Dielectric-top membranes in plane-wave backgrounds

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1st INPP Demokritos-APCTP meeting
National Center for Scientific Research Demokritos, 04 October 2024

based on my work with M. Axenides and M. Floratos, PLB **773** (2017) 265 [arxiv:1707.02878], PRD **97** (2018) 126019 [arxiv:1712.06544], PRD **104** (2021) 106002 [arxiv:2109.01088] (with D. Katsinis), as well as work in progress

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Section 1

Introduction

Plane-fronted gravitational waves with parallel rays (pp-waves)

- Plane-fronted (gravitational) waves with parallel rays (or pp-waves) are solutions of the 4-dimensional (vacuum) Einstein equations. In Brinkmann coordinates,

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Plane-fronted means that pp-waves can be completely covered by 2d wave fronts orthogonal to the wave vector k. The wave fronts are planes which propagate parallel to each other in the direction of $k =$ constant ("parallel rays").

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- By choosing $H(u,x,y)$, Brinkmann metric also solves Einstein-Maxwell theory... Plane waves are special pp-waves:

$$H(u,x,y) = a(u)\left(x^2 - y^2\right) + 2b(u)\,xy + c(u)\left(x^2 + y^2\right), \qquad (\text{in vacuum, } c(u) = 0),$$

gravitational analogs of plane electromagnetic waves... providing the field very far from finite gravity sources...

Pp-waves & plane-waves in $d+1$ dimensions

- Most general metric of a $d+1$ dimensional spacetime with a covariantly constant null Killing vector k:

$$ds^2 = -2dx^+dx^- - F(x^+,x^i)dx^+dx^+ + 2A_j(x^+,x^i)dx^+dx^j + g_{jk}(x^+,x^i)dx^j dx^k, \quad x^\pm \equiv \frac{1}{\sqrt{2}}\left(x^0 \pm x^d\right),$$

where $i,j = 1,2,\ldots d-1$ & $F(u,x^i)$, $A_j(u,x^i)$, $g_{jk}(u,x^i)$ are determined from the sugra equations of motion...

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- For $A_j = 0$, $g_{jk} = \delta_{jk}$, we retrieve the $d+1$ dimensional Brinkmann metric:

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- Homogeneous and isotropic plane-waves have $\mu_{ij} = \mu$:

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- The $d + 1$ dimensional Brinkmann metric can be written in a form which resembles linearized gravity:

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- Brinkmann spacetimes are α'-exact solutions of supergravity/string theory (with or without flux terms)...

  Amati-Klimčík (1988), Horowitz-Steif (1990)

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- Penrose-Güven limits preserve susy... maximally susy backgrounds of 11d/IIB sugra $AdS_{4/5/7} \times S^{7/5/4}$, give rise to two maximally susy homogeneous plane-wave solutions in 10 & 11d...

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$$ds^2 = -2dx^+dx^- - \left[\frac{\mu^2}{9}\sum_{i=1}^{3} x_i x_i + \frac{\mu^2}{36}\sum_{j=1}^{6} y_j y_j\right] dx^+dx^+ + \sum_{i=1}^{3} dx_i dx_i + \sum_{j=1}^{6} dy_j dy_j, \qquad F_{123+} = \mu.$$

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- BMN sector of AdS_5/CFT_4: Penrose limit of IIB string theory on $AdS_5 \times S^5 \leftrightarrow$ BMN limit of $\mathcal{N} = 4$ SYM...

  Berenstein-Maldacena-Nastase (2002)

M-theory on a plane wave

- The matrix model of Berenstein, Maldacena and Nastase (BMN),

$$H = H_0 + \frac{R}{2} \cdot \mathrm{Tr}\left[\sum_{i=1}^{3} \frac{m^2}{9} \mathbf{X}_i^2 + \sum_{j=4}^{9} \frac{m^2}{36} \mathbf{X}_j^2 + \sum_{i,j,k=1}^{3} \frac{2m}{3} i\epsilon_{ijk} \mathbf{X}_i \mathbf{X}_j \mathbf{X}_k - \frac{m}{2} i\Psi^T \gamma_{123} \Psi\right],$$

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Banks-Fischler-Shenker-Susskind (1996)

where the vectors $\mathbf{X}_A$ and 16d Majorana spinor Ψ are $N \times N$ Hermitian matrices... γ_A are the 9d (16×16) Euclidean Dirac matrices, R is the DLCQ compactification radius, and $m \equiv \mu/R$...

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- The BMN matrix model constitutes a deformation of the BMN matrix model by mass terms and a Myers term...

M-theory on a plane wave from membranes

- The mass terms of the BMN matrix model lift the flat directions of BFSS, making the supermembrane spectrum discrete... On the other hand, the Myers term allows for static fuzzy sphere solutions:

$$\mathbf{X}_i = r \cdot J_i, \quad i = 1, 2, 3 \qquad \& \qquad \mathbf{X}_j = 0, \quad j = 4, \ldots, 9,$$

where the matrices J_i furnish a N-dimensional representation of $\mathfrak{su}(2)$. The radii,

$$r = 0, \qquad r = \frac{\mu}{3}, \qquad r = \frac{\mu}{6},$$

correspond to the max susy vacuum $\mathbf{X}_i = 0$, a 1/2-BPS solution and, an unstable, non-susy, positive-energy solution...

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- Equivalently, the light-cone (super)membrane on the maximally susy plane-wave background can be seen as the continuum (matrix dimensionality $N \to \infty$) limit of the BMN matrix model...

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$$\mathbf{X}_i = r \cdot J_i, \quad i = 1,2,3 \qquad \& \qquad \mathbf{X}_j = 0, \quad j = 4,\ldots,9,$$

  where the matrices J_i furnish a N-dimensional representation of $\mathfrak{su}(2)$. The radii,

$$r = 0, \qquad r = \frac{\mu}{3}, \qquad r = \frac{\mu}{6},$$

  correspond to the max susy vacuum $\mathbf{X}_i = 0$, a 1/2-BPS solution and, an unstable, non-susy, positive-energy solution...
- The BMN matrix model describes the discrete light-cone quantization (DLCQ) of M-theory on the KG background...
- As shown by Dasgupta, Sheikh-Jabbari, Van Raamsdonk (2002), the BMN matrix model can be derived by regularizing the light-cone (super)membrane in the 11d maximally susy KG background...
- Equivalently, the light-cone (super)membrane on the maximally susy plane-wave background can be seen as the continuum (matrix dimensionality $N \to \infty$) limit of the BMN matrix model...
- In the following we are going to work exclusively with the classical bosonic membrane on the 11-dimensional maximally susy plane-wave background...

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- We are going to construct solutions of spinning membranes, based on some tools and techniques that were introduced for flat space...
- Let us first briefly review the corresponding membrane action...

Subsection 3

Membranes in the light-cone gauge

Bosonic membrane in a curved background

- Bosonic membranes in curved backgrounds are described by the Dirac-Nambu-Goto (DNG) action:

$$\mathcal{S}_{\text{DNG}} = -T \int d\tau d^2\sigma \left\{ \sqrt{-h} + \dot{X}^m \partial_1 X^n \partial_2 X^r A_{rnm}(X) \right\}, \qquad T \equiv \frac{1}{(2\pi)^2 \, \ell_{11}^3},$$

where $(m, n, r, s = 0, \ldots, 10)$,

$$h_{ij} \equiv G_{mn} \partial_i X^m \partial_j X^n \quad \text{(induced metric)} \quad h \equiv \det h_{ij} \qquad \& \qquad F_{mnrs} = 4\partial_{[m} A_{nrs]} \quad \text{(field strength)},$$

and A_{nrs} is the (antisymmetric) 3-form field of 11-dimensional supergravity...

The light-cone gauge

- In the light-cone gauge, we write:

$$X^{\pm} = \frac{1}{\sqrt{2}}\left(X^0 \pm X^{10}\right) \qquad \& \qquad X^+ = \tau.$$

Goldstone-Hoppe (1982)

- The light-cone Hamiltonian is then written as follows ($G_{--} = G_{a-} = 0$):

$$H = T\int d^2\sigma \Bigg\{ \frac{1}{2}\frac{G_{+-}}{P_- - C_-}\left[\left(P_a - C_a - \frac{P_- - C_-}{G_{+-}}G_{a+}\right)^2 + \frac{1}{2}G_{ab}G_{cd}\{X^a, X^c\}\{X^b, X^d\}\right] -$$

$$-\frac{1}{2}\frac{P_- - C_-}{G_{+-}}G_{++} - C_+ + \frac{1}{P_- - C_-}\left[C_- C_{+-} - \{X^a, X^b\}P_a C_{+-b}\right]\Bigg\},$$

de Wit-Peeters-Plefka (1998)

where $(a, b, c, d = 1, \ldots, 9)$,

$$C_{\pm} \equiv C_{\pm ab} - \partial_1 X^a \partial_2 X^b, \quad C_{+-} \equiv -C_{+-a}\{X^-, X^a\}, \quad C_a \equiv -\left(C_{-ab}\{X^b, X^-\} + C_{abc}\partial_1 X^b \partial_2 X^c\right).$$

Poisson brackets

The Poisson bracket is defined as:

$$\{f\,,\,g\} \equiv \frac{\epsilon_{rs}}{\sqrt{w\left(\boldsymbol{\sigma}\right)}}\,\partial_r f\;\partial_s g = \frac{1}{\sqrt{w\left(\boldsymbol{\sigma}\right)}}\left(\partial_1 f\;\partial_2 g - \partial_2 f\;\partial_1 g\right),$$

where $d^2\sigma = \sqrt{w\left(\boldsymbol{\sigma}\right)}\,d\sigma_1\,d\sigma_2$. In a flat space-sheet, $w\left(\boldsymbol{\sigma}\right) = 1$.

