NATIONAL CENTRE FOR SCIENTIFIC RESEARCH "DEMOKRITOS"
INSTITUTE OF NUCLEAR AND PARTICLE PHYSICS

H.F.R.I.
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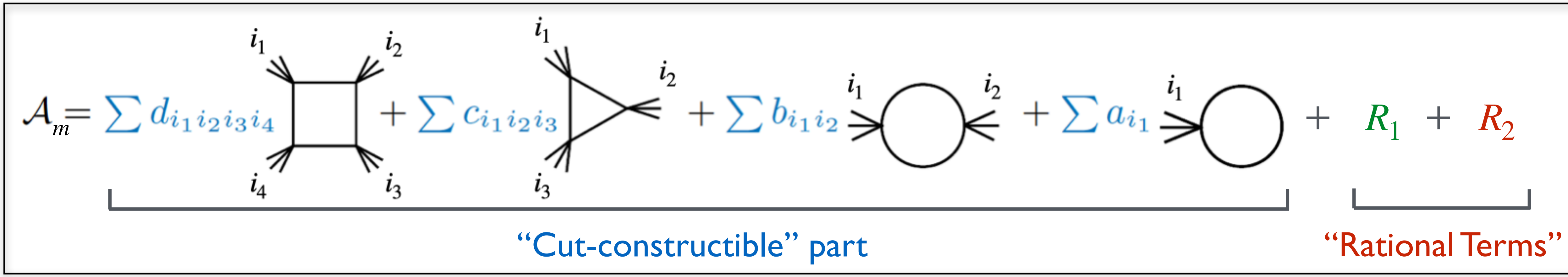
Towards numerical computation of dimensionally regularised QCD helicity amplitudes

Giuseppe Bevilacqua
NCSR "Demokritos"

APCTP meeting and HOCTOOLS-II mini-workshop
October 1, 2024

In collaboration with C. Papadopoulos, D. Canko, N. Dokmetzoglou and A. Spourdalakis
Work in progress

One-loop reduction in a nutshell

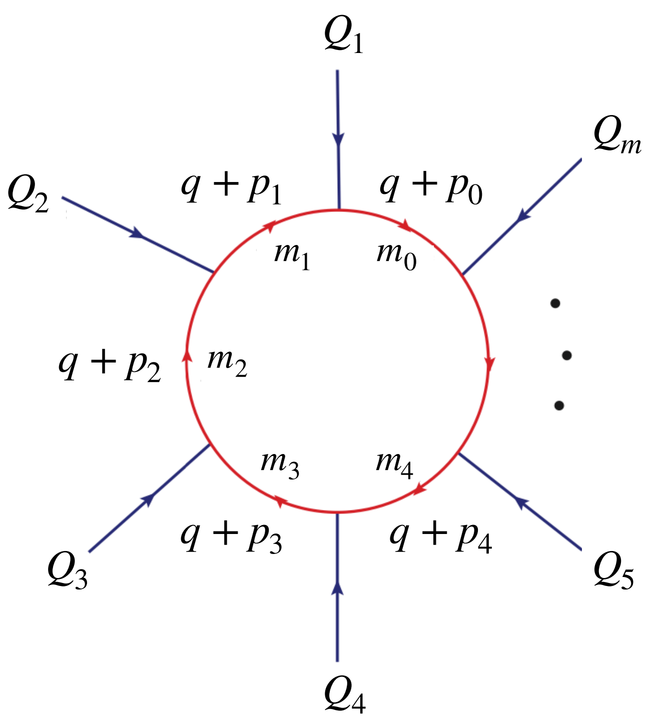


$$\mathcal{A}_m = \int \frac{d^d\bar{q}}{(2\pi)^d} \frac{\bar{N}(\bar{q})}{\bar{D}_0 \bar{D}_1 \ldots \bar{D}_{m-1}}$$

- Key input of *integrand-level* reduction is the numerator
  - Typically, ME generators provide numerators in $d = 4 \;\rightarrow\; N(q)$
  - Mismatch with the d-dimensional quantity appearing in loop integrand $\rightarrow\; \bar{N}(\bar{q})$
  - Rational Terms (R_1, R_2) compensate for the mismatch in D_i's and $N(q)$
- Achieving numerical computation of d-dimensional amplitudes has potential in performance and bookkeeping ($\rightarrow$ no need for R_1, R_2)
- Beyond 1-loop: more natural treatment of the reduction problem

Basic notation at 1-loop

$$\mathscr{A}_m = \int \frac{d^d\bar{q}}{(2\pi)^d} \frac{\bar{N}(\bar{q})}{\bar{D}_0 \bar{D}_1 \dots \bar{D}_{m-1}}$$



$\bar{N}(\bar{q})$ ⟶ *Numerator*

$\bar{D}_i \equiv (\bar{q}+p_i)^2 - m_i^2$ ⟶ *Propagators*

$$\bar{q}^2 = q^2 + \tilde{q}^2, \quad \tilde{q}^2 \equiv \mu$$

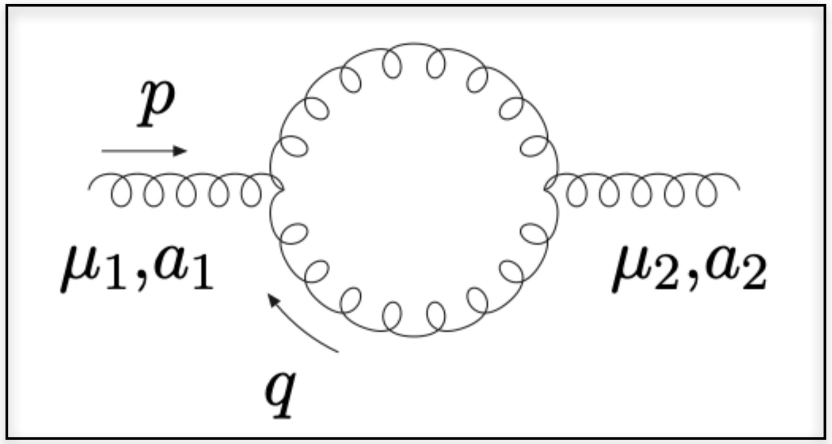
$$\bar{\gamma}^\mu = \gamma^\mu + \tilde{\gamma}^\mu \quad \hookrightarrow \quad \bar{\gamma}^\mu \bar{\gamma}_\mu = d = 4 - 2\epsilon$$

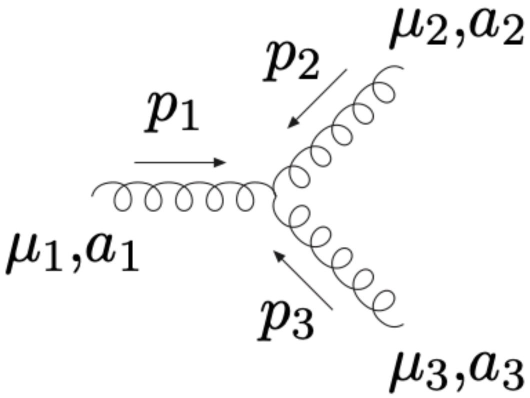
$$\bar{g}^{\mu\nu} = g^{\mu\nu} + \tilde{g}^{\mu\nu} \quad \hookrightarrow \quad \bar{g}^{\mu\nu}\bar{g}_{\mu\nu} = d = 4 - 2\epsilon$$

$$\bar{D}_i^2 = D_i^2 + \mu \qquad \bar{N}(\bar{q}) = N(q) + \tilde{N}(\tilde{q})$$

- Dimensional regularisation → *'t Hooft-Veltman scheme*
  - physical momenta (Q_i) in $d = 4$ dimensions
  - loop momentum ($\bar{q}$) in $d = 4 - 2\epsilon$ dimensions

A simple example: 1-loop gluon self-energy





$$= g\, f^{a_1 a_2 a_3}\, \underbrace{V_{\mu_1\mu_2\mu_3}(p_1,p_2,p_3)}_{\substack{\equiv \\ g_{\mu_1\mu_2}(p_2-p_1)_{\mu_3} \\ +\, g_{\mu_2\mu_3}(p_3-p_2)_{\mu_1} \\ +\, g_{\mu_3\mu_1}(p_1-p_3)_{\mu_2}}}$$

$$\bar{N}(\bar{q}) = \bar{N}^{\mu_1\mu_2}(\bar{q})\, \varepsilon_{\mu_1}(p)\, \varepsilon_{\mu_2}(p) = N(q) + \tilde{N}(\bar{q})$$

$$\hookrightarrow \quad \bar{N}^{\mu_1\mu_2}(\bar{q}) \to V^{\mu_1}{}_{\bar{\beta}\bar{\gamma}}(p, -\bar{q}-p, \bar{q})\, V^{\mu_2\,\bar{\gamma}\bar{\beta}}(-p, -\bar{q}, \bar{q}+p) =$$

$$= \left(5p^2 + 2p\cdot q + 2q^2\right) g^{\mu_1\mu_2} - 2\, p^{\mu_1}p^{\mu_2} + 5\, p^{\mu_1}q^{\mu_2} + 5\, q^{\mu_1}p^{\mu_2} + 10\, q^{\mu_1}q^{\mu_2} \longrightarrow N(q)$$

$$- (d-4)\left(p^{\mu_1}p^{\mu_2} + 2p^{\mu_1}q^{\mu_2} + 2q^{\mu_1}p^{\mu_2} + 4q^{\mu_1}q^{\mu_2}\right) + 2\,\mu\, g^{\mu_1\mu_2} \longrightarrow \tilde{N}(\bar{q})$$

