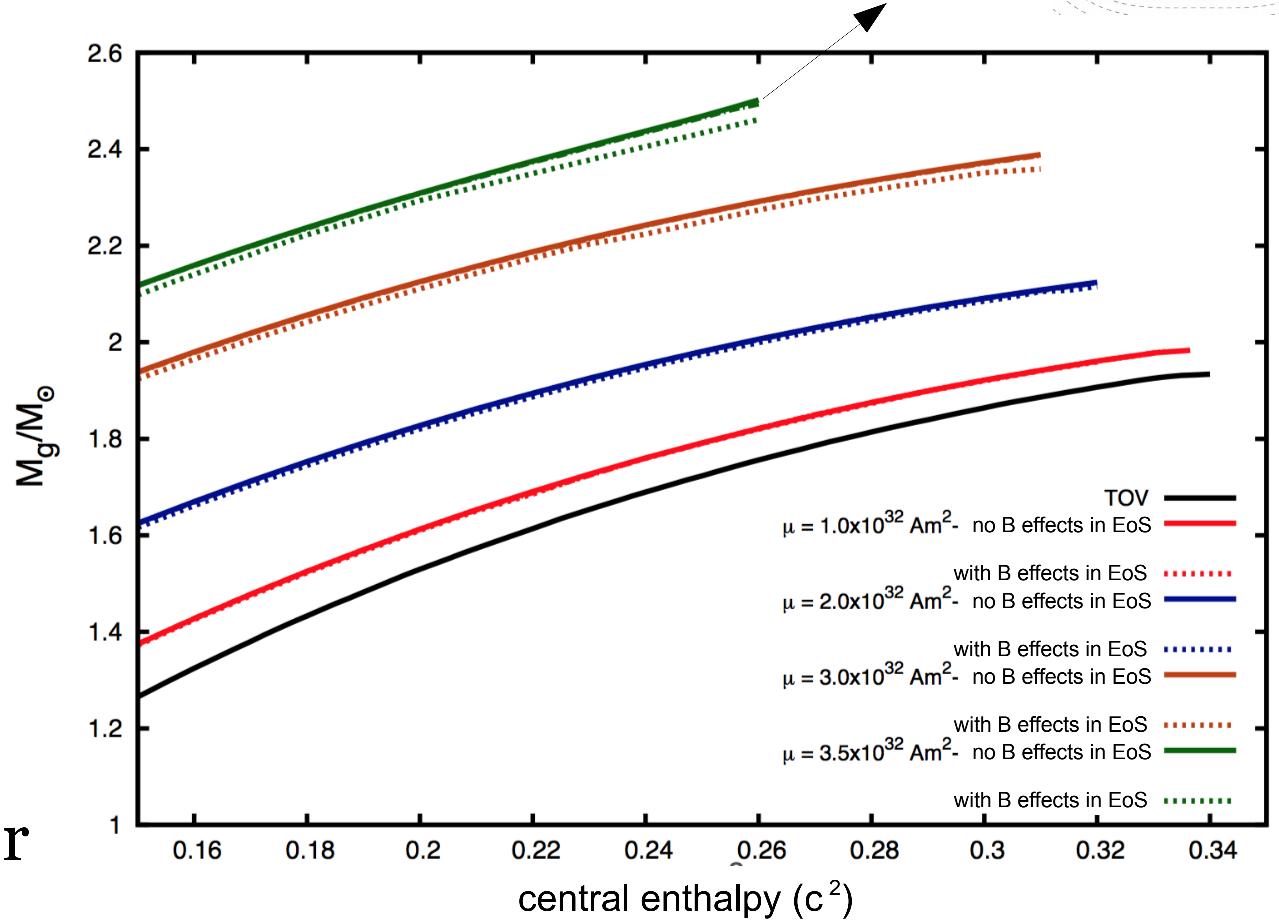
- we assume an axisymmetric poloidal magnetic field