Section 2

Spherical Euler-top membranes in flat backgrounds

M. Axenides, E. Floratos, L. Perivolaropoulos
Metastability of spherical membranes in supermembrane and matrix theory
JHEP **11** (2000) 020 [arXiv:hep-th/0007198]

M. Axenides, E. Floratos
Euler-top dynamics of Nambu-Goto p-branes
JHEP **03** (2007) 093 [arXiv:hep-th/0608017]

Light-cone gauge in flat space

- In a flat background

$$G_{+-} = -1, \quad G_{ab} = \delta_{ab}, \qquad G_{++} = G_{--} = G_{a\pm} = 0, \qquad C_{\pm} = C_{+-} = C_{a} = 0,$$

therefore the light-cone Hamiltonian becomes ($P_- = -1$):

$$H = \frac{T}{2}\int d^2\sigma \left[P^2 + \frac{1}{2}\{X^i, X^j\}^2\right].$$

- The corresponding equations of motion and the Gauss law constraint become:

$$\ddot{X}^i = \{\{X^i, X^j\}, X^j\} \qquad \& \qquad \sum_{i=1}^{9}\{\dot{X}^i, X^i\} = 0.$$

Euler-top membranes in flat space

- Consider the ansatz:

$$\boxed{X^i = R^{ij}\,(\tau)\,X_0^j\,(\sigma)\,, \qquad R \equiv \exp\,(\Omega\,\tau)\,, \qquad \Omega^T = -\Omega}\,.$$

If we define the angular momentum and moment of inertia matrices of the membrane as

$$\mathrm{I}^{ij} = T\int d^2\sigma\, X^i X^j \qquad \& \qquad \mathrm{L}^{ij} = T\int d^2\sigma\,\left(\dot{X}^i X^j - \dot{X}^j X^i\right),$$

we can prove that the energy of the membrane is given by

$$\boxed{E = -\frac{3}{4}\cdot\frac{\mathrm{Tr}\,[\Omega\cdot\mathrm{L}]^2}{2\,\mathrm{Tr}\,[\Omega^2\cdot\mathrm{I}]}}\,,$$

Axenides-Floratos (2006)

which is the generalization of the familiar from point-particle mechanics Euler-top Hamiltonian:

$$E = \frac{\ell_x^2}{2\mathrm{I}_x} + \frac{\ell_y^2}{2\mathrm{I}_y} + \frac{\ell_z^2}{2\mathrm{I}_z}\,.$$

The spherical ansatz

- Consider the following spherical configuration:

$$X_i \equiv x_i(\tau) \cdot e_1, \qquad i = 1, 2, \ldots, q_1$$

$$Y_j \equiv X_{q_1+j} = y_j(\tau) \cdot e_2, \qquad j = 1, 2, \ldots, q_2, \qquad \& \qquad q_1 + q_2 + q_3 = 9$$

$$Z_j \equiv X_{q_2+k} = z_k(\tau) \cdot e_3, \qquad k = 1, 2, \ldots, q_3,$$

Collins-Tucker (1976)

that breaks the manifest $\mathfrak{so}(9)$ symmetry of the action to $\mathfrak{so}(q_1) \times \mathfrak{so}(q_2) \times \mathfrak{so}(q_3)$. We have defined:

$$(e_1, e_2, e_3) = (\cos\phi \sin\theta, \sin\phi \sin\theta, \cos\theta), \qquad \phi \in [0, 2\pi), \quad \theta \in [0, \pi]$$

$$\{e_i, e_j\} = \epsilon_{ijk}\, e_k, \qquad \int e_i\, e_j\, d^2\sigma = \frac{4\pi}{3}\, \delta_{ij}$$

and the membrane area element is given by:

$$d^2\sigma = d\sigma_1\, d\sigma_2 = \sin\theta\, d\phi\, d\theta \qquad \& \qquad \sqrt{w(\theta)} = \sin\theta.$$

The spherical ansatz

- Here's the energy of the bubble:

$$E = \frac{2\pi T}{3}\left[\dot{\mathbf{x}}^2 + \dot{\mathbf{y}}^2 + \dot{\mathbf{z}}^2 + \mathbf{x}^2\mathbf{y}^2 + \mathbf{y}^2\mathbf{z}^2 + \mathbf{z}^2\mathbf{x}^2\right].$$

- The corresponding equations of motion are:

$$\ddot{x}_i + \left(\mathbf{y}^2 + \mathbf{z}^2\right) x_i = 0, \qquad \ddot{y}_j + \left(\mathbf{z}^2 + \mathbf{x}^2\right) y_j = 0, \qquad \ddot{z}_k + \left(\mathbf{y}^2 + \mathbf{x}^2\right) z_k = 0,$$

while the Gauss law constraint

$$\sum_{i=1}^{q_1}\{\dot{x}^i, x^i\} + \sum_{j=1}^{q_2}\{\dot{y}^j, y^j\} + \sum_{k=1}^{q_3}\{\dot{z}^k, z^k\} = 0,$$

is automatically satisfied by this ansatz.

The spherical ansatz

- Let us switch to the notation:

$$r_x^2 \equiv \mathbf{x}^2 = \sum_{i=1}^{q_1} x_i x_i, \qquad r_y^2 \equiv \mathbf{y}^2 = \sum_{j=1}^{q_2} y_j y_j, \qquad r_z^2 \equiv \mathbf{z}^2 = \sum_{k=1}^{q_3} z_k z_k$$

$$(\ell_x)_{ij} \equiv \dot{x}_i x_j - x_i \dot{x}_j \Big|_{\mathfrak{so}(q_1)}, \qquad (\ell_y)_{ij} \equiv \dot{y}_i y_j - y_i \dot{y}_j \Big|_{\mathfrak{so}(q_2)}, \qquad (\ell_z)_{ij} \equiv \dot{z}_i z_j - z_i \dot{z}_j \Big|_{\mathfrak{so}(q_3)}$$

conserved

$$\dot{\mathbf{x}}^2 \equiv \sum_{i=1}^{q_1} \dot{x}_i \dot{x}_i = \dot{r}_x^2 + \frac{\ell_x^2}{r_x^2}, \qquad \dot{\mathbf{y}}^2 \equiv \sum_{j=1}^{q_2} \dot{y}_j \dot{y}_j = \dot{r}_y^2 + \frac{\ell_y^2}{r_y^2}, \qquad \dot{\mathbf{z}}^2 \equiv \sum_{k=1}^{q_3} \dot{z}_k \dot{z}_k = \dot{r}_z^2 + \frac{\ell_z^2}{r_z^2},$$

which allows to write the energy of the membrane as follows:

$$E = \frac{2\pi T}{3}\left(E_{\text{kin}} + V_{\text{eff}}\right), \qquad E_{\text{kin}} \equiv \dot{r}_x^2 + \dot{r}_y^2 + \dot{r}_z^2 \quad \& \quad V_{\text{eff}} \equiv \frac{\ell_x^2}{r_x^2} + \frac{\ell_y^2}{r_y^2} + \frac{\ell_z^2}{r_z^2} + r_x^2 r_y^2 + r_y^2 r_z^2 + r_z^2 r_x^2,$$

where

$$\ell_x^2 = \frac{1}{2}(\ell_x)_{ij}(\ell_x)_{ij}, \qquad \ell_y^2 = \frac{1}{2}(\ell_y)_{ij}(\ell_y)_{ij}, \qquad \ell_z^2 = \frac{1}{2}(\ell_z)_{ij}(\ell_z)_{ij}.$$

Spherical Euler tops

- As shown by Axenides-Floratos (2006), the radii $r_x = x_0^2$, $r_y = y_0^2$, $r_z = z_0^2$ of the Euler top solutions

$$\mathbf{x}(\tau) = e^{\Omega_x \tau} \cdot \mathbf{x}_0, \qquad \mathbf{y}(\tau) = e^{\Omega_y \tau} \cdot \mathbf{y}_0, \qquad \mathbf{z}(\tau) = e^{\Omega_z \tau} \cdot \mathbf{z}_0,$$

can be determined for all antisymmetric matrices Ω_x, Ω_y, Ω_z in terms of the corresponding angular momenta ℓ_x, ℓ_y, ℓ_z, by minimizing the effective potential:

$$\boxed{V_{\text{eff}} = \frac{\ell_x^2}{r_x^2} + \frac{\ell_y^2}{r_y^2} + \frac{\ell_z^2}{r_z^2} + r_x^2 r_y^2 + r_y^2 r_z^2 + r_z^2 r_x^2},$$

i.e. by solving

$$\frac{dV_{\text{eff}}}{dr_x} = -\frac{2\ell_x^2}{r_x^3} + 2r_x\left(r_y^2 + r_z^2\right) = \frac{dV_{\text{eff}}}{dr_y} = -\frac{2\ell_y^2}{r_y^3} + 2r_y\left(r_z^2 + r_x^2\right) = \frac{dV_{\text{eff}}}{dr_z} = -\frac{2\ell_z^2}{r_z^3} + 2r_z\left(r_x^2 + r_y^2\right) = 0.$$

- Equivalently we can plug the above ansatz into the equations of motion in order to determine the relation between the radii r_x, r_y, r_z and the components of the matrices Ω_x, Ω_y, Ω_z.

Symmetric & axially symmetric Euler spheres

- For a single radius $r = r_x = r_y = r_z$, $\ell = \ell_x = \ell_y = \ell_z$ the effective potential becomes:

$$V_{\text{eff}} \equiv \frac{3\ell}{r^2} + 3r^4,$$

finding

$$r_{(\min)} = \frac{\ell^{1/3}}{2^{1/6}}, \qquad V_{\text{eff}(\min)} = \frac{9\ell^{4/3}}{4^{1/3}}.$$

- The axially symmetric (two-radii) $r_\alpha = r_x = r_y$, $\ell_\alpha = \ell_x = \ell_y$ effective potential is:

$$V_{\text{eff}} \equiv \frac{2\ell_\alpha^2}{r_\alpha^2} + \frac{\ell_z^2}{r_z^2} + r_\alpha^4 + 2r_\alpha^2 r_z^2$$

with

$$r^2_{\alpha(\min)} = \frac{2\ell_\alpha^{4/3}}{\left(\ell_z + \sqrt{\ell_z^2 + 8\ell_\alpha^2}\right)^{2/3}}, \quad r^2_{z(\min)} = \frac{\ell_z}{2\ell_\alpha^{2/3}}\left(\ell_z + \sqrt{\ell_z^2 + 8\ell^2}\right)^{1/3}$$

$$V_{\text{eff}(\min)} = \frac{6\ell_\alpha^{2/3}}{\left(\ell_z + \sqrt{\ell_z^2 + 8\ell_\alpha^2}\right)^{4/3}}\left[\ell_z\left(\ell_z + \sqrt{\ell_z^2 + 8\ell_\alpha^2}\right) + 2\ell_\alpha^2\right].$$

Section 3

Spherical dielectric tops in plane-wave backgrounds

M. Axenides, E. Floratos, D. Katsinis, GL
M-theory as a dynamical system generator
[arXiv:2007.07028]

M. Axenides, E. Floratos, GL
to appear

Light-cone gauge in the plane-wave background

- In the maximally supersymmetric plane background,

$$G_{+-}=-1,\qquad G_{ab}=\delta_{ab},\qquad G_{++}=-\frac{\mu^2}{9}\sum_{i=1}^{3}x^ix^i-\frac{\mu^2}{36}\sum_{j=1}^{6}y^jy^j,\qquad G_{--}=G_{a\pm}=0$$

$$C_{-}=C_{+-}=C_a=0,\qquad C_{+}=\frac{\mu}{3}\epsilon_{ijk}\partial_1x^i\partial_2x^jx^k,$$

the light-cone Hamiltonian becomes (for $P_{-}=-1$):

$$H=\frac{T}{2}\int d^2\sigma\left[\pi_i^2+\frac{1}{2}\left\{x^i,x^j\right\}^2+\frac{1}{2}\left\{y^i,y^j\right\}^2+\left\{x^i,y^j\right\}^2+\frac{\mu^2x^2}{9}+\frac{\mu^2y^2}{36}-\frac{\mu}{3}\epsilon_{ijk}\left\{x^i,x^j\right\}x^k\right].$$