$\hookrightarrow \quad \tilde{N}(\bar{q}) \propto \mu\,,\, (d-4) \quad \to$ *evanescent terms* (vanish in $d = 4$)

Computing evanescent terms

$$\mathscr{A}_m = \int \frac{d^d\bar{q}}{(2\pi)^d} \frac{\bar{N}(\bar{q})}{\bar{D}_0\,\bar{D}_1\ldots\bar{D}_{m-1}} \quad \rightarrow \; N(q) + \tilde{N}(q)$$

- Goal: compute $\tilde{N}(q)$ within a *numerical, recursive* framework (built on $d = 4$)

$$\boxed{\tilde{N}(q) = \mathscr{E}[\,N(q)\,]}$$

- Possible solution: *Four-Dimensional Formulation* (FDF)

[Fazio, Mastrolia, Mirabella and Torres Bobadilla, Eur.Phys.J.C 74 (2014) 12, 3197]

↪ • 4-dimensional d.o.f of gauge bosons carried out by μ

• $(d-4)$ dimensional d.o.f carried out by *scalar* particles $\Rightarrow$ *extra Feynman rules*

Computing evanescent terms

$$\tilde{N}(q) = \mathscr{E}[\, N(q) \,]$$

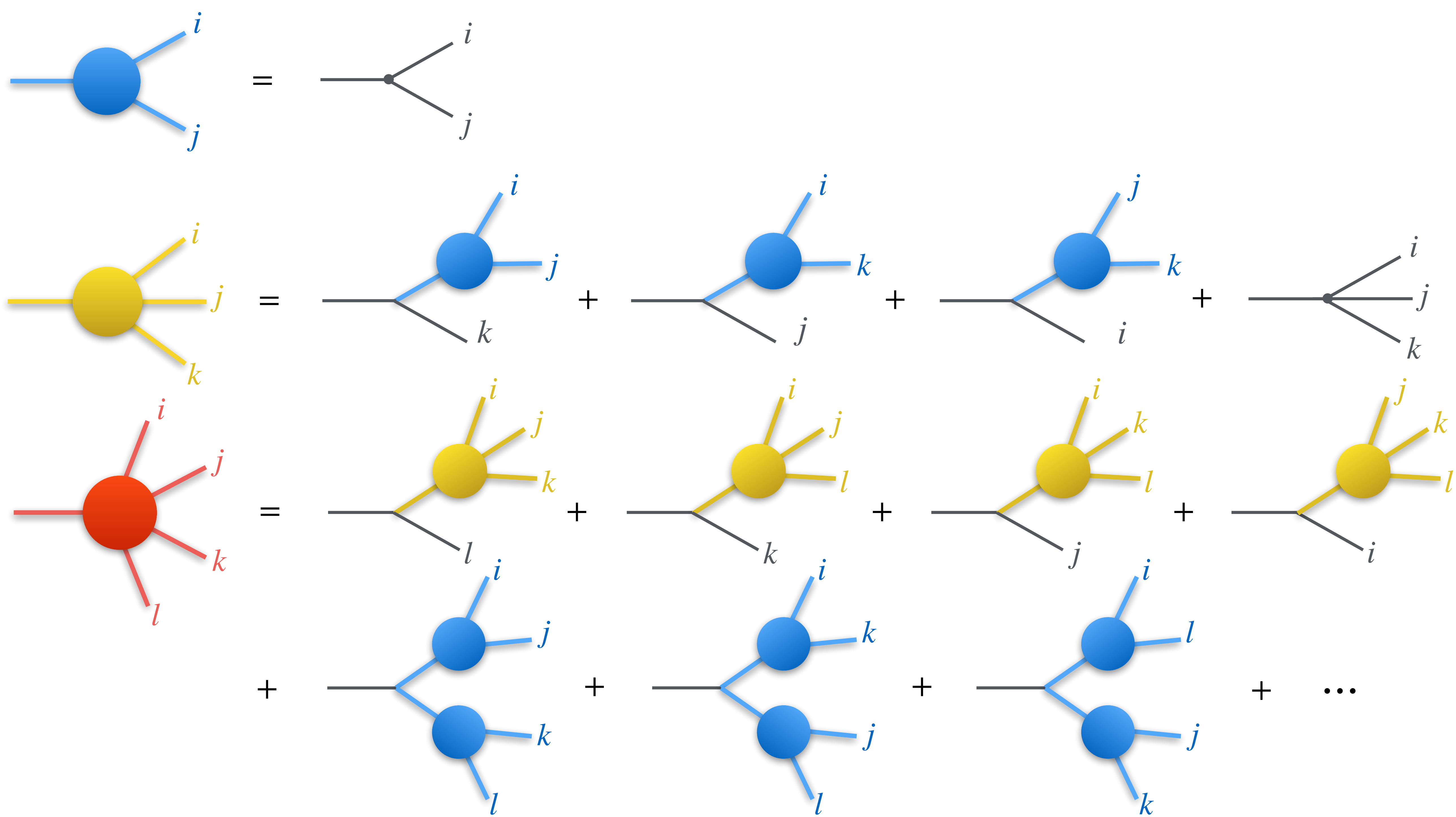
- We are exploring an *alternative* formulation which does not require extra Feynman rules

- Basic idea: obtain evanescent terms via *modified recursion relations*

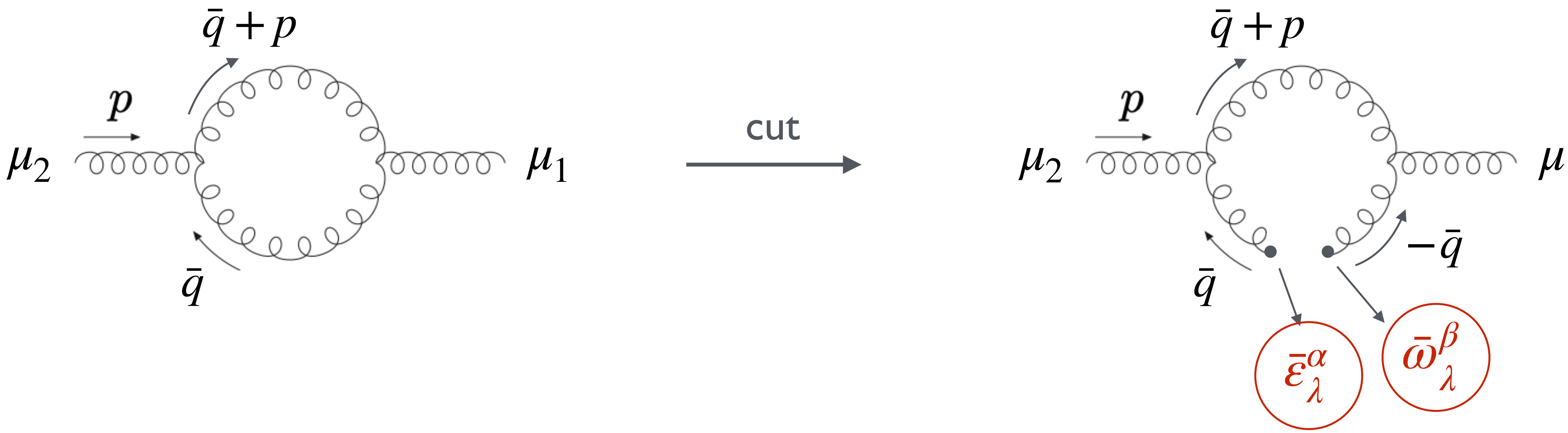
Before going to technical details, let's briefly sketch how recursion relations are organised within the HELAC framework