M.Bocquet, S.Bonazzola, E.Gourgoulhon, J.Novak Astron.Astrophys. 1995



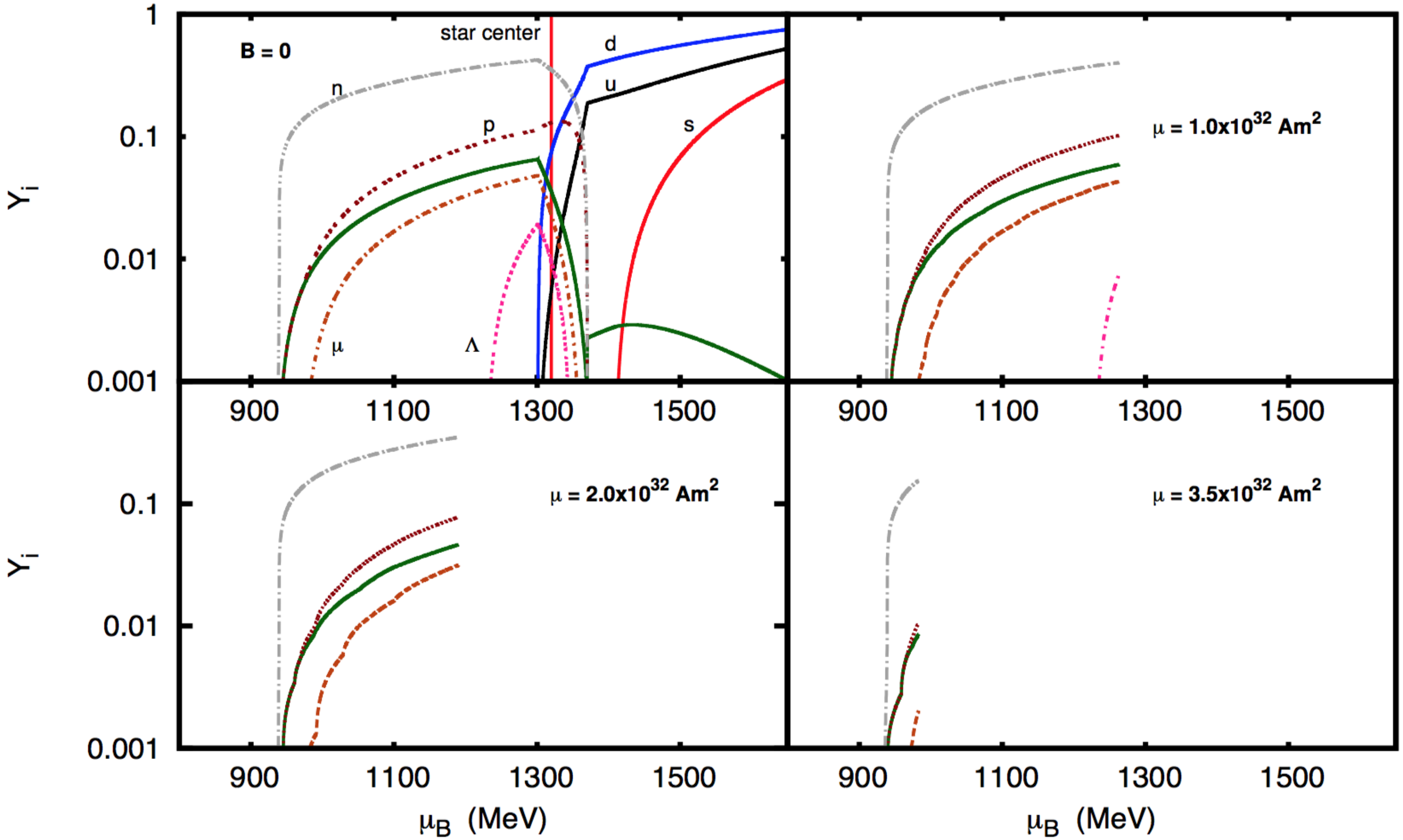
- anisotropic energy-momentum tensor due to:
  - pure electromagnetic contribution
  - contribution from B effects in EoS with AMM