Light-cone gauge in the plane-wave background

- In the maximally supersymmetric plane background,

$$G_{+-} = -1, \qquad G_{ab} = \delta_{ab}, \qquad G_{++} = -\frac{\mu^2}{9}\sum_{i=1}^{3} x^i x^i - \frac{\mu^2}{36}\sum_{j=1}^{6} y^j y^j, \qquad G_{--} = G_{a\pm} = 0$$

$$C_- = C_{+-} = C_a = 0, \qquad C_+ = \frac{\mu}{3}\epsilon_{ijk}\partial_1 x^i \partial_2 x^j x^k,$$

which can also be expressed as a sum of squares:

$$H = \frac{T}{2}\int d^2\sigma \left[\pi^2 + \left(\frac{\mu}{3}x_i - \frac{1}{2}\epsilon_{ijk}\left\{x_j, x_k\right\}\right)^2 + \frac{1}{2}\left\{y_i, y_j\right\}^2 + \frac{\mu^2}{36}y_j y_j + \left\{x_i, y_j\right\}^2\right].$$

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- The corresponding equations of motion and the Gauss law constraint read:

$$\ddot{x}_i = \left\{\left\{x_i, x_j\right\}, x_j\right\} + \left\{\left\{x_i, y_j\right\}, y_j\right\} - \frac{\mu^2}{9}x_i + \frac{\mu}{2}\epsilon_{ijk}\left\{x_j, x_k\right\}, \qquad \sum_{i=1}^{3}\left\{\dot{x}^i, x^i\right\} + \sum_{j=1}^{6}\left\{\dot{y}^j, y^j\right\} = 0$$

$$\ddot{y}_i = \left\{\left\{y_i, y_j\right\}, y_j\right\} + \left\{\left\{y_i, x_j\right\}, x_j\right\} - \frac{\mu^2}{36}y_i.$$

The spherical ansatz

Here's the generalization of the flat spherical ansatz to the maximally supersymmetric plane-wave background:

$$x_i \equiv x_{1i} = \tilde{x}_{1i}(\tau)\, e_1(\sigma), \quad i = 1, \ldots, q_1, \qquad y_i \equiv y_{1i} = \tilde{y}_{1i}(\tau)\, e_1(\sigma), \quad i = 1, \ldots, s_1$$

$$x_{q_1+j} \equiv x_{2j} = \tilde{x}_{2j}(\tau)\, e_2(\sigma), \quad j = 1, \ldots, q_2, \qquad y_{s_1+j} \equiv y_{2j} = \tilde{y}_{2j}(\tau)\, e_2(\sigma), \quad j = 1, \ldots, s_2$$

$$x_{q_1+q_2+k} \equiv x_{3k} = \tilde{x}_{3k}(\tau)\, e_3(\sigma), \quad k = 1, \ldots, q_3, \qquad y_{s_1+s_2+k} \equiv y_{3k} = \tilde{y}_{3k}(\tau)\, e_3(\sigma), \quad k = 1, \ldots, s_3,$$

where

$$q_1 + q_2 + q_3 = 3 \qquad \& \qquad s_1 + s_2 + s_3 = 6,$$

and again,

$$(e_1, e_2, e_3) = (\cos\phi \sin\theta, \sin\phi \sin\theta, \cos\theta), \qquad \phi \in [0, 2\pi), \quad \theta \in [0, \pi]$$

$$\{e_i, e_j\} = \epsilon_{ijk}\, e_k, \qquad \int e_i\, e_j\, d^2\sigma = \frac{4\pi}{3}\, \delta_{ij}$$

$$d^2\sigma = d\sigma_1\, d\sigma_2 = \sin\theta\, d\phi\, d\theta \qquad \& \qquad \sqrt{w(\theta)} = \sin\theta.$$

The spherical ansatz

Now switch to the notation:

$$r_{xx}^2 \equiv \tilde{x}_1^2 = \sum_{i=1}^{q_1} \tilde{x}_{1i}\tilde{x}_{1i}, \qquad r_{xy}^2 \equiv \tilde{x}_2^2 = \sum_{i=1}^{q_2} \tilde{x}_{2i}\tilde{x}_{2i}, \qquad r_{xz}^2 \equiv \tilde{x}_3^2 = \sum_{i=1}^{q_3} \tilde{x}_{3i}\tilde{x}_{3i}$$

$$(\ell_{xx})_{ij} \equiv \dot{\tilde{x}}_{1i}\tilde{x}_{1j} - \tilde{x}_{1i}\dot{\tilde{x}}_{1j}\Big|_{\mathfrak{so}(q_1)}, \quad (\ell_{xy})_{ij} \equiv \dot{\tilde{x}}_{2i}\tilde{x}_{2j} - \tilde{x}_{2i}\dot{\tilde{x}}_{2j}\Big|_{\mathfrak{so}(q_2)}, \quad (\ell_{xz})_{ij} \equiv \dot{\tilde{x}}_{3i}\tilde{x}_{3j} - \tilde{x}_{3i}\dot{\tilde{x}}_{3j}\Big|_{\mathfrak{so}(q_3)} \quad \text{conserved}$$

$$\dot{\tilde{x}}_1^2 \equiv \sum_{i=1}^{q_1} \dot{\tilde{x}}_{1i}\dot{\tilde{x}}_{1i} = \dot{r}_{xx}^2 + \frac{\ell_{xx}^2}{r_{xx}^2}, \qquad \dot{\tilde{x}}_2^2 \equiv \sum_{i=1}^{q_2} \dot{\tilde{x}}_{2i}\dot{\tilde{x}}_{2i} = \dot{r}_{xy}^2 + \frac{\ell_{xy}^2}{r_{xy}^2}, \qquad \dot{\tilde{x}}_3^2 \equiv \sum_{i=1}^{q_3} \dot{\tilde{x}}_{3i}\dot{\tilde{x}}_{3i} = \dot{r}_{xz}^2 + \frac{\ell_{xz}^2}{r_{xz}^2},$$

and similarly for the six coordinates y:

$$r_{yx}^2 \equiv \tilde{y}_1^2 = \sum_{j=1}^{s_1} \tilde{y}_{1j}\tilde{y}_{1j}, \qquad r_{yy}^2 \equiv \tilde{y}_2^2 = \sum_{j=1}^{s_2} \tilde{y}_{2j}\tilde{y}_{2j}, \qquad r_{yz}^2 \equiv \tilde{y}_3^2 = \sum_{j=1}^{s_3} \tilde{y}_{3j}\tilde{y}_{3j}$$

$$(\ell_{yx})_{ij} \equiv \dot{\tilde{y}}_{1i}\tilde{y}_{1j} - \tilde{y}_{1i}\dot{\tilde{y}}_{1j}\Big|_{\mathfrak{so}(s_1)}, \quad (\ell_{yy})_{ij} \equiv \dot{\tilde{y}}_{2i}\tilde{y}_{2j} - \tilde{y}_{2i}\dot{\tilde{y}}_{2j}\Big|_{\mathfrak{so}(s_2)}, \quad (\ell_{yz})_{ij} \equiv \dot{\tilde{y}}_{3i}\tilde{y}_{3j} - \tilde{y}_{3i}\dot{\tilde{y}}_{3j}\Big|_{\mathfrak{so}(s_3)} \quad \text{conserved}$$

$$\dot{\tilde{y}}_1^2 \equiv \sum_{j=1}^{s_1} \dot{\tilde{y}}_{1j}\dot{\tilde{y}}_{1j} = \dot{r}_{yx}^2 + \frac{\ell_{yx}^2}{r_{yx}^2}, \qquad \dot{\tilde{y}}_2^2 \equiv \sum_{j=2}^{s_2} \dot{\tilde{y}}_{2j}\dot{\tilde{y}}_{2j} = \dot{r}_{yy}^2 + \frac{\ell_{yy}^2}{r_{yy}^2}, \qquad \dot{\tilde{y}}_3^2 \equiv \sum_{j=1}^{s_3} \dot{\tilde{y}}_{3j}\dot{\tilde{y}}_{3j} = \dot{r}_{yz}^2 + \frac{\ell_{yz}^2}{r_{yz}^2}.$$

The spherical ansatz

- Here's the resulting effective potential:

$$V_{\text{eff}} = \frac{2\pi T}{3}\Bigg[\frac{\ell_{xx}^2}{r_{xx}^2} + \frac{\ell_{xy}^2}{r_{xy}^2} + \frac{\ell_{xz}^2}{r_{xz}^2} + \frac{\ell_{yx}^2}{r_{yx}^2} + \frac{\ell_{yy}^2}{r_{yy}^2} + \frac{\ell_{yz}^2}{r_{yz}^2} + r_{xx}^2 r_{xy}^2 + r_{xy}^2 r_{xz}^2 + r_{xz}^2 r_{xx}^2 + r_{yx}^2 r_{yy}^2 + r_{yy}^2 r_{yz}^2 +$$

$$+ r_{yz}^2 r_{yx}^2 + r_{xx}^2\left(r_{yy}^2 + r_{yz}^2\right) + r_{xy}^2\left(r_{yz}^2 + r_{yx}^2\right) + r_{xz}^2\left(r_{yx}^2 + r_{yy}^2\right) + \frac{\mu^2}{9}\left(r_{xx}^2 + r_{xy}^2 + r_{xz}^2\right) +$$

$$+ \frac{\mu^2}{36}\left(r_{yx}^2 + r_{yy}^2 + r_{yz}^2\right) - 2\mu\,\epsilon_{ijk}\tilde{x}_{1i}\tilde{x}_{2j}\tilde{x}_{3k}\Bigg].$$

The spherical ansatz

- Here's the resulting effective potential:

$$V_{\rm eff} = \frac{2\pi T}{3}\left[\frac{\ell_{xx}^2}{r_{xx}^2} + \frac{\ell_{xy}^2}{r_{xy}^2} + \frac{\ell_{xz}^2}{r_{xz}^2} + \frac{\ell_{yx}^2}{r_{yx}^2} + \frac{\ell_{yy}^2}{r_{yy}^2} + \frac{\ell_{yz}^2}{r_{yz}^2} + r_{xx}^2 r_{xy}^2 + r_{xy}^2 r_{xz}^2 + r_{xz}^2 r_{xx}^2 + r_{yx}^2 r_{yy}^2 + r_{yy}^2 r_{yz}^2 + \right.$$
$$+ r_{yz}^2 r_{yx}^2 + r_{xx}^2\left(r_{yy}^2 + r_{yz}^2\right) + r_{xy}^2\left(r_{yz}^2 + r_{yx}^2\right) + r_{xz}^2\left(r_{yx}^2 + r_{yy}^2\right) + \frac{\mu^2}{9}\left(r_{xx}^2 + r_{xy}^2 + r_{xz}^2\right) +$$
$$\left. + \frac{\mu^2}{36}\left(r_{yx}^2 + r_{yy}^2 + r_{yz}^2\right) - 2\mu\,\epsilon_{ijk}\tilde{x}_{1i}\tilde{x}_{2j}\tilde{x}_{3k}\right].$$

- The effective potential is made up of four basic types of terms: • (1) kinetic/angular momentum terms (repulsive), • (2) quartic interaction terms (attractive), • (3) mass terms (attractive), and • (4) cubic Myers terms (repulsive).