Dyson-Schwinger recursion in a nutshell

Computing scattering amplitudes without Feynman diagrams

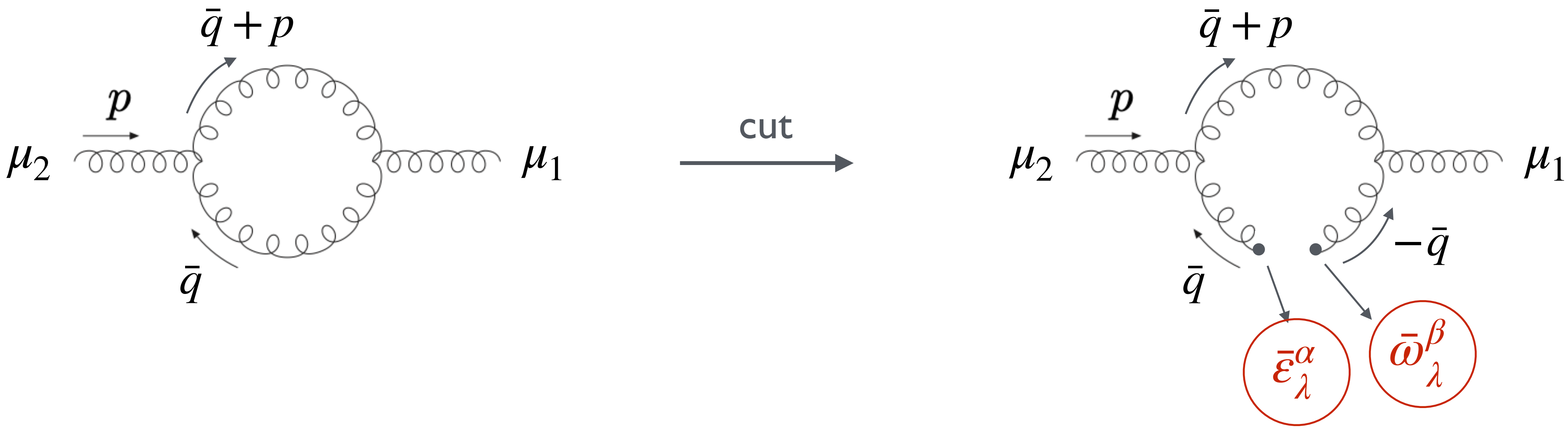


- Cutting loop propagator → Tree-level process with two extra particles



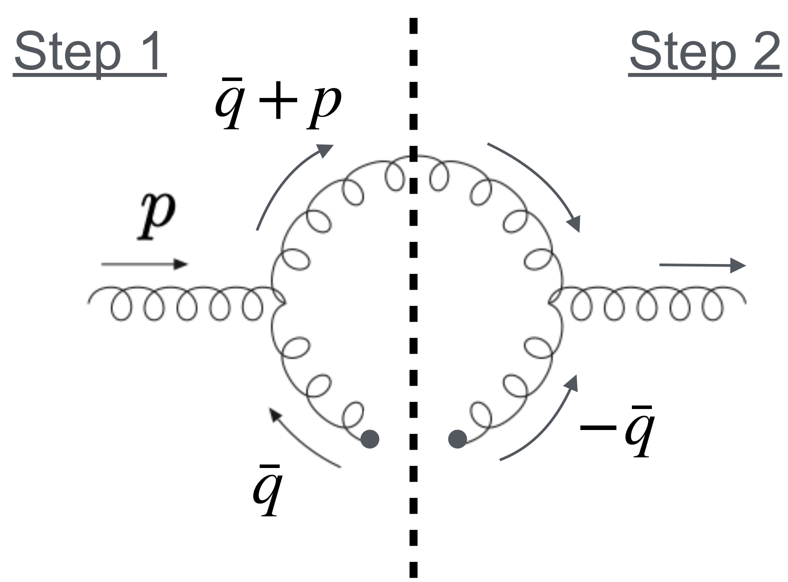
$$\hookrightarrow \quad \boxed{\sum_{\lambda} \bar{\varepsilon}^{\alpha}_{\lambda}\, \bar{\omega}^{\beta}_{\lambda} \rightarrow \bar{g}^{\alpha\beta}}$$

- Cutting loop propagator → Tree-level process with two extra particles



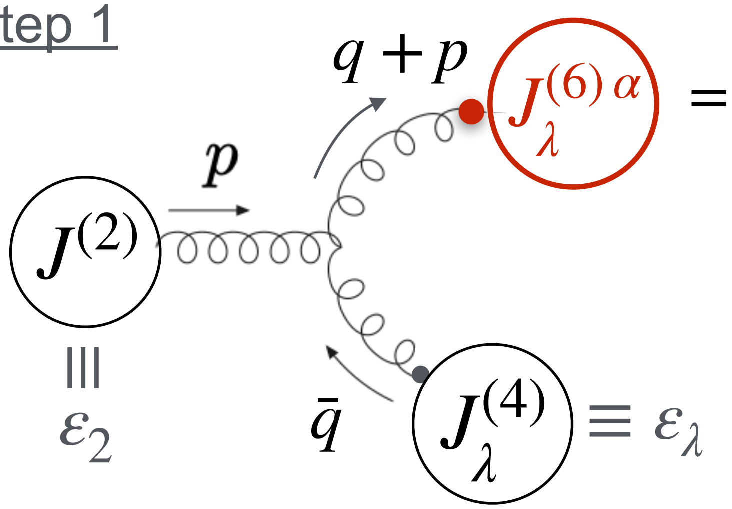
$$\hookrightarrow \quad \sum_{\lambda} \bar{\varepsilon}^{\alpha}_{\lambda}\, \bar{\omega}^{\beta}_{\lambda} \rightarrow \bar{g}^{\alpha\beta}$$

- Recursive calculation of tree-level process



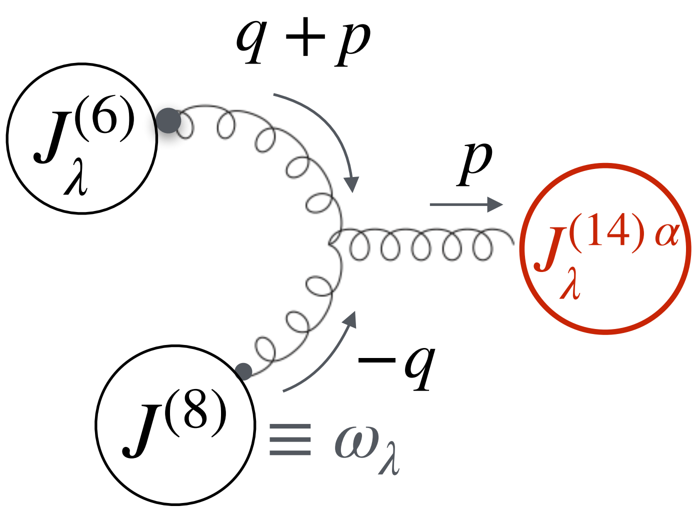
Recursive calculation in $d = 4$

Step 1



$$J_\lambda^{(6)\,\alpha} = \left((-q-2p)\cdot\varepsilon_\lambda\right) J^{(2)\,\alpha} + \left(-p\cdot J^{(2)}\right)\varepsilon_\lambda^\alpha + \left(J^{(2)}\cdot\varepsilon_\lambda\right)(p-q)^\alpha$$

Step 2



$$J_\lambda^{(14)\,\alpha} = \left((-q-2p)\cdot\omega_\lambda\right) J^{(6)\,\alpha} + \left((-q-p)\cdot J^{(6)}\right)\omega_\lambda^\alpha + \left(J^{(6)}\cdot\omega_\lambda\right)(p+2q)^\alpha$$

$\hookrightarrow$ $$N(q) = \sum_\lambda \left(J_\lambda^{(14)}\cdot J^{(1)}\right)$$ $\rightarrow$ Numerator in $d = 4$

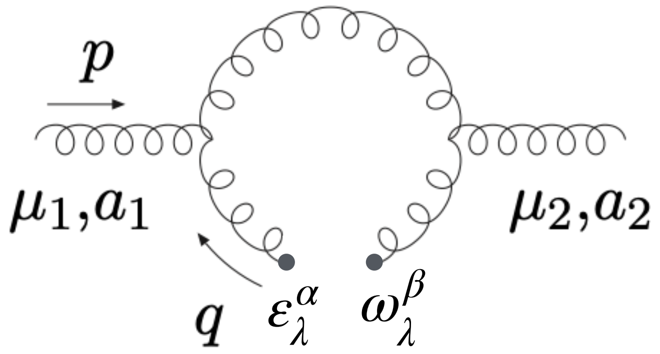
Moving to d dimensions

- Obtaining the evanescent terms $\bar{N}(\bar{q})$ for the considered case is simple in the context of *symbolic* calculations. Term-by-term, apply the following rules:

$$\mathcal{E}[q^2 X] = \mu X$$

$$\mathcal{E}[\sum_\lambda (\varepsilon_\lambda \cdot \omega_\lambda) X] = (d-4) X$$

$$\mathcal{E}[\sum_\lambda (\varepsilon_\lambda \cdot q)(\omega_\lambda \cdot q) X] = \mu X$$



- In *numerical* calculations one has access to *currents* ($J^{(N)}$), not to individual analytic terms

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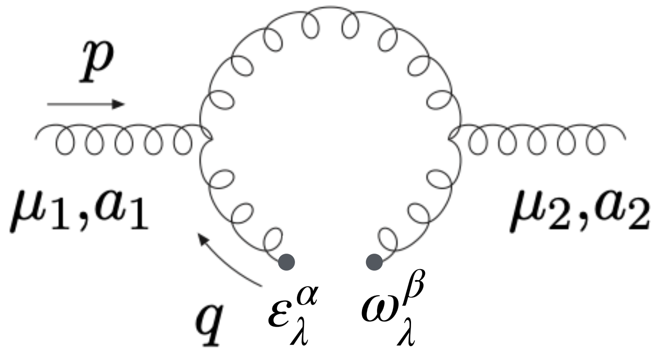


- In *numerical* calculations one has access to *currents* ($J^{(N)}$), not to individual analytic terms

  ↪ to apply the above rules, partial information of $J^{(N)}$'s *substructures* is required.
  For the considered case, the substructures read: $q^\alpha \quad \varepsilon_\lambda^\alpha \quad (\varepsilon_\lambda^\alpha \cdot q) \quad (\varepsilon_\lambda^\alpha \cdot q)\, q^\alpha$

Outline

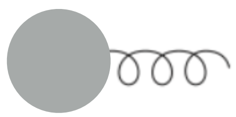
- Warm-up example: pure-gluon QCD at one loop
- Including fermions: recursion formulae for massless QCD
- Steps towards 2-loop [work in progress]

I. Warm-up example

Pure gluon QCD at 1-loop

Anatomy of substructures (1-loop)