- B in EoS not fixed but determined self-consistently for different magnetic dipole moments μ



B.Franzon, VD, S.Schramm Mon.Not.Roy. Astron.Soc. 2016

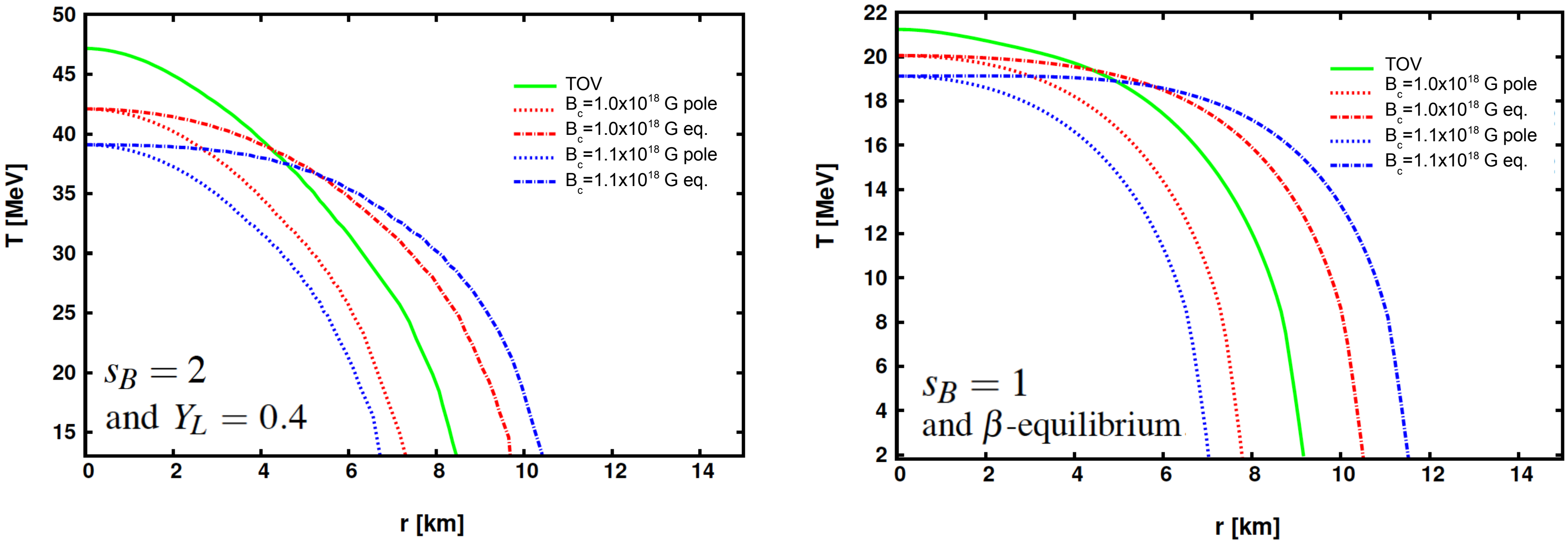
b) Star population at fixed baryon mass calculated from EoS including B and AMM effects



- different composition for different magnetic dipole moments μ's (different internal magnetic field distributions)

- magnetic field decay accompanied by deconfinement?

c) snapshots of temporal evolution of magnetized at fixed baryonic mass for hadronic stars (without B and AMM effects)



B.Franzon, VD, S.Schramm Phys.Rev.D 2016

- different magnetic field distributions

- magnetic field influences temperature distribution in star

- different effect in different regions of star

- detailed temporal evolution necessary!

Homework !

What happens in the extremely high magnetic field limit?

Hint: Begin by calculating what happens with the Landau levels in the case without AMM effects.