The spherical ansatz

- Here's the resulting effective potential:

$$V_{\text{eff}} = \frac{2\pi T}{3}\left[\frac{\ell_{xx}^2}{r_{xx}^2} + \frac{\ell_{xy}^2}{r_{xy}^2} + \frac{\ell_{xz}^2}{r_{xz}^2} + \frac{\ell_{yx}^2}{r_{yx}^2} + \frac{\ell_{yy}^2}{r_{yy}^2} + \frac{\ell_{yz}^2}{r_{yz}^2} + r_{xx}^2 r_{xy}^2 + r_{xy}^2 r_{xz}^2 + r_{xz}^2 r_{xx}^2 + r_{yx}^2 r_{yy}^2 + r_{yy}^2 r_{yz}^2 + \right.$$

$$+ r_{yz}^2 r_{yx}^2 + r_{xx}^2\left(r_{yy}^2 + r_{yz}^2\right) + r_{xy}^2\left(r_{yz}^2 + r_{yx}^2\right) + r_{xz}^2\left(r_{yx}^2 + r_{yy}^2\right) + \frac{\mu^2}{9}\left(r_{xx}^2 + r_{xy}^2 + r_{xz}^2\right) +$$

$$\left. + \frac{\mu^2}{36}\left(r_{yx}^2 + r_{yy}^2 + r_{yz}^2\right) - 2\mu\,\epsilon_{ijk}\tilde{x}_{1i}\tilde{x}_{2j}\tilde{x}_{3k}\right].$$

- The effective potential is made up of four basic types of terms: • (1) kinetic/angular momentum terms (repulsive), • (2) quartic interaction terms (attractive), • (3) mass terms (attractive), and • (4) cubic Myers terms (repulsive).

  The last two types of terms (i.e. mass terms and Myers terms) are μ-dependent and therefore absent from the flat space case ($\mu \to 0$), which was studied in Axenides-Floratos (2006)...

The spherical ansatz

- Here's the resulting effective potential:

$$V_{\text{eff}} = \frac{2\pi T}{3}\left[\frac{\ell_{xx}^2}{r_{xx}^2} + \frac{\ell_{xy}^2}{r_{xy}^2} + \frac{\ell_{xz}^2}{r_{xz}^2} + \frac{\ell_{yx}^2}{r_{yx}^2} + \frac{\ell_{yy}^2}{r_{yy}^2} + \frac{\ell_{yz}^2}{r_{yz}^2} + r_{xx}^2 r_{xy}^2 + r_{xy}^2 r_{xz}^2 + r_{xz}^2 r_{xx}^2 + r_{yx}^2 r_{yy}^2 + r_{yy}^2 r_{yz}^2 + \right.$$

$$+ r_{yz}^2 r_{yx}^2 + r_{xx}^2\left(r_{yy}^2 + r_{yz}^2\right) + r_{xy}^2\left(r_{yz}^2 + r_{yx}^2\right) + r_{xz}^2\left(r_{yx}^2 + r_{yy}^2\right) + \frac{\mu^2}{9}\left(r_{xx}^2 + r_{xy}^2 + r_{xz}^2\right) +$$

$$\left. + \frac{\mu^2}{36}\left(r_{yx}^2 + r_{yy}^2 + r_{yz}^2\right) - 2\mu\, \epsilon_{ijk}\tilde{x}_{1i}\tilde{x}_{2j}\tilde{x}_{3k}\right].$$

- The effective potential is made up of four basic types of terms: • (1) kinetic/angular momentum terms (repulsive), • (2) quartic interaction terms (attractive), • (3) mass terms (attractive), and • (4) cubic Myers terms (repulsive).

  The last two types of terms (i.e. mass terms and Myers terms) are μ-dependent and therefore absent from the flat space case ($\mu \to 0$), which was studied in Axenides-Floratos (2006)...

  In either case ($\mu = 0$ or $\mu \neq 0$), it is the equilibration between attraction and repulsion which determines the minima of the effective potential...

The spherical ansatz

- Here's the resulting effective potential:

$$V_{\text{eff}} = \frac{2\pi T}{3}\Bigg[\frac{\ell_{xx}^2}{r_{xx}^2} + \frac{\ell_{xy}^2}{r_{xy}^2} + \frac{\ell_{xz}^2}{r_{xz}^2} + \frac{\ell_{yx}^2}{r_{yx}^2} + \frac{\ell_{yy}^2}{r_{yy}^2} + \frac{\ell_{yz}^2}{r_{yz}^2} + r_{xx}^2 r_{xy}^2 + r_{xy}^2 r_{xz}^2 + r_{xz}^2 r_{xx}^2 + r_{yx}^2 r_{yy}^2 + r_{yy}^2 r_{yz}^2 +$$

$$+ r_{yz}^2 r_{yx}^2 + r_{xx}^2\left(r_{yy}^2 + r_{yz}^2\right) + r_{xy}^2\left(r_{yz}^2 + r_{yx}^2\right) + r_{xz}^2\left(r_{yx}^2 + r_{yy}^2\right) + \frac{\mu^2}{9}\left(r_{xx}^2 + r_{xy}^2 + r_{xz}^2\right) +$$

$$+ \frac{\mu^2}{36}\left(r_{yx}^2 + r_{yy}^2 + r_{yz}^2\right) - 2\mu\,\epsilon_{ijk}\tilde{x}_{1i}\tilde{x}_{2j}\tilde{x}_{3k}\Bigg].$$

- The effective potential is made up of four basic types of terms: • (1) kinetic/angular momentum terms (repulsive), • (2) quartic interaction terms (attractive), • (3) mass terms (attractive), and • (4) cubic Myers terms (repulsive).

  The last two types of terms (i.e. mass terms and Myers terms) are μ-dependent and therefore absent from the flat space case ($\mu \to 0$), which was studied in Axenides-Floratos (2006)...

  In either case ($\mu = 0$ or $\mu \neq 0$), it is the equilibration between attraction and repulsion which determines the minima of the effective potential... Yet another interesting aspect of these systems is the existence of closed periodic orbits which do not correspond to critical points...

The spherical ansatz

- Here's the resulting effective potential:

$$V_{\text{eff}} = \frac{2\pi T}{3}\left[\frac{\ell_{xx}^2}{r_{xx}^2} + \frac{\ell_{xy}^2}{r_{xy}^2} + \frac{\ell_{xz}^2}{r_{xz}^2} + \frac{\ell_{yx}^2}{r_{yx}^2} + \frac{\ell_{yy}^2}{r_{yy}^2} + \frac{\ell_{yz}^2}{r_{yz}^2} + r_{xx}^2 r_{xy}^2 + r_{xy}^2 r_{xz}^2 + r_{xz}^2 r_{xx}^2 + r_{yx}^2 r_{yy}^2 + r_{yy}^2 r_{yz}^2 + \right.$$

$$+ r_{yz}^2 r_{yx}^2 + r_{xx}^2\left(r_{yy}^2 + r_{yz}^2\right) + r_{xy}^2\left(r_{yz}^2 + r_{yx}^2\right) + r_{xz}^2\left(r_{yx}^2 + r_{yy}^2\right) + \frac{\mu^2}{9}\left(r_{xx}^2 + r_{xy}^2 + r_{xz}^2\right) +$$

$$\left. + \frac{\mu^2}{36}\left(r_{yx}^2 + r_{yy}^2 + r_{yz}^2\right) - 2\mu\,\epsilon_{ijk}\tilde{x}_{1i}\tilde{x}_{2j}\tilde{x}_{3k}\right].$$

- As in flat spacetime, minimization of the effective potential leads to (dielectric) top solutions of the form:

$$\tilde{\mathbf{x}}_1(\tau) = e^{\Omega_{xx}\tau}\cdot\tilde{\mathbf{x}}_{10}, \qquad \tilde{\mathbf{x}}_2(\tau) = e^{\Omega_{xy}\tau}\cdot\tilde{\mathbf{x}}_{20}, \qquad \tilde{\mathbf{x}}_3(\tau) = e^{\Omega_{xz}\tau}\cdot\tilde{\mathbf{x}}_{30}$$

$$\tilde{\mathbf{y}}_1(\tau) = e^{\Omega_{yx}\tau}\cdot\tilde{\mathbf{y}}_{10}, \qquad \tilde{\mathbf{y}}_2(\tau) = e^{\Omega_{yy}\tau}\cdot\tilde{\mathbf{y}}_{20}, \qquad \tilde{\mathbf{y}}_3(\tau) = e^{\Omega_{yz}\tau}\cdot\tilde{\mathbf{y}}_{30}.$$

- We can identify 3 cases, based on the ways we can distribute the 3 spatial coordinates x_i into 3 groups:

  Case I: $x_1, x_2, x_3 \sim e_1$ Case II: $x_1, x_2 \sim e_1$ & $x_3 \sim e_3$ Case III: $x_1 \sim e_1,\ x_2 \sim e_2,\ x_3 \sim e_3$.

- In each case, we obtain a set of different effective potentials and (dielectric or not) membrane tops.

Case I: $q_1 = 3,\ q_2 = q_3 = 0$

- For $q_1 = 3,\ q_2 = q_3 = 0$ the spherical ansatz for the x-coordinates takes the form:

$$(x_1, x_2, x_3) = (\tilde{x}_1(\tau)\, e_1\,,\ \tilde{x}_2(\tau)\, e_1\,,\ \tilde{x}_3(\tau)\, e_1) \qquad \& \qquad r_x^2 \equiv \sum_{i=1}^{3} \tilde{x}_i(\tau)\, \tilde{x}_i(\tau)$$

$$(\ell_x)_{ij} \equiv \dot{\tilde{x}}_i(\tau)\, \tilde{x}_j(\tau) - \tilde{x}_i(\tau)\, \dot{\tilde{x}}_j(\tau)\,.$$

- The effective potential becomes:

$$V_{\text{eff}} = \frac{2\pi T}{3}\left[\frac{\ell_x^2}{r_x^2} + \frac{\ell_{yx}^2}{r_{yx}^2} + \frac{\ell_{yy}^2}{r_{yy}^2} + \frac{\ell_{yz}^2}{r_{yz}^2} + r_{yx}^2 r_{yy}^2 + r_{yy}^2 r_{yz}^2 + r_{yz}^2 r_{yx}^2 + r_x^2\left(r_{yy}^2 + r_{yz}^2\right) + \frac{\mu^2 r_x^2}{9} + \right.$$

$$\left. + \frac{\mu^2}{36}\left(r_{yx}^2 + r_{yy}^2 + r_{yz}^2\right)\right].$$

- Completely symmetric (single-radius) configuration: $r = r_x = r_{yx} = r_{yy} = r_{yz}\,,\ \ell = \ell_x = \ell_{yx} = \ell_{yy} = \ell_{yz}$.
- There are 5 different axially symmetric (2-radii) configurations.
- There are 4 configurations with 3 different radii.