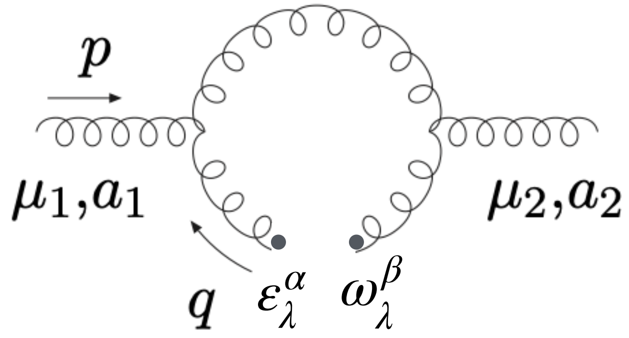
- **Vector-current** decomposition at 1-loop* *pure gluon QCD



$$J^{(N)\,\alpha} \equiv C_q^{(N)} q^\alpha + C_\varepsilon^{(N)} \varepsilon_\lambda^\alpha + (\varepsilon_\lambda \cdot q)\,[\, \underbrace{C_{\varepsilon q,q}^{(N)}\, q^\alpha + X_{\varepsilon q}^{(N)}}_{\equiv\, J_{\varepsilon q}^{(N)\,\alpha}} \,] + R^\alpha$$

R^α → [remainder]

[$C_q^{(N)}$, $C_\varepsilon^{(N)}$, $C_{\varepsilon q,q}^{(N)}$ → *scalars* ; $J_{\varepsilon q}^{(N)\,\alpha}$ → *vector*]



Anatomy of substructures (1-loop)

- Vector-current decomposition at 1-loop*   *pure gluon QCD

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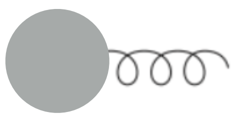
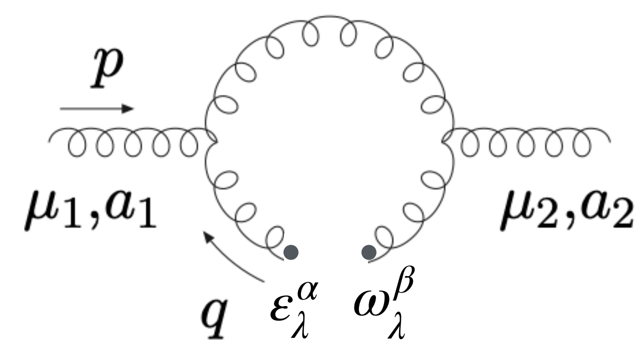


[$C_q^{(N)}$, $C_\varepsilon^{(N)}$, $C_{\varepsilon q,q}^{(N)}$ → *scalars* ; $J_{\varepsilon q}^{(N)\,\alpha}$ → *vector*]



- Keeping track of $C_q^{(N)}$, $C_\varepsilon^{(N)}$, $C_{\varepsilon q,q}^{(N)}$, $J_{\varepsilon q}^{(N)\,\alpha}$ *at every step of the recursion*, one can obtain evanescent terms for all currents in a pure 4-dimensional framework: $\tilde{J}^{(N)} = \mathscr{E}[J^{(N)}]$

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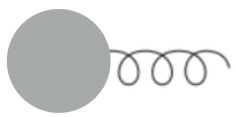
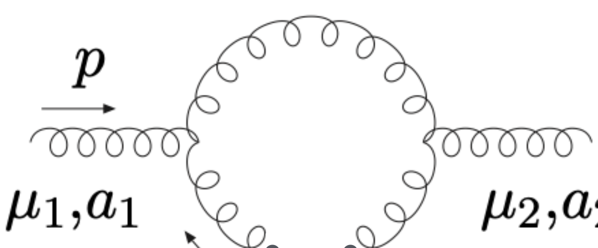
- Vector-current decomposition at 1-loop* — *pure gluon QCD

$$J^{(N)\,\alpha} \equiv C_q^{(N)} q^\alpha + C_\varepsilon^{(N)} \varepsilon_\lambda^\alpha + (\varepsilon_\lambda \cdot q)\,[\,\underbrace{C_{\varepsilon q,q}^{(N)}\, q^\alpha + X_{\varepsilon q}^{(N)}}_{\equiv\, J_{\varepsilon q}^{(N)\,\alpha}}\,] + R^\alpha$$



[$C_q^{(N)}$, $C_\varepsilon^{(N)}$, $C_{\varepsilon q,q}^{(N)}$ $\rightarrow$ *scalars* ; $J_{\varepsilon q}^{(N)\,\alpha}$ $\rightarrow$ *vector*]



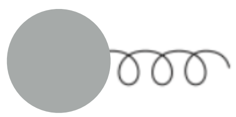
- Keeping track of $C_q^{(N)}$, $C_\varepsilon^{(N)}$, $C_{\varepsilon q,q}^{(N)}$, $J_{\varepsilon q}^{(N)\,\alpha}$ *at every step of the recursion*, one can obtain evanescent terms for all currents in a pure 4-dimensional framework: $\tilde{J}^{(N)} = \mathscr{E}[J^{(N)}]$

- $C_q^{(N)}$, $C_\varepsilon^{(N)}$, $C_{\varepsilon q,q}^{(N)}$, $J_{\varepsilon q}^{(N)\,\alpha}$ obey recursion relations, similarly to the currents $J^{(N)}$

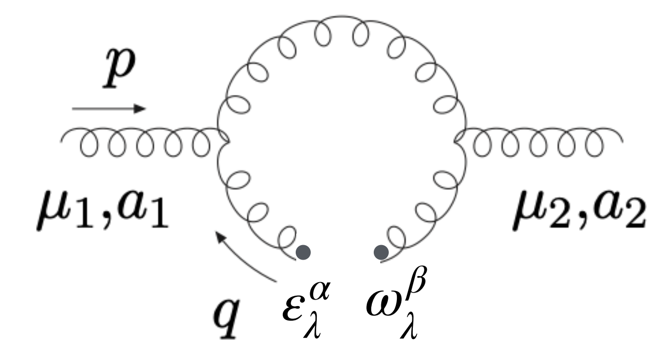
Anatomy of substructures (1-loop)

- Vector-current decomposition at 1-loop* *pure gluon QCD

$$J^{(N)\,\alpha} \equiv C_q^{(N)} q^\alpha + C_\varepsilon^{(N)} \varepsilon_\lambda^\alpha + (\varepsilon_\lambda \cdot q)\,[\,\underbrace{C_{\varepsilon q,q}^{(N)}\, q^\alpha + X_{\varepsilon q}^{(N)}}_{\equiv\, J_{\varepsilon q}^{(N)\,\alpha}}\,] + R^\alpha$$



[$C_q^{(N)}$, $C_\varepsilon^{(N)}$, $C_{\varepsilon q,q}^{(N)}$ $\to$ *scalars* ; $J_{\varepsilon q}^{(N)\,\alpha}$ $\to$ *vector*]

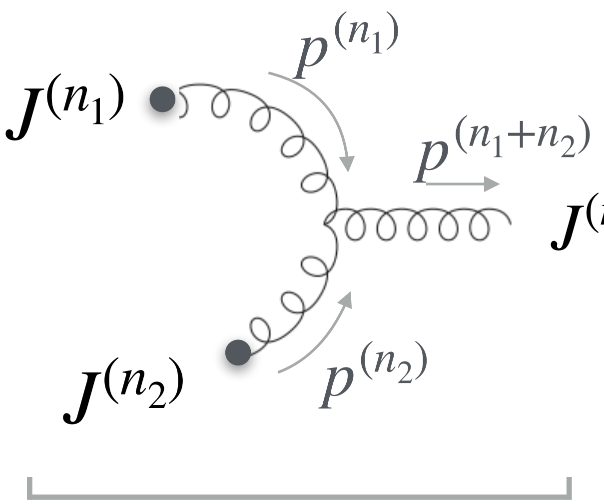


- Keeping track of $C_q^{(N)}$, $C_\varepsilon^{(N)}$, $C_{\varepsilon q,q}^{(N)}$, $J_{\varepsilon q}^{(N)\,\alpha}$ *at every step of the recursion*, one can obtain evanescent terms for all currents in a pure 4-dimensional framework: $\tilde{J}^{(N)} = \mathscr{E}[J^{(N)}]$

- $C_q^{(N)}$, $C_\varepsilon^{(N)}$, $C_{\varepsilon q,q}^{(N)}$, $J_{\varepsilon q}^{(N)\,\alpha}$ obey recursion relations, similarly to the currents $J^{(N)}$

- $C_q^{(N)}$, $C_\varepsilon^{(N)}$, $C_{\varepsilon q,q}^{(N)}$, $J_{\varepsilon q}^{(N)\,\alpha}$ = 0 for *tree-level* (*i.e.* non q-dependent) currents

Recursion relations (1-loop)



$$J^{(n_1+n_2)\,\alpha} \equiv \left(J^{(n_2)}\cdot(2p^{(n_1)}+p^{(n_2)})\right) J^{(n_1)\,\alpha} - \left(J^{(n_1)}\cdot(p^{(n_1)}+2\,p^{(n_2)})\right) J^{(n_2)\,\alpha} + \left(J^{(n_1)}\cdot J^{(n_2)}\right)\left(p^{(n_2)}-p^{(n_1)}\right)^{\alpha}$$