Case II: $q_1 = 2,\ q_2 = 0,\ q_3 = 1$

- For $q_1 = 2,\ q_2 = 0,\ q_3 = 1$ our ansatz is written:

$$(x_1, x_2, x_3) = (\tilde{x}_{11}(\tau)\, e_1\,,\ \tilde{x}_{12}(\tau)\, e_1\,,\ r_{xz}(\tau)\, e_3) \qquad \& \qquad r_{xx}^2 \equiv \tilde{x}_{11}^2(\tau) + \tilde{x}_{12}^2(\tau)$$

$$(\ell_{xx})_{ij} \equiv \dot{\tilde{x}}_{1i}(\tau)\, \tilde{x}_{1j}(\tau) - \tilde{x}_{1i}(\tau)\, \dot{\tilde{x}}_{1j}(\tau)\,.$$

- The effective potential is given by:

$$V_{\rm eff} = \frac{2\pi T}{3}\left[\frac{\ell_{xx}^2}{r_{xx}^2} + \frac{\ell_{yx}^2}{r_{yx}^2} + \frac{\ell_{yy}^2}{r_{yy}^2} + \frac{\ell_{yz}^2}{r_{yz}^2} + r_{xx}^2 r_{xz}^2 + r_{yx}^2 r_{yy}^2 + r_{yy}^2 r_{yz}^2 + r_{yz}^2 r_{yx}^2 + r_{xx}^2\left(r_{yy}^2 + r_{yz}^2\right) + \right.$$

$$\left. + r_{xz}^2\left(r_{yz}^2 + r_{yx}^2\right) + \frac{\mu^2}{9}\left(r_{xx}^2 + r_{xz}^2\right) + \frac{\mu^2}{36}\left(r_{yx}^2 + r_{yy}^2 + r_{yz}^2\right)\right].$$

- Completely symmetric configuration: $r = r_{xx} = r_{xz} = r_{yx} = r_{yy} = r_{yz},\ \ell = \ell_{xx} = \ell_{yx} = \ell_{yy} = \ell_{yz}$.
- There are 13 different axially symmetric (2-radii) configurations.
- There exist 21 three-radii configurations.

Example 1

- Take for example a type II configuration with all the $SO(6)$ variables y_i set to zero:

$$x_1 = x(\tau) \cdot e_1, \quad x_2 = y(\tau) \cdot e_1, \quad x_3 = z(\tau) \cdot e_2 \qquad \& \qquad y_i = 0, \quad i = 1, \ldots, 6,$$

The corresponding effective potential reads,

$$V_{\text{eff}} = \frac{2\pi T}{3} \left[\frac{\ell^2}{x^2 + y^2} + (x^2 + y^2)\, z^2 + \frac{\mu^2}{9} (x^2 + y^2 + z^2) \right],$$

where we have set $\ell_{x1} = \ell$. The corresponding minimization condition $\nabla V_{\text{eff}} = 0$ leads to

$$x\, z^2 + \frac{\mu^2 x}{9} - \frac{x\, \ell^2}{(x^2 + y^2)^2} = y\, z^2 + \frac{\mu^2 y}{9} - \frac{y\, \ell^2}{(x^2 + y^2)^2} = z\,(x^2 + y^2) + \frac{\mu^2 z}{9} = 0,$$

which has the following solution

$$x^2 + y^2 = \frac{3\ell}{\mu} \qquad \& \qquad z = 0.$$

To agree with the form of the above ansatz we can choose, for instance,

$$x(\tau) = \sqrt{\frac{3\ell}{\mu}} \cos\frac{\mu\tau}{3}, \qquad y(\tau) = \sqrt{\frac{3\ell}{\mu}} \sin\frac{\mu\tau}{3}, \qquad z(\tau) = 0.$$

Example 1

- Alternatively we could have directly inserted the ansatz into the light-cone equations of motion,

$$\ddot{x}\cdot e_1 = -x\,z^2\cdot e_1 - \frac{\mu^2 x}{9}\cdot e_1 + \mu\,y\,z\cdot e_3$$

$$\ddot{y}\cdot e_1 = -y\,z^2\cdot e_1 - \frac{\mu^2 y}{9}\cdot e_1 + \mu\,x\,z\cdot e_3$$

$$\ddot{z}\cdot e_2 = -z\left(x^2+y^2\right)\cdot e_2 - \frac{\mu^2 z}{9}\cdot e_2,$$

from which it can be seen that any solution of the type

$$\tilde{\mathbf{x}}_1(\tau) = e^{\Omega_{xx}\tau}\cdot\tilde{\mathbf{x}}_{10}, \qquad \tilde{\mathbf{x}}_2(\tau) = e^{\Omega_{xy}\tau}\cdot\tilde{\mathbf{x}}_{20}, \qquad \tilde{\mathbf{x}}_3(\tau) = e^{\Omega_{xz}\tau}\cdot\tilde{\mathbf{x}}_{30},$$

will satisfy

$$x^2+y^2 = \frac{3\ell}{\mu} \qquad \& \qquad z=0.$$

Example 2

- Another interesting type II solution is the following:

$$x_1 = x\,(\tau)\cdot e_1\,, \quad x_2 = y\,(\tau)\cdot e_2\,, \quad x_3 = 0 \qquad \& \qquad y_i = 0, \quad i = 1,\ldots,6,$$

where again all the $SO\,(6)$ variables y_i and the $SO(3)$ coordinate x_2 are zero... The effective potential becomes,

$$V_{\text{eff}} = \frac{2\pi T}{3}\left[x^2 y^2 + \frac{\mu^2}{9}\left(x^2 + y^2\right)\right],$$

so that there is only one trivial critical point at $x = y = 0$, which is obtained by minimizing the effective potential:

$$x\,y^2 + \frac{\mu^2 x}{9} = y\,x^2 + \frac{\mu^2 y}{9} = 0.$$

Potentials of the above form (which are in fact generalizations of the YM potential $x^2y^2/2$) have a very interesting and rich set of (stable) periodic orbits... See e.g. Contopoulos-Harsoula (2023)...

Case III: $q_1 = q_2 = q_3 = 1$

- For $q_1 = q_2 = q_3 = 1$ the spherical ansatz becomes:

$$(x_1, x_2, x_3) = \left(r_{xx}(\tau)\, e_1\,,\ \ r_{xy}(\tau)\, e_2\,,\ \ r_{xz}(\tau)\, e_3\right).$$

- In this case the effective potential reads:

$$V_{\text{eff}} = \frac{2\pi T}{3}\Bigg[\frac{\ell_{yx}^2}{r_{yx}^2} + \frac{\ell_{yy}^2}{r_{yy}^2} + \frac{\ell_{yz}^2}{r_{yz}^2} + r_{xx}^2 r_{xy}^2 + r_{xy}^2 r_{xz}^2 + r_{xz}^2 r_{xx}^2 + r_{yx}^2 r_{yy}^2 + r_{yy}^2 r_{yz}^2 + r_{yz}^2 r_{yx}^2 +$$

$$+ r_{xx}^2\left(r_{yy}^2 + r_{yz}^2\right) + r_{xy}^2\left(r_{yz}^2 + r_{yx}^2\right) + r_{xz}^2\left(r_{yx}^2 + r_{yy}^2\right) + \frac{\mu^2}{9}\left(r_{xx}^2 + r_{xy}^2 + r_{xz}^2\right) +$$

$$+ \frac{\mu^2}{36}\left(r_{yx}^2 + r_{yy}^2 + r_{yz}^2\right) - 2\mu r_{xx} r_{xy} r_{xz}\Bigg].$$

- Completely symmetric configuration: $r = r_{xx} = r_{xy} = r_{xz} = r_{yx} = r_{yy} = r_{yz}$, $\ell = \ell_{yx} = \ell_{yy} = \ell_{yz}$.
- There are 9 different axially symmetric (2-radii) configurations.
- There are 10 configurations with 3 different radii.

An example

- Consider the following (out the 9 in total) axially symmetric configuration of case III:

$$r_x = r_{xx} = r_{xy} = r_{xz} \qquad \& \qquad r_y = r_{yx} = r_{yy} = r_{yz} \qquad \& \qquad \ell_y = \ell_{yx} = \ell_{yy} = \ell_{yz}$$

with effective potential:

$$V_{\text{eff}} = 2\pi T\left[\frac{\ell_y^2}{r_y^2} + r_x^4 + 2r_x^2 r_y^2 + \frac{\mu^2 r_x^2}{9} + \frac{\mu^2 r_y^2}{36} - \frac{2\mu}{3} r_x^3\right].$$

- The minimization conditions read:

$$\frac{dV_{\text{eff}}}{dr_{r_x}} = r_x\left(r_x^2 - \frac{\mu}{2} r_x + r_y^2 + \frac{\mu^2}{18}\right) = \frac{dV_{\text{eff}}}{dr_{r_y}} = r_y^6 + \left(r_x^2 + \frac{\mu^2}{72}\right) r_y^4 - \frac{\ell_y^2}{2} = 0.$$

- We obtain the following selection rule:

$$r_y \leq \frac{\mu}{12} \qquad \& \qquad r_x \geq \frac{144^2 \ell_y^2}{\mu^5} + \frac{\mu}{12}$$

e.g. in the marginal case $r_x = \mu/4$, $r_y = \mu/12$, $\ell_y = \mu^3/144\sqrt{6}$, it's $V_{\text{eff(min)}} = 7\pi T\mu^4/1296$.

- We also find the static solutions $r_y = 0$, $r_x = \mu/3$ (BPS) and $r_x = \mu/6$ (for which $V_{\text{eff(min)}} = \pi T\mu^4/648$)...