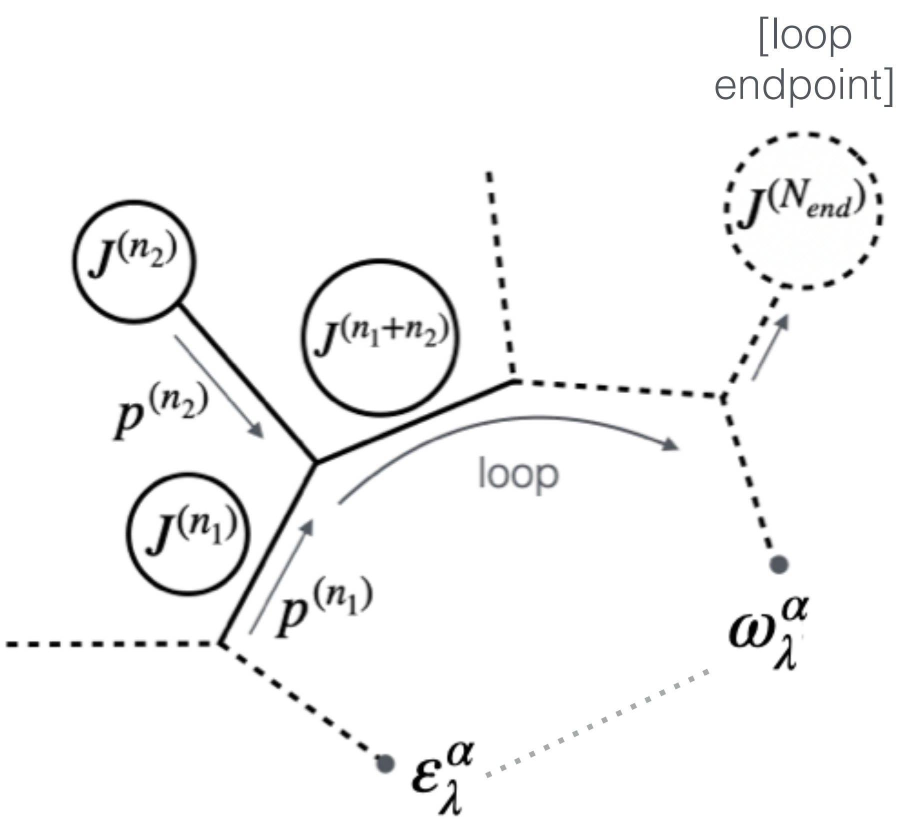
$$\equiv \mathrm{V}^{\alpha}_{ggg}\left(J^{(n_1)},J^{(n_2)}\right)$$

Recursion relations for $C_q^{(N)}$, $C_\varepsilon^{(N)}$, $C_{\varepsilon q,q}^{(N)}$, $J_{\varepsilon q}^{(N)\,\alpha}$ are derived from the vertex function

Initial conditions:

$$C_q^{(1)} = 0 \qquad C_\varepsilon^{(1)} = 1$$

$$C_{\varepsilon q,q}^{(1)} = 0 \qquad J_{\varepsilon q}^{(1)\,\alpha} = 0$$



$$C_q^{(n_1+n_2)} = C_q^{(n_1)}\left[J^{(n_2)}\cdot(2\,p^{(n_1)}+p^{(n_2)})\right] - \left(1+\delta_{(n_1+n_2)\,N_{end}}\right)\left(J^{(n_1)}\cdot J^{(n_2)}\right)$$

$$C_\varepsilon^{(n_1+n_2)} = C_\varepsilon^{(n_1)}\left[J^{(n_2)}\cdot(2\,p^{(n_1)}+p^{(n_2)})\right]$$

$$J_{\varepsilon q}^{(n_1+n_2)} = \mathrm{V}^{\alpha}_{ggg}\left(J_{\varepsilon q}^{(n_1)},J^{(n_2)}\right) - \left(C_\varepsilon^{(n_1)} + \mu\, C_{\varepsilon q,q}^{(n_1)}\right) J^{(n_2)\,\alpha}$$

$$C_{\varepsilon q,q}^{(n_1+n_2)} = C_{\varepsilon q,q}^{(n_1)}\left[J^{(n_2)}\cdot(2\,p^{(n_1)}+p^{(n_2)})\right] - \left(1+\delta_{(n_1+n_2)\,N_{end}}\right)\left(J_{q\varepsilon}^{(n_1)}\cdot J^{(n_2)}\right)$$

$$\mathscr{E}\left[J^{(n_1+n_2)\,\alpha}\right] = \mu\left[\left(-1+2\,\delta_{(n_1+n_2)N_{end}}\right)C_q^{(n_1)}J^{(n_2)} + \delta_{(n_1+n_2)N_{end}}\left(J_{\varepsilon q}^{(n_1)} + C_{\varepsilon q,q}^{(n_1)}\left(p^{(n_2)}-p^{(n_1)}\right)^{\alpha}\right)\right] + (d-4)\left[\delta_{(n_1+n_2)N_{end}}\,C_\varepsilon^{(n_1)}\left(p^{(n_2)}-p^{(n_1)}\right)^{\alpha}\right]$$

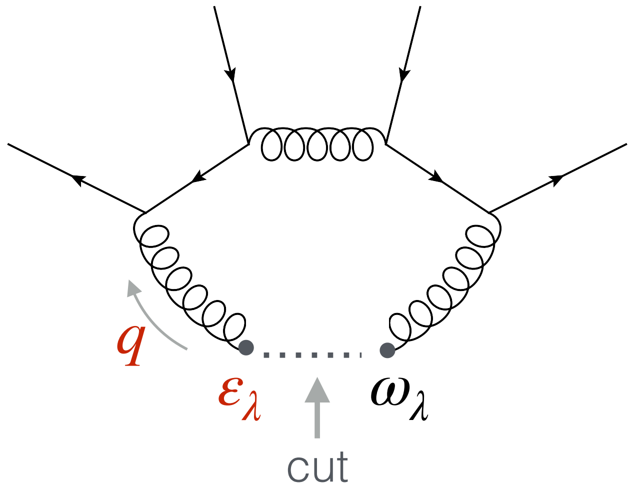
II. Including fermions

Massless QCD at 1-loop

Including fermionic contributions (1-loop)

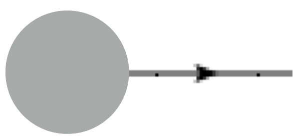
Substructures:

$$\not{q} \quad \not{\varepsilon}_\lambda \quad (\varepsilon_\lambda \cdot q) \quad (\varepsilon_\lambda \cdot q)\not{q} \quad \not{q}\not{\varepsilon}_\lambda$$

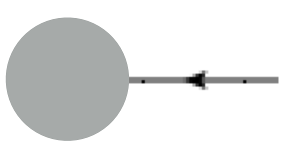


- Fermion-current decomposition at 1-loop

$$J^{(N)} \equiv \not{q}\, \psi_q{}^{(N)} + \not{\varepsilon}_\lambda\, \psi_\varepsilon{}^{(N)} + (\varepsilon_\lambda \cdot q)\, [\, \underbrace{\not{q}\, \psi_{\varepsilon q,q}{}^{(N)} + X_{\varepsilon q}{}^{(N)}}_{\equiv\, \psi_{\varepsilon q}{}^{(N)}}\,] + \not{q}\not{\varepsilon}_\lambda\, \psi^{(N)}_{q/\varepsilon} + R^{(N)}$$



$$\bar{J}^{(N)} \equiv \bar{\psi}_q{}^{(N)}\not{q} + \bar{\psi}_\varepsilon{}^{(N)}\not{\varepsilon}_\lambda + (\varepsilon_\lambda \cdot q)\, [\, \underbrace{\bar{\psi}_{\varepsilon q,q}{}^{(N)}\not{q} + \bar{X}_{\varepsilon q}{}^{(N)}}_{\equiv\, \bar{\psi}_{\varepsilon q}{}^{(N)}}\,] + \bar{\psi}^{(N)}_{q/\varepsilon}\, \not{\varepsilon}_\lambda\not{q} + \bar{R}^{(N)}$$



[$\psi_q{}^{(N)}$, $\psi_\varepsilon{}^{(N)}$, $\psi_{\varepsilon q,q}{}^{(N)}$, $\psi_{\varepsilon q}{}^{(N)}$, $\psi^{(N)}_{q/\varepsilon}$ $\rightarrow$ spinors]

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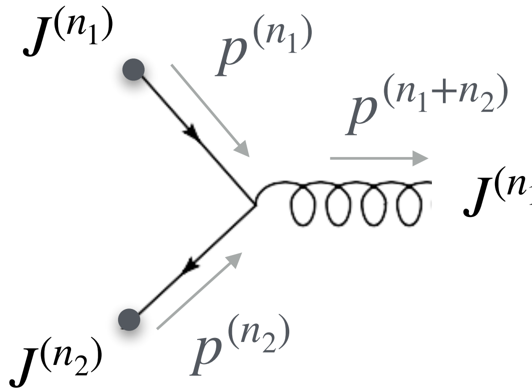
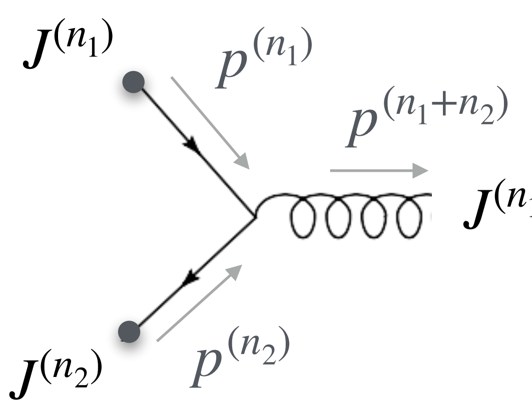
Including fermionic contributions (1-loop)

- **Back to vector current:**



$$J^{(n_1+n_2)\,\alpha} \equiv \bar{J}^{(n_2)}\,\gamma^{\alpha}\,J^{(n_1)}$$

- Back to vector current:

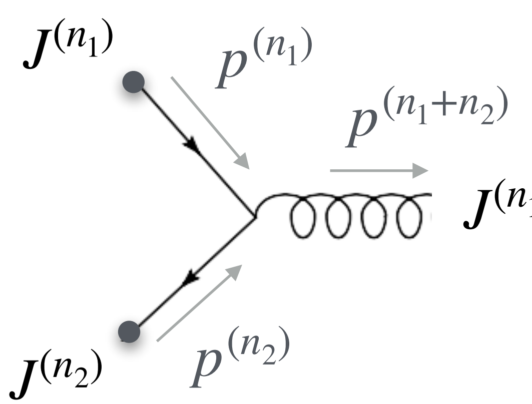


$$J^{(n_1+n_2)\,\alpha} \equiv \bar{J}^{(n_2)}\,\gamma^\alpha\,J^{(n_1)} \longrightarrow$$

critical structures

$L^{(n_1)} = 1$, $L^{(n_2)} = 0$	$L^{(n_1)} = 0$, $L^{(n_2)} = 1$
$\bar{J}^{(n_2)}\,\gamma^\alpha\,\not{q}\,\psi_q^{(n_1)}$	$\bar{\psi}_q^{(n_2)}\,\not{q}\,\gamma^\alpha\,J^{(n_1)}$
$\bar{J}^{(n_2)}\,\gamma^\alpha\,\not{\varepsilon}_\lambda\,\psi_\varepsilon^{(n_1)}$	$\bar{\psi}_\varepsilon^{(n_2)}\,\not{\varepsilon}_\lambda\,\gamma^\alpha\,J^{(n_1)}$
$\bar{J}^{(n_2)}\,\gamma^\alpha\,\not{q}\,\not{\varepsilon}_\lambda\,\psi_{q/\varepsilon}^{(n_1)}$	$\bar{\psi}_{q/\varepsilon}^{(n_2)}\,\not{\varepsilon}_\lambda\not{q}\,\gamma^\alpha\,J^{(n_1)}$
$\bar{J}^{(n_2)}\,\gamma^\alpha\,(\varepsilon_\lambda\cdot q)\,\psi_{\varepsilon q}^{(n_1)}$	$\bar{\psi}_{\varepsilon q}^{(n_2)}\,\gamma^\alpha\,(\varepsilon_\lambda\cdot q)\,J^{(n_1)}$
$\bar{J}^{(n_2)}\,\gamma^\alpha\,(\varepsilon_\lambda\cdot q)\,\not{q}\,\psi_{\varepsilon q,q}^{(n_1)}$	$\bar{\psi}_{\varepsilon q,q}^{(n_2)}\,\not{q}\,(\varepsilon_\lambda\cdot q)\,\gamma^\alpha\,J^{(n_1)}$

- Back to vector current:



$$J^{(n_1+n_2)\,\alpha} \equiv \bar{J}^{(n_2)}\gamma^\alpha J^{(n_1)}$$

critical structures

$L^{(n_1)} = 1$, $L^{(n_2)} = 0$	$L^{(n_1)} = 0$, $L^{(n_2)} = 1$
$\bar{J}^{(n_2)}\gamma^\alpha \not{q}\, \psi_q^{(n_1)}$	$\bar{\psi}_q^{(n_2)} \not{q}\, \gamma^\alpha J^{(n_1)}$
$\bar{J}^{(n_2)}\gamma^\alpha \not{\varepsilon}_\lambda\, \psi_\varepsilon^{(n_1)}$	$\bar{\psi}_\varepsilon^{(n_2)} \not{\varepsilon}_\lambda\, \gamma^\alpha J^{(n_1)}$
$\bar{J}^{(n_2)}\gamma^\alpha \not{q}\not{\varepsilon}_\lambda\, \psi_{q/\varepsilon}^{(n_1)}$	$\bar{\psi}_{q/\varepsilon}^{(n_2)} \not{\varepsilon}_\lambda \not{q}\, \gamma^\alpha J^{(n_1)}$
$\bar{J}^{(n_2)}\gamma^\alpha (\varepsilon_\lambda \cdot q)\, \psi_{\varepsilon q}^{(n_1)}$	$\bar{\psi}_{\varepsilon q}^{(n_2)} \gamma^\alpha (\varepsilon_\lambda \cdot q)\, J^{(n_1)}$
$\bar{J}^{(n_2)}\gamma^\alpha (\varepsilon_\lambda \cdot q) \not{q}\, \psi_{\varepsilon q,q}^{(n_1)}$	$\bar{\psi}_{\varepsilon q,q}^{(n_2)} \not{q} (\varepsilon_\lambda \cdot q)\, \gamma^\alpha J^{(n_1)}$

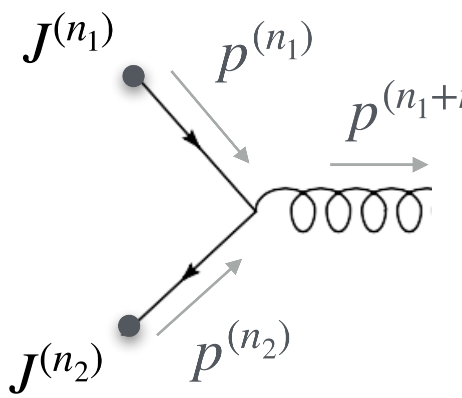
- $\mathscr{E}[\,(\bar{J}^{(n_2)}\gamma^\alpha \not{q}\, \psi_q^{(n_1)})\, q_\alpha\,] = \mu\, \bar{J}^{(n_2)} \psi_q^{(n_1)}$
- $\mathscr{E}[\,(\bar{J}^{(n_2)}\gamma^\alpha \not{\varepsilon}_\lambda\, \psi_\varepsilon^{(n_1)})\, \omega_{\lambda\,\alpha}\,] = (d-4)\, \bar{J}^{(n_2)} \psi_\varepsilon^{(n_1)}$
- $\mathscr{E}[\,(\bar{J}^{(n_2)}\gamma^\alpha (\varepsilon_\lambda \cdot q)\, \psi_{\varepsilon q}^{(n_1)})\; (\omega_\lambda \cdot q)] = \mu\, \bar{J}^{(n_2)} \gamma^\alpha \psi_{\varepsilon q}^{(n_1)}$

$\vdots$

- $\mathscr{E}[\,(\bar{J}^{(n_2)}\gamma^\alpha \not{q}\not{\varepsilon}_\lambda\, \psi_{q/\varepsilon}^{(n_1)})\, q_\alpha] = \mu\, \bar{J}^{(n_2)} \not{\varepsilon}_\lambda \psi_{q/\varepsilon}^{(n_1)} = \mu\, (\bar{J}^{(n_2)}\gamma^\alpha \psi_{q/\varepsilon}^{(n_1)})\, \varepsilon_{\lambda\,\alpha}$

Including fermionic contributions (1-loop)

- Back to vector current:



$$J^{(n_1+n_2)\,\alpha} \equiv \bar{J}^{(n_2)}\,\gamma^\alpha\,J^{(n_1)}$$

critical structures

$L^{(n_1)} = 1$, $L^{(n_2)} = 0$	$L^{(n_1)} = 0$, $L^{(n_2)} = 1$
$\bar{J}^{(n_2)}\,\gamma^\alpha\,\not{q}\;\psi_q^{(n_1)}$	$\bar{\psi}_q^{(n_2)}\,\not{q}\;\gamma^\alpha\,J^{(n_1)}$
$\bar{J}^{(n_2)}\,\gamma^\alpha\,\not{\varepsilon}_\lambda\;\psi_\varepsilon^{(n_1)}$	$\bar{\psi}_\varepsilon^{(n_2)}\,\not{\varepsilon}_\lambda\;\gamma^\alpha\,J^{(n_1)}$
$\bar{J}^{(n_2)}\,\gamma^\alpha\,\not{q}\,\not{\varepsilon}_\lambda\;\psi_{q/\varepsilon}^{(n_1)}$	$\bar{\psi}_{q/\varepsilon}^{(n_2)}\,\not{\varepsilon}_\lambda\not{q}\;\gamma^\alpha\,J^{(n_1)}$
$\bar{J}^{(n_2)}\,\gamma^\alpha\,(\varepsilon_\lambda\cdot q)\;\psi_{\varepsilon q}^{(n_1)}$	$\bar{\psi}_{\varepsilon q}^{(n_2)}\,\gamma^\alpha\,(\varepsilon_\lambda\cdot q)\;J^{(n_1)}$
$\bar{J}^{(n_2)}\,\gamma^\alpha\,(\varepsilon_\lambda\cdot q)\,\not{q}\;\psi_{\varepsilon q,q}^{(n_1)}$	$\bar{\psi}_{\varepsilon q,q}^{(n_2)}\,\not{q}\,(\varepsilon_\lambda\cdot q)\;\gamma^\alpha\,J^{(n_1)}$