Section 4

Static dielectric membranes in $SO(3)$

M. Axenides, E. Floratos, GL
M2-brane dynamics in the classical limit of the BMN matrix model
PLB **773** (2017) 265 [arxiv:1707.02878]

M. Axenides, E. Floratos, GL
Multipole stability of spinning M2-branes in the classical limit of the BMN matrix model
PRD **97** (2018) 126019 [arxiv:1712.06544]

M. Axenides, E. Floratos, GL
to appear

The $SO(3)$ solution

Setting all $SO(6)$ coordinates of type III configurations to zero and switching to dimensionless time $t \equiv \mu\tau$,

$$x_i = \mu u_i e_i, \quad i = 1,2,3 \qquad \& \qquad y_i = \mu v_i = 0, \quad i = 1,\ldots 6,$$

the membrane equations of motion become:

$$\ddot{u}_1 + \left(u_2^2 + u_3^2 + \frac{1}{9}\right) u_1 = u_2 u_3, \qquad \ddot{u}_2 + \left(u_1^2 + u_3^2 + \frac{1}{9}\right) u_2 = u_1 u_3, \qquad \ddot{u}_3 + \left(u_1^2 + u_2^2 + \frac{1}{9}\right) u_3 = u_1 u_2$$

$$\ddot{v}_i = 0, \quad i = 1,\ldots,6.$$

The dynamics is fully specified in terms of the Hamiltonian...

$$H = \frac{4\pi T \mu^4}{3} \cdot \mathcal{H}, \quad \mathcal{H} \equiv \frac{1}{2}\left[p_1^2 + p_2^2 + p_3^2 + u_1^2 u_2^2 + u_2^2 u_3^2 + u_1^2 u_3^2 + \frac{1}{9}\left(u_1^2 + u_2^2 + u_3^2\right) - 2u_1 u_2 u_3\right],$$

and Hamilton's equations of motion:

$$p_i = \dot{u}_i, \qquad \dot{p}_i = -\frac{\partial \mathcal{H}}{\partial u_i},$$

which evidently imply the above Lagrangian equations of motion...

The $SO(3)$ solution

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$$x_i = \mu u_i e_i, \quad i = 1,2,3 \qquad \& \qquad y_i = \mu v_i = 0, \quad i = 1, \ldots 6,$$

the membrane equations of motion become:

$$\ddot{u}_1 + \left(u_2^2 + u_3^2 + \frac{1}{9}\right) u_1 = u_2 u_3, \qquad \ddot{u}_2 + \left(u_1^2 + u_3^2 + \frac{1}{9}\right) u_2 = u_1 u_3, \qquad \ddot{u}_3 + \left(u_1^2 + u_2^2 + \frac{1}{9}\right) u_3 = u_1 u_2$$

$$\ddot{v}_i = 0, \quad i = 1, \ldots, 6.$$

The effective potential energy of the static membrane is given by

$$V_{\text{eff}} = \frac{2\pi T \mu^4}{3}\left[\left(u_1^2 u_2^2 + u_2^2 u_3^2 + u_1^2 u_3^2\right) + \frac{1}{9}\left(u_1^2 + u_2^2 + u_3^2\right) - 2u_1 u_2 u_3\right].$$

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$$x_i = \mu u_i e_i, \quad i = 1,2,3 \qquad \& \qquad y_i = \mu v_i = 0, \quad i = 1,\ldots 6,$$

the membrane equations of motion become:

$$\ddot{u}_1 + \left(u_2^2 + u_3^2 + \frac{1}{9}\right) u_1 = u_2 u_3, \qquad \ddot{u}_2 + \left(u_1^2 + u_3^2 + \frac{1}{9}\right) u_2 = u_1 u_3, \qquad \ddot{u}_3 + \left(u_1^2 + u_2^2 + \frac{1}{9}\right) u_3 = u_1 u_2$$

$$\ddot{v}_i = 0, \quad i = 1,\ldots,6.$$

The effective potential energy of the static membrane is given by

$$V_{\text{eff}} = \frac{2\pi T\mu^4}{3}\left[\left(u_1^2 u_2^2 + u_2^2 u_3^2 + u_1^2 u_3^2\right) + \frac{1}{9}\left(u_1^2 + u_2^2 + u_3^2\right) - 2u_1 u_2 u_3\right].$$

This potential turns out to be a special case of the so-called generalized 3-dimensional Hénon-Heiles potential,

$$V_{\text{HH}} = \frac{1}{2}\left(u_1^2 + u_2^2 + u_3^2\right) + K_3\, u_1 u_2 u_3 + K_0\left(u_1^2 + u_2^2 + u_3^2\right)^2 + K_4\left(u_1^4 + u_2^4 + u_3^4\right) \quad \text{(Efstathiou-Sadovskii, 2004)}.$$

For $K_3 = -9$, $K_0 = -K_4 = 9/4$, V_{HH} obviously reduces to the above effective potential V_{eff}.

$SO(3)$ extrema

The extrema of the potential solve the equilibrium conditions:

$$\partial_i V_{\text{eff}} = 0 \Rightarrow \left(u_2^2 + u_3^2 + \frac{1}{9}\right) u_1 = u_2 u_3$$

$$\left(u_3^2 + u_1^2 + \frac{1}{9}\right) u_2 = u_3 u_1$$

$$\left(u_1^2 + u_2^2 + \frac{1}{9}\right) u_3 = u_1 u_2 .$$

Here are the corresponding roots:

$$\mathbf{u}_0 = 0, \qquad \mathbf{u}_{1/6} = \frac{1}{6} \cdot (\pm 1, \pm 1, \pm 1) , \qquad \mathbf{u}_{1/3} = \frac{1}{3} \cdot (\pm 1, \pm 1, \pm 1) ,$$

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The extrema of the potential solve the equilibrium conditions:

$$\partial_i V_{\text{eff}} = 0 \Rightarrow \left(u_2^2 + u_3^2 + \frac{1}{9}\right) u_1 = u_2 u_3$$

$$\left(u_3^2 + u_1^2 + \frac{1}{9}\right) u_2 = u_3 u_1$$

$$\left(u_1^2 + u_2^2 + \frac{1}{9}\right) u_3 = u_1 u_2 .$$

Here are the corresponding roots:

$$\mathbf{u}_0 = 0, \qquad \mathbf{u}_{1/6} = \frac{1}{6} \cdot (\pm 1, \pm 1, \pm 1)\,, \qquad \mathbf{u}_{1/3} = \frac{1}{3} \cdot (\pm 1, \pm 1, \pm 1)\,,$$

- The extrema are nine in total because the product of their components must be non-negative...

$SO(3)$ extrema

The extrema of the potential solve the equilibrium conditions:

$$\partial_i V_{\text{eff}} = 0 \Rightarrow \left(u_2^2 + u_3^2 + \frac{1}{9}\right) u_1 = u_2 u_3$$

$$\left(u_3^2 + u_1^2 + \frac{1}{9}\right) u_2 = u_3 u_1$$

$$\left(u_1^2 + u_2^2 + \frac{1}{9}\right) u_3 = u_1 u_2 .$$

Here are the corresponding roots:

$$\mathbf{u}_0 = 0, \qquad \mathbf{u}_{1/6} = \frac{1}{6} \cdot (\pm 1, \pm 1, \pm 1) , \qquad \mathbf{u}_{1/3} = \frac{1}{3} \cdot (\pm 1, \pm 1, \pm 1) ,$$

- The extrema are nine in total because the product of their components must be non-negative...
- V_{eff} has the symmetry of a tetrahedron T_d that is generated by the 4 critical points $\mathbf{u}_{1/3}$ and $\mathbf{u}_{1/6}$...

$SO(3)$ extrema

The extrema of the potential solve the equilibrium conditions:

$$\partial_i V_{\text{eff}} = 0 \Rightarrow \left(u_2^2 + u_3^2 + \frac{1}{9}\right) u_1 = u_2 u_3$$

$$\left(u_3^2 + u_1^2 + \frac{1}{9}\right) u_2 = u_3 u_1$$

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- The extrema are nine in total because the product of their components must be non-negative...
- V_{eff} has the symmetry of a tetrahedron T_d that is generated by the 4 critical points $\mathbf{u}_{1/3}$ and $\mathbf{u}_{1/6}$...
- $\mathbf{u}_0$ (point-like membrane) and $\mathbf{u}_{1/3}$ (Myers dielectric sphere) are global minima, while $\mathbf{u}_{1/6}$ is a saddle point...

$SO(3)$ extrema

- The value of the effective potential at the extremal points is

$$V_{\text{eff}}(0) = V_{\text{eff}}\left(\frac{1}{3}\right) = 0, \qquad V_{\text{eff}}\left(\frac{1}{6}\right) = \frac{2\pi T \mu^4}{6^4}.$$

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- When the u_i in are not all equal, the equations of motion are so complicated that exact time-dependent solutions can only be found numerically...
- When all the $SO(3)$ membrane coordinates u_i are equal, an analytic solution is possible...

Subsection 2

Spherically symmetric membrane

The spherically symmetric membrane

- The ansatz for the fully symmetric membrane in $SO(3)$ reads:

$$u \equiv u_1 = u_2 = u_3.$$

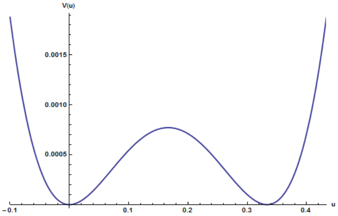
- The membrane Hamiltonian is that of a double-well oscillator:

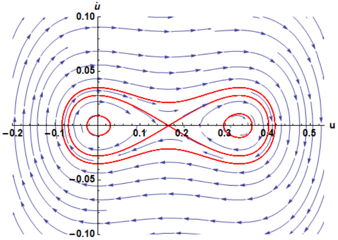
$$H = 2\pi T \mu^4 \left[p^2 + u^2 \left(u - \frac{1}{3} \right)^2 \right].$$

- Here are the corresponding equations of motion:

$$\dot{u} = p, \qquad \dot{p} = -u \left[2u^2 + \frac{1}{9} - u \right].$$

- Define $\mathcal{E} \equiv E/2\pi T \mu^4$, $\mathcal{E}_c \equiv 6^{-4}$. There are three kinds of orbits: • (1) oscillations of small energies ($\mathcal{E} < \mathcal{E}_c$) around either of the two stable global minima ($u_0 = 0, 1/3$) • (2) oscillations of larger energies ($\mathcal{E} > \mathcal{E}_c$) around the local maximum ($u_0 = 1/6$) • (3) two homoclinic orbits through the unstable equilibrium point at $u_0 = 1/6$ with energy equal to the potential height ($\mathcal{E} = \mathcal{E}_c$).





The spherically symmetric membrane

- The orbits may be computed from the initial conditions:

$$\dot{u}_0(0) = 0, \qquad u_0(0) = \frac{1}{6} \pm \sqrt{\frac{1}{6^2} + \sqrt{\mathcal{E}}},$$

where the $\pm$ signs correspond to the right/left side of the double-well potential.