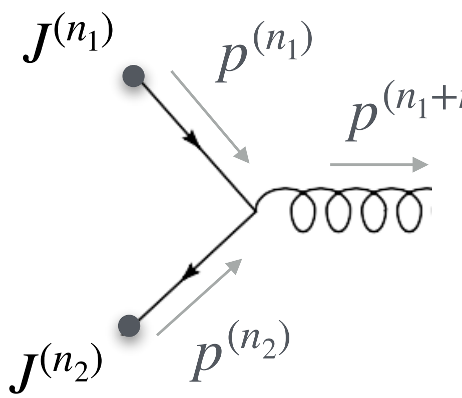
- $\mathcal{E}[\,(\bar{J}^{(n_2)}\,\gamma^\alpha\,\not{q}\;\psi_q^{(n_1)})\,q_\alpha\,] = \mu\;\underbrace{\bar{J}^{(n_2)}\,\psi_q^{(n_1)}}_{\hookrightarrow\; C_q^{(n_1+n_2)}}$
- $\mathcal{E}[\,(\bar{J}^{(n_2)}\,\gamma^\alpha\,\not{\varepsilon}_\lambda\;\psi_\varepsilon^{(n_1)})\,\omega_{\lambda\,\alpha}\,] = (d-4)\;\underbrace{\bar{J}^{(n_2)}\,\psi_\varepsilon^{(n_1)}}_{\hookrightarrow\; C_\varepsilon^{(n_1+n_2)}}$
- $\mathcal{E}[\,(\bar{J}^{(n_2)}\,\gamma^\alpha\,(\varepsilon_\lambda\cdot q)\;\psi_{\varepsilon q}^{(n_1)})\;(\omega_\lambda\cdot q)] = \mu\;\underbrace{\bar{J}^{(n_2)}\,\gamma^\alpha\,\psi_{\varepsilon q}^{(n_1)}}_{\hookrightarrow\; J_{\varepsilon q}^{(n_1+n_2)\,\alpha}}$

  $\vdots$

- $\mathcal{E}[\,(\bar{J}^{(n_2)}\,\gamma^\alpha\,\not{q}\,\not{\varepsilon}_\lambda\;\psi_{q/\varepsilon}^{(n_1)})\,q_\alpha] = \mu\;\bar{J}^{(n_2)}\,\not{\varepsilon}_\lambda\,\psi_{q/\varepsilon}^{(n_1)} = \mu\;(\bar{J}^{(n_2)}\gamma^\alpha\,\psi_{q/\varepsilon}^{(n_1)})\,\varepsilon_{\lambda\,\alpha}$

Including fermionic contributions (1-loop)

- Back to vector current:



$$J^{(n_1+n_2)\,\alpha} \equiv \bar{J}^{(n_2)}\,\gamma^\alpha\,J^{(n_1)} \;\longrightarrow$$

critical structures

$L^{(n_1)} = 1$, $L^{(n_2)} = 0$	$L^{(n_1)} = 0$, $L^{(n_2)} = 1$
$\bar{J}^{(n_2)}\,\gamma^\alpha\,\not{q}\;\psi_q^{(n_1)}$	$\bar{\psi}_q^{(n_2)}\,\not{q}\,\gamma^\alpha\,J^{(n_1)}$
$\bar{J}^{(n_2)}\,\gamma^\alpha\,\not{\varepsilon}_\lambda\;\psi_\varepsilon^{(n_1)}$	$\bar{\psi}_\varepsilon^{(n_2)}\,\not{\varepsilon}_\lambda\,\gamma^\alpha\,J^{(n_1)}$
$\bar{J}^{(n_2)}\,\gamma^\alpha\,\not{q}\,\not{\varepsilon}_\lambda\;\psi_{q/\varepsilon}^{(n_1)}$	$\bar{\psi}_{q/\varepsilon}^{(n_2)}\,\not{\varepsilon}_\lambda\not{q}\,\gamma^\alpha\,J^{(n_1)}$
$\bar{J}^{(n_2)}\,\gamma^\alpha\,(\varepsilon_\lambda\cdot q)\;\psi_{\varepsilon q}^{(n_1)}$	$\bar{\psi}_{\varepsilon q}^{(n_2)}\,\gamma^\alpha\,(\varepsilon_\lambda\cdot q)\;J^{(n_1)}$
$\bar{J}^{(n_2)}\,\gamma^\alpha\,(\varepsilon_\lambda\cdot q)\,\not{q}\;\psi_{\varepsilon q,q}^{(n_1)}$	$\bar{\psi}_{\varepsilon q,q}^{(n_2)}\,\not{q}\,(\varepsilon_\lambda\cdot q)\,\gamma^\alpha\,J^{(n_1)}$

- $\mathscr{E}[\,(\bar{J}^{(n_2)}\gamma^\alpha\,\not{q}\;\psi_q^{(n_1)})\,q_\alpha\,] = \mu\,\underbrace{\bar{J}^{(n_2)}\,\psi_q^{(n_1)}}_{\hookrightarrow\;C_q^{(n_1+n_2)}}$

- $\mathscr{E}[\,(\bar{J}^{(n_2)}\gamma^\alpha\,\not{\varepsilon}_\lambda\;\psi_\varepsilon^{(n_1)})\,\omega_{\lambda\,\alpha}\,] = (d-4)\,\underbrace{\bar{J}^{(n_2)}\,\psi_\varepsilon^{(n_1)}}_{\hookrightarrow\;C_\varepsilon^{(n_1+n_2)}}$

- $\mathscr{E}[\,(\bar{J}^{(n_2)}\gamma^\alpha\,(\varepsilon_\lambda\cdot q)\;\psi_{\varepsilon q}^{(n_1)})\;(\omega_\lambda\cdot q)] = \mu\,\underbrace{\bar{J}^{(n_2)}\,\gamma^\alpha\,\psi_{\varepsilon q}^{(n_1)}}_{\hookrightarrow\;J_{\varepsilon q}^{(n_1+n_2)\,\alpha}}$

  ⋮

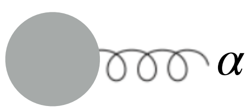
- $\mathscr{E}[\,(\bar{J}^{(n_2)}\gamma^\alpha\,\not{q}\,\not{\varepsilon}_\lambda\;\psi_{q/\varepsilon}^{(n_1)})\,q_\alpha] = \mu\,\bar{J}^{(n_2)}\,\not{\varepsilon}_\lambda\,\psi_{q/\varepsilon}^{(n_1)} = \mu\,\underbrace{(\bar{J}^{(n_2)}\gamma^\alpha\,\psi_{q/\varepsilon}^{(n_1)})}_{\equiv\;T_{q/\varepsilon}^{(n_1+n_2)\,\alpha}}\,\varepsilon_{\lambda\,\alpha}$

Fermions induce new substructures in vector currents

Including fermionic contributions (1-loop)

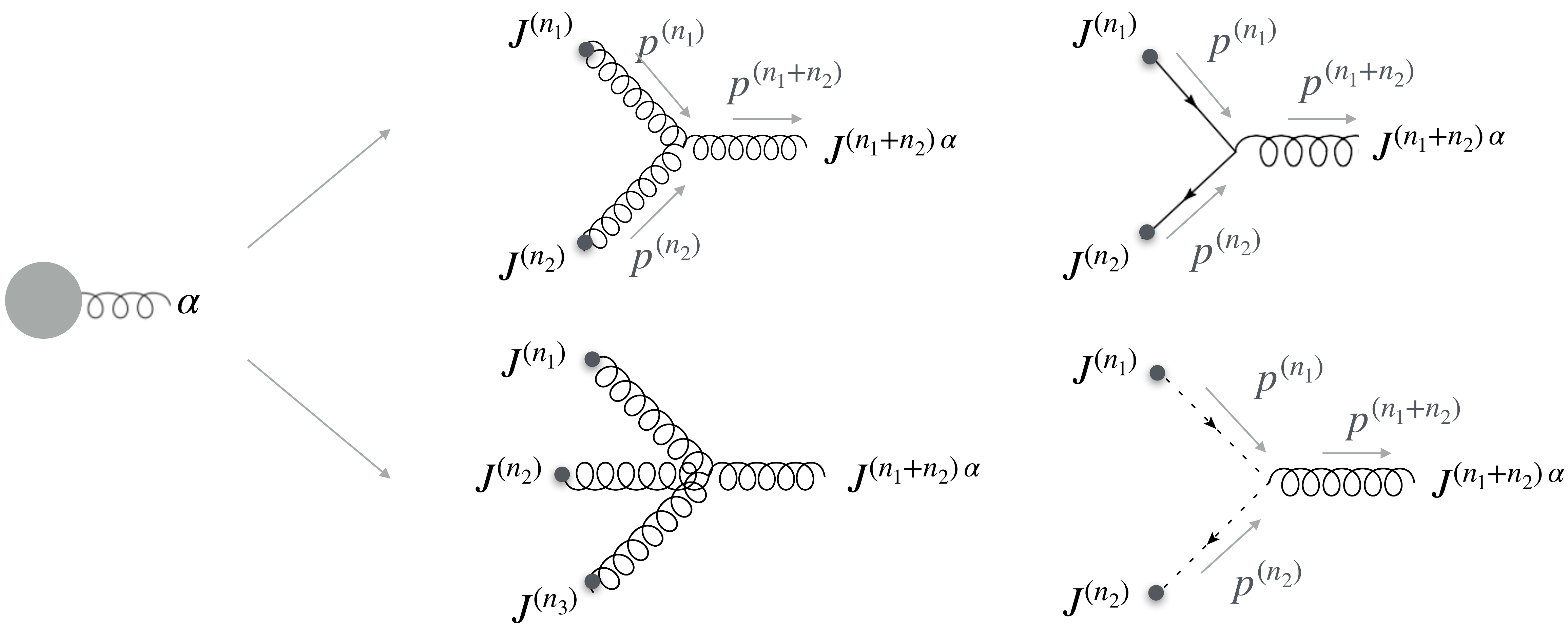
- General vector-current decomposition at 1-loop:

$$J^{(N)\,\alpha} \equiv C_q^{(N)} q^\alpha + C_\varepsilon^{(N)} \varepsilon_\lambda^\alpha + (\varepsilon_\lambda \cdot q)\,[\,\underbrace{C_{\varepsilon q,q}^{(N)}\, q^\alpha + X_{\varepsilon q}^{(N)}}_{\equiv\, J_{\varepsilon q}^{(N)\,\alpha}}\,] + T_{q/\varepsilon}^{(N)\,\alpha} + T^{(N)} + R^\alpha$$