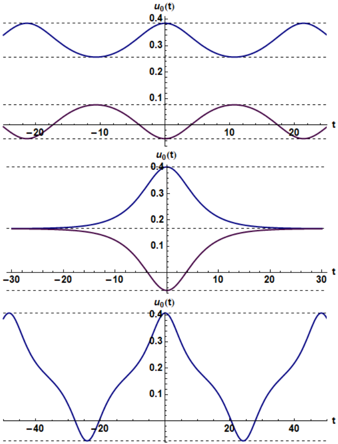
- Integrating the energy integral we find the solution:

$$u_0(t) = \frac{1}{6} \pm \sqrt{\frac{1}{6^2} + \sqrt{\mathcal{E}}} \cdot cn\left[\sqrt{2\sqrt{\mathcal{E}}} \cdot t \middle| \frac{1}{2}\left(1 + \frac{1}{36\sqrt{\mathcal{E}}}\right)\right],$$

where only the plus sign should be kept for $\mathcal{E} \geq \mathcal{E}_c$.

- For the critical energy $\mathcal{E} = \mathcal{E}_c$ the homoclinic orbit is obtained:

$$u_0(t) = \frac{1}{6} \pm \frac{1}{3\sqrt{2}} \cdot \text{sech}\left(\frac{t}{3\sqrt{2}}\right).$$

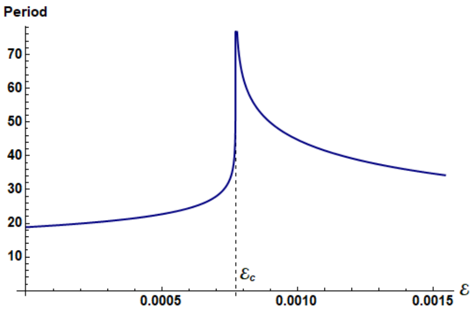


The spherically symmetric membrane

- The period as a function of the energy is given in terms of the complete elliptic integral of the first kind:

$$T(\mathcal{E}) = 2\sqrt{\frac{2}{\sqrt{\mathcal{E}}}} \cdot \mathbf{K}\left(\frac{1}{2}\left(1 + \frac{1}{36\sqrt{\mathcal{E}}}\right)\right),$$

it becomes infinite for the homoclinic orbit $\mathcal{E} = \mathcal{E}_c$. For more, see e.g. Brizard-Westland (2017).



Subsection 3

Leading order stability

Leading order stability analysis

The full type III equations of motion around each of the $SO(3)$ extremal points read:

$$\ddot{u}_1 + \left(u_2^2 + u_3^2 + \frac{r_{y2}^2}{\mu^2} + \frac{r_{y3}^2}{\mu^2} + \frac{1}{9}\right) u_1 = u_2 u_3, \quad \ddot{v}_i + \left(\frac{r_{y2}^2}{\mu^2} + \frac{r_{y3}^2}{\mu^2} + u_2^2 + u_3^2 + \frac{1}{36}\right) v_i = 0$$

$$\ddot{u}_2 + \left(u_3^2 + u_1^2 + \frac{r_{y3}^2}{\mu^2} + \frac{r_{y1}^2}{\mu^2} + \frac{1}{9}\right) u_2 = u_3 u_1, \quad \ddot{v}_j + \left(\frac{r_{y3}^2}{\mu^2} + \frac{r_{y1}^2}{\mu^2} + u_3^2 + u_1^2 + \frac{1}{36}\right) v_j = 0$$

$$\ddot{u}_3 + \left(u_1^2 + u_2^2 + \frac{r_{y1}^2}{\mu^2} + \frac{r_{y2}^2}{\mu^2} + \frac{1}{9}\right) u_3 = u_1 u_2, \quad \ddot{v}_k + \left(\frac{r_{y1}^2}{\mu^2} + \frac{r_{y2}^2}{\mu^2} + v_1^2 + v_2^2 + \frac{1}{36}\right) v_k = 0,$$

where we have set $t \equiv \mu\tau$ and

$$x_i = \mu u_i e_i, \quad i = 1,2,3 \qquad \& \qquad \begin{aligned} y_i &= \mu v_i e_1, \quad i = 1, \ldots, s_1 \\ y_j &= \mu v_j e_2, \quad j = s_1 + 1, \ldots, s_1 + s_2 \\ y_k &= \mu v_k e_3, \quad k = s_1 + s_2 + 1, \ldots, s_1 + s_2 + s_3. \end{aligned}$$

Radial stability analysis

- By radially perturbing each of the 9 critical points as:

$$u_i = u_i^0 + \delta u_i(t), \quad i = 1, 2, 3, \qquad \& \qquad v_j = \delta v_j(t), \quad j = 1, \dots, 6,$$

we obtain the following system of fluctuation equations

$$\delta \ddot{\mathbf{u}} = -\begin{bmatrix} 2u_0^2 + \frac{1}{9} & 2u_1^0 u_2^0 - u_3^0 & 2u_1^0 u_3^0 - u_2^0 \\ 2u_2^0 u_1^0 - u_3^0 & 2u_0^2 + \frac{1}{9} & 2u_2^0 u_3^0 - u_1^0 \\ 2u_3^0 u_1^0 - u_2^0 & 2u_3^0 u_2^0 - u_1^0 & 2u_0^2 + \frac{1}{9} \end{bmatrix} \cdot \delta \mathbf{u} \quad \& \quad \delta \ddot{\mathbf{v}} = -\left(2u_0^2 + \frac{1}{36}\right) \cdot \delta \mathbf{v},$$

where we have defined

$$u_0^2 \equiv \left(u_1^0\right)^2 = \left(u_2^0\right)^2 = \left(u_3^0\right)^2,$$

for the common value of the square of each extremum's components. Then we plug the particular solution

$$\begin{bmatrix} \delta \mathbf{u} \\ \delta \mathbf{v} \end{bmatrix} = e^{\lambda t} \boldsymbol{\xi},$$

we solve the resulting eigenvalue/eigenvector problem...

Radial stability analysis

- By radially perturbing each of the 9 critical points as:

$$u_i = u_i^0 + \delta u_i(t), \quad i = 1, 2, 3, \qquad \& \qquad v_j = \delta v_j(t), \quad j = 1, \ldots, 6.$$

This way we confirm the conclusion we derived above from the corresponding Hessian matrix, i.e. that $\mathbf{u}_0$ and $\mathbf{u}_{1/3}$ are global minima (positive-definite Hessian) and $\mathbf{u}_{1/6}$ is a saddle point (indefinite Hessian):

extremum	location	eigenvalues $r = \lambda^2$ (degeneracy)	stability
$\mathbf{u}_0$	0	$-\frac{1}{9}$ (3), $-\frac{1}{36}$ (6)	center (stable)
$\mathbf{u}_{1/6}$	$\left(\pm\frac{1}{6}, \pm\frac{1}{6}, \pm\frac{1}{6}\right)$	$\frac{1}{18}$ (1), $-\frac{5}{18}$ (2), $-\frac{1}{12}$ (6)	saddle point
$\mathbf{u}_{1/3}$	$\left(\pm\frac{1}{3}, \pm\frac{1}{3}, \pm\frac{1}{3}\right)$	$-\frac{1}{9}$ (1), $-\frac{4}{9}$ (2), $-\frac{1}{4}$ (6)	center (stable)

Axenides-Floratos-GL (2017a)

where the negative eigenvalues $r = \lambda^2 < 0$ correspond to stable directions and the positive eigenvalues $r = \lambda^2 > 0$ lead to stable/unstable directions (depending on the sign of the real eigenvalue λ)...

Angular stability analysis

- We may also perform more general (angular/multipole) perturbations of the following form:

$$x_i(t) = x_i^0 + \delta x_i(t), \qquad i = 1, 2, 3,$$

where δx_i is expanded in spherical harmonics $Y_{jm}(\theta, \phi)$:

$$x_i(t) = \mu u_i(t) e_i, \qquad x_i^0 = \mu u_i^0 e_i, \qquad \delta x_i(t) = \mu \cdot \sum_{j=1}^{\infty} \sum_{m=-j}^{j} \eta_i^{jm}(t) Y_{jm}(\theta, \phi).$$

- For the critical points $\mathbf{u}_0$, $\mathbf{u}_{1/6}$, $\mathbf{u}_{1/3}$ we find the eigenvalues (Axenides-Floratos-GL, 2017b):

$$\mathbf{u}_0: \quad \lambda_P^2 = \lambda_\pm^2 = -\frac{1}{9}, \qquad \lambda_\theta^2 = -\frac{1}{36}$$

$$\mathbf{u}_{1/6}: \lambda_P^2 = 0, \qquad \lambda_+^2 = -\frac{1}{36}(j+1)(j+4), \qquad \lambda_-^2 = -\frac{j(j-3)}{36}, \qquad \lambda_\theta^2 = -\frac{1}{36}(j^2+j+1)$$

$$\mathbf{u}_{1/3}: \lambda_P^2 = 0, \qquad \lambda_+^2 = -\frac{1}{36}(j+1)^2, \qquad \lambda_-^2 = -\frac{j^2}{9}, \qquad \lambda_\theta^2 = -\frac{1}{36}(2j+1)^2,$$

with multiplicities $d_P = 2j+1$, $d_+ = 2j+3$, $d_- = 2j-1$ and $d_\theta = 6(2j+1)$, respectively.

Angular stability analysis

- The critical point $\mathbf{u}_0$ (point-like membrane) is obviously stable.
- $\mathbf{u}_{1/3}$ has a zero mode of degeneracy $2d_P$ while all its other eigenvalues are stable for $j = 1, 2, \ldots$
- $\mathbf{u}_{1/6}$ has one $2d_P$-degenerate zero mode for every j and a 10-fold degenerate zero mode for $j = 3$. It is unstable for $j = 1$ (2-fold degenerate) and $j = 2$ (6-fold degenerate).
- The above results were first obtained by (Dasgupta, Sheikh-Jabbari, Van Raamsdonk (2002)) from the BMN matrix model point-of-view.
- In the flat-space limit ($\mu \to 0$), we recover the results of (Axenides-Floratos-Perivolaropoulos, 2000, 2001).

Section 5

The $SO(3) \times SO(6)$ symmetric membrane

M. Axenides, E. Floratos, GL
M2-brane dynamics in the classical limit of the BMN matrix model
PLB **773** (2017) 265 [arxiv:1707.02878]

M. Axenides, E. Floratos, GL
Multipole stability of spinning M2-branes in the classical limit of the BMN matrix model
PRD **97** (2018) 126019 [arxiv:1712.06544]

M. Axenides, E. Floratos, GL
to appear

The $SO(3) \times SO(6)$ sector

- Similar analysis can be carried out in the $SO(3) \times SO(6)$ sector where the equations of motion become:

$$\ddot{u}_1 + \left(u_2^2 + u_3^2 + \frac{r_{y2}^2}{\mu^2} + \frac{r_{y3}^2}{\mu^2} + \frac{1}{9}\right) u_1 = u_2 u_3, \quad \ddot{v}_i + \left(\frac{r_{y2}^2}{\mu^2} + \frac{r_{y3}^2}{\mu^2} + u_2^2 + u_3^2 + \frac{1}{36}\right) v_i = 0, \quad i, j, k = 1, 2, 3,$$

$$\ddot{u}_2 + \left(u_3^2 + u_1^2 + \frac{r_{y3}^2}{\mu^2} + \frac{r_{y1}^2}{\mu^2} + \frac{1}{9}\right) u_2 = u_3 u_1, \quad \ddot{v}_j + \left(\frac{r_{y3}^2}{\mu^2} + \frac{r_{y1}^2}{\mu^2} + u_3^2 + u_1^2 + \frac{1}{36}\right) v_j = 0$$

$$\ddot{u}_3 + \left(u_1^2 + u_2^2 + \frac{r_{y1}^2}{\mu^2} + \frac{r_{y2}^2}{\mu^2} + \frac{1}{9}\right) u_3 = u_1 u_2, \quad \ddot{v}_k + \left(\frac{r_{y1}^2}{\mu^2} + \frac{r_{y2}^2}{\mu^2} + v_1^2 + v_2^2 + \frac{1}{36}\right) v_k = 0.$$

- A solution of these equations of motion is

$$u_i^0 = u_0, \qquad v_j^0(t) = v_0 \cos\left(\omega t + \varphi_j\right), \qquad w_j^0(t) \equiv v_{j+3}^0(t) = v_0 \sin\left(\omega t + \varphi_k\right),$$

where (u_0, v_0) are the critical points of the corresponding (axially symmetric) potential

$$V \equiv \frac{V_{\text{eff}}}{2\pi T \mu^4} = u^4 + 2u^2 v^2 + v^4 + \frac{u^2}{9} + \frac{v^2}{36} - \frac{2u^3}{3} + \frac{\ell^2}{v^2}, \qquad \ell\mu^3 \equiv \ell_1 = \ell_2 = \ell_3.$$

- It can be proven that the critical points (u_0, v_0) always lie within the interval:

$$\frac{1}{6} \leq u_0 \leq \frac{1}{3} \qquad \& \qquad 0 \leq v_0 \leq \frac{1}{12}.$$

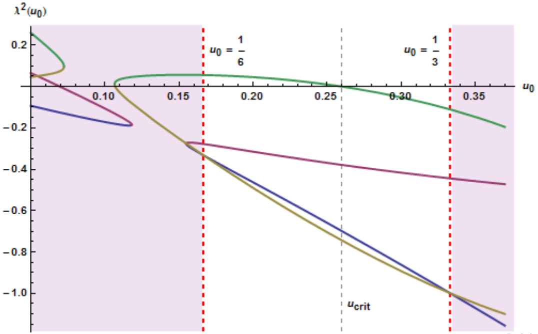
Radial stability analysis

- Setting,

$$u_i = u_i^0 + \delta u_i\,(t)\,, \qquad v_i = v_i^0\,(t) + \delta v_i'\,(t)\,, \qquad w_i = w_i^0\,(t) + \delta w_i'\,(t)\,,$$

we find the radial eigenvalues

$$\lambda_{1\pm}^2 = \frac{1}{9} - \frac{5u_0}{2} \pm \sqrt{\frac{1}{9^2} - \frac{u_0}{9} - \frac{5u_0^2}{12} + 4u_0^3}, \qquad \lambda_{2\pm}^2 = \frac{5}{18} - \frac{5u_0}{2} \pm \sqrt{\frac{5^2}{18^2} - \frac{35u_0}{18} + \frac{163u_0^2}{12} - 20u_0^3}.$$



Angular stability analysis

- Going further, we again set out to perform angular/multipole perturbations of the form:

$$x_i = x_i^0 + \delta x_i, \quad i = 1,2,3 \qquad \& \qquad y_k = y_k^0 + \delta y_k, \quad k = 1,\ldots,6,$$

where the $\delta x_i,\ \delta y_k$ are expanded in spherical harmonics $Y_{jm}(\theta,\phi)$ as:

$$\delta x_i = \mu \cdot \sum_{j,m} \eta_i^{jm}(\tau)\, Y_{jm}(\theta,\phi) \qquad \delta y_k = \mu \cdot \sum_{j,m} \epsilon_k^{jm}(\tau)\, Y_{jm}(\theta,\phi) \qquad \delta y_l = \mu \cdot \sum_{j,m} \zeta_l^{jm}(\tau)\, Y_{jm}(\theta,\phi),$$

and $i = 1,2,3,\ k = 1,3,5$ and $l = 2,4,6$.

- One of the eigenvalues always vanishes, two others are given by the following analytic expression

$$\lambda_P^2 = \frac{1}{2}\left(j^2+j+2\right)u_0 - \frac{1}{18}\left(1 + j\,(j+1) \pm 3\sqrt{144\left(j^2+j-2\right)u_0^3 - 12\left(j^2+j-14\right)u_0^2 - 24u_0 + 1}\right),$$

while 6 more eigenvalues $\lambda_\pm$ are also known in closed forms but are too complicated to be included here.

- The corresponding multiplicities of the eigenvalues are $d_P = 2j+1,\ d_+ = 2j+3,\ d_- = 2j-1$.

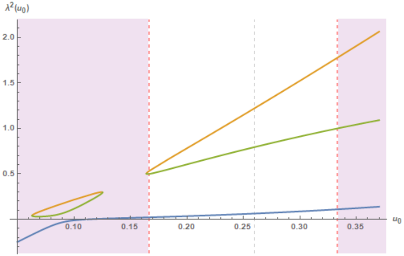
Lowest-lying modes

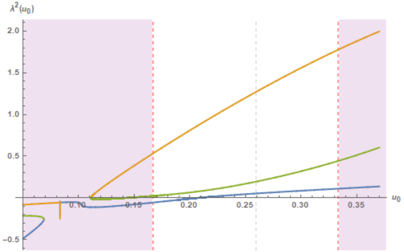
- For $j = 1$ four eigenvalues vanish, while two others coincide with those found from radial perturbations:

$$\lambda_P^2 = 4u_0 + \frac{1}{3}, \qquad \lambda_+^2 = \frac{5u_0}{2} - \frac{1}{9} \pm \sqrt{\frac{1}{9^2} - \frac{u_0}{9} - \frac{5u_0^2}{12} + 4u_0^3}$$

$$\lambda_-^2 = \frac{5u_0}{2} - \frac{5}{18} \pm \sqrt{\frac{5^2}{18^2} - \frac{35u_0}{18} + \frac{163u_0^2}{12} - 20u_0^3}.$$

- For $j = 2$ there's one zero eigenvalue while $\lambda_P > 0$. We can also plot the eigenvalues of $\lambda_\pm$:





Multipole stability

- The nonzero $j = 1$ eigenvalues are all positive/stable in the interval $\frac{1}{6} \leq u_0 \leq \frac{1}{3}$, $0 \leq v_0 \leq \frac{1}{12}$, except $\lambda^2_{-(-)}$ which is positive/stable only for $u_{\text{crit}} < u_0 < 1/3$, where $u_{\text{crit}} \equiv \frac{1}{60}\left(11 + \sqrt{21}\right)$.
- For $j = 2$, the λ_P, λ_+ and one of the λ_- eigenvalues are positive/stable. The remaining λ_- eigenvalue is negative/unstable in the interval $\frac{1}{6} \leq u_0 \leq 0.207245 < u_{\text{crit}}$.
- Here's a summary of the angular/multipole spectrum:

eigenvalues	$j = 1$	$j = 2$	$j \geq 3$	degeneracy
λ^2_P	$0, 0, +$	$0, +, +$	$0, +, +$	$d_P = 2j + 1$
λ^2_+	$0, +, +$	$+, +, +$	$+, +, +$	$d_+ = 2j + 3$
λ^2_-	$0, +, \{0, \pm\}$ (positive for $u_0 > u_{\text{crit}}$)	$+, +, \{0, \pm\}$ (positive for $u_0 > 0.207245$)	$+, +, +$	$d_- = 2j - 1$

Instability cascade

- By examining higher orders in perturbation theory beyond the linear level (in the interval $1/6 \leq u_0 \leq u_{\text{crit}}$) we expect to obtain a cascade of instabilities that originates from the $j = 1, 2$ sectors and propagates towards the higher multipoles...

$$x_i = \sum_{n=0}^{\infty} \varepsilon^n \delta x_i^n = x_i^0 + \sum_{n=1}^{\infty} \varepsilon^n \delta x_i^n, \quad i = 1, 2, 3$$

$$y_i = \sum_{n=0}^{\infty} \varepsilon^n \delta y_i^n = y_i^0 + \sum_{n=1}^{\infty} \varepsilon^n \delta y_i^n, \quad i = 1, \ldots, 6.$$

- This is due to the fact that the various (constant j) multipoles at a given order in perturbation theory couple to all the j's of the previous orders through an effective forcing term that arises in the corresponding fluctuation equation...

$$\delta x_i^n = \mu \cdot \sum_{j,m} \eta_i^{njm}(\tau)\, Y_{jm}(\theta, \phi), \qquad \eta_i^{njm}(0) = 0, \qquad i = 1, 2, 3$$

$$\delta y_i^n = \mu \cdot \sum_{j,m} \theta_i^{njm}(\tau)\, Y_{jm}(\theta, \phi), \qquad \theta_i^{njm}(0) = 0, \qquad i = 1, \ldots, 6.$$

- E.g. the lowest order instabilities ($j = 1, 2$) couple to all the modes (having different j's) of the first order...

Section 6

Conclusions

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- Radial & angular perturbation analysis in the $SO(3) \times SO(6)$ case in (Axenides-Floratos-GL, 2017a, 2017b). Studied instabilities...

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- Radial & angular perturbation analysis in the $SO(3) \times SO(6)$ case in (Axenides-Floratos-GL, 2017a, 2017b). Studied instabilities...
- Analysis of higher orders in perturbation theory... instability cascade... (Axenides-Floratos-Katsinis-GL, 2020, 2021).

Conclusions

- The spherically symmetric membrane in $SO(3)$ is integrable (Axenides-Floratos-GL, 2017a).
- Radial & angular perturbation analysis for the elliptic $SO(3)$ membrane was carried out in (Axenides-Floratos-GL, 2017a, 2017b). Found instabilities...
- Radial & angular perturbation analysis in the $SO(3) \times SO(6)$ case in (Axenides-Floratos-GL, 2017a, 2017b). Studied instabilities...
- Analysis of higher orders in perturbation theory... instability cascade... (Axenides-Floratos-Katsinis-GL, 2020, 2021).

Ευχαριστώ!