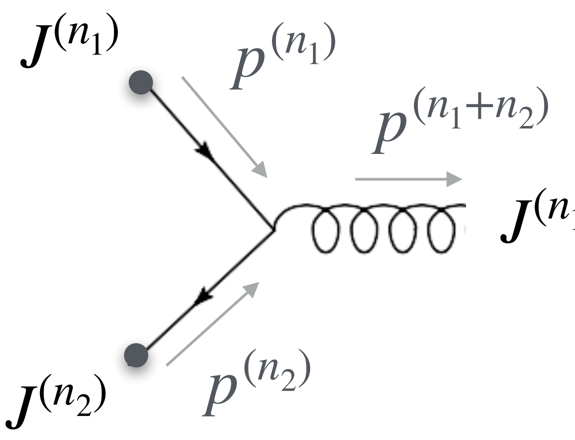


$T_{q/\varepsilon}^{(N)\,\alpha} \equiv \bar{J}^{(n_2)} \gamma^\alpha \psi_{q/\varepsilon}^{(n_1)}$; $T^{(N)} \equiv \bar{J}^{(n_2)} J^{(n_1)}$

[$C_q^{(N)}$, $C_\varepsilon^{(N)}$, $C_{\varepsilon q,q}^{(N)}$, $T^{(N)}$ → *scalars* ; $J_{\varepsilon q}^{(N)\,\alpha}$, $T_{q/\varepsilon}^{(N)\,\alpha}$ → *vectors*]

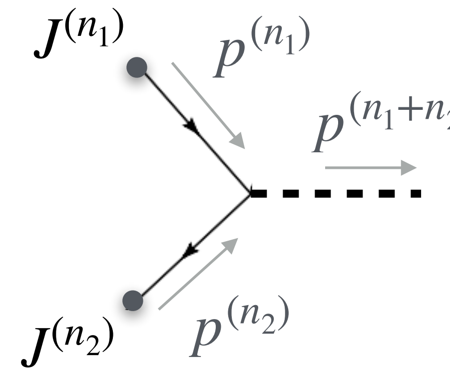


Including fermionic contributions (1-loop)



$$J^{(n_1+n_2)\,\alpha} \equiv \bar{J}^{(n_2)}\,\gamma^{\alpha}\,J^{(n_1)}$$

$$\equiv \mathrm{V}^{\alpha}_{\mathrm{qqg}}\left(J^{(n_1)}, \bar{J}^{(n_2)}\right)$$



$$\equiv \bar{J}^{(n_2)}\,J^{(n_1)}$$

$$\equiv \mathrm{S}_{\mathrm{qqs}}\left(J^{(n_1)}, \bar{J}^{(n_2)}\right)$$

$$C_q^{(n_1+n_2)} = \theta[L^{(n_1)}]\left[\mathrm{S}_{\mathrm{qqs}}\left(\psi_q^{(n_1)}, \bar{J}^{(n_2)}\right)\right] + \theta[L^{(n_2)}]\left[\mathrm{S}_{\mathrm{qqs}}\left(J^{(n_1)}, \bar{\psi}_q^{(n_1)}\right)\right]$$

$$C_\varepsilon^{(n_1+n_2)} = \theta[L^{(n_1)}]\left[\mathrm{S}_{\mathrm{qqs}}\left(\psi_\varepsilon^{(n_1)}, \bar{J}^{(n_2)}\right)\right] + \theta[L^{(n_2)}]\left[\mathrm{S}_{\mathrm{qqs}}\left(J^{(n_1)}, \bar{\psi}_\varepsilon^{(n_1)}\right)\right]$$

$$C_{\varepsilon q,q}^{(n_1+n_2)} = \theta[L^{(n_1)}]\left[\mathrm{S}_{\mathrm{qqs}}\left(\psi_{\varepsilon q,q}^{(n_1)}, \bar{J}^{(n_2)}\right)\right] + \theta[L^{(n_2)}]\left[\mathrm{S}_{\mathrm{qqs}}\left(J^{(n_1)}, \bar{\psi}_{\varepsilon q,q}^{(n_1)}\right)\right]$$

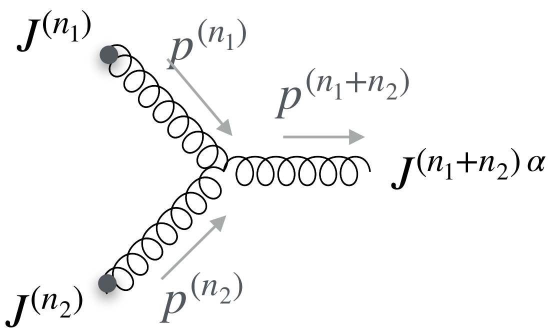
$$J_{\varepsilon q}^{(n_1+n_2)\,\alpha} = \theta[L^{(n_1)}]\left[\mathrm{S}_{\mathrm{qqs}}\left(\psi_{\varepsilon q}^{(n_1)}, \bar{J}^{(n_2)}\right)\right] + \theta[L^{(n_2)}]\left[\mathrm{S}_{\mathrm{qqs}}\left(J^{(n_1)}, \bar{\psi}_{\varepsilon q}^{(n_1)}\right)\right]$$

$$T_{q/\varepsilon}^{(n_1+n_2)\,\alpha} = \theta[L^{(n_1)}]\left[V^{\alpha}_{\mathrm{qqg}}\left(\psi_{q/\varepsilon}^{(n_1)}, \bar{J}^{(n_2)}\right)\right] + \theta[L^{(n_2)}]\left[V^{\alpha}_{\mathrm{qqg}}\left(J^{(n_1)}, \bar{\psi}_{q/\varepsilon}^{(n_2)}\right)\right]$$

$$T^{(n_1+n_2)} = \left(\theta[L^{(n_1)}] + \theta[L^{(n_2)}]\right)\left[\mathrm{S}_{\mathrm{qqs}}\left(J^{(n_1)}, \bar{J}^{(n_2)}\right)\right]$$

$$\mathscr{E}\left[J^{(n_1+n_2)\,\alpha}\right] = 0$$

Including fermionic contributions (1-loop)



$$\equiv \mathrm{V}^{\alpha}_{\mathrm{ggg}}\left(J^{(n_1)}, \bar{J}^{(n_2)}\right)$$

$$\equiv \left(J^{(n_2)} \cdot (2p^{(n_1)} + p^{(n_2)})\right) J^{(n_1)\,\alpha} - \left(J^{(n_1)} \cdot (p^{(n_1)} + 2\,p^{(n_2)})\right) J^{(n_2)\,\alpha} + \left(J^{(n_1)} \cdot J^{(n_2)}\right) \left(p^{(n_2)} - p^{(n_1)}\right)^{\alpha}$$

$$C_q^{(n_1+n_2)} = \theta[L^{(n_1)}] \left[C_q^{(n_1)} \left(J^{(n_2)} \cdot (2\,p^{(n_1)} + p^{(n_2)})\right) \right] + \theta[L^{(n_2)}] \left[-C_q^{(n_2)} \left(J^{(n_1)} \cdot (p^{(n_1)} + 2\,p^{(n_2)})\right) \right] + \left(\theta[L^{(n_2)}]\,(-1)^{L^{(n_2)}+1} - \theta[L^{(n_1)}]\,(-1)^{L^{(n_1)}+1} \right) \left[(J^{(n_1)} \cdot J^{(n_2)}) \right]$$

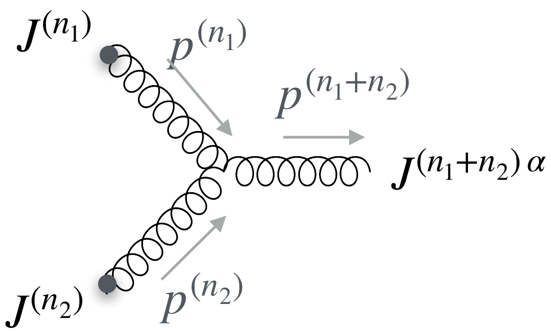
$$C_\varepsilon^{(n_1+n_2)} = \theta[L^{(n_1)}] \left[C_\varepsilon^{(n_1)} \left(J^{(n_2)} \cdot (2\,p^{(n_1)} + p^{(n_2)})\right) \right] + \theta[L^{(n_2)}] \left[-C_\varepsilon^{(n_2)} \left(J^{(n_1)} \cdot (p^{(n_1)} + 2\,p^{(n_2)})\right) \right]$$

$$C_{\varepsilon q,q}^{(n_1+n_2)} = \theta[L^{(n_1)}] \left[C_{q\varepsilon,q}^{(n_1)} \left(J^{(n_2)} \cdot (2\,p^{(n_1)} + p^{(n_2)})\right) \right] + \theta[L^{(n_2)}] \left[-C_{q\varepsilon,q}^{(n_2)} \left(J^{(n_1)} \cdot (p^{(n_1)} + 2\,p^{(n_2)})\right) \right] + \left(\theta[L^{(n_2)}]\,(-1)^{L^{(n_2)}+1} - \theta[L^{(n_1)}]\,(-1)^{L^{(n_1)}+1} \right) \left[\theta[L^{(n_1)}]\,(J_{q\varepsilon}^{(n_1)} \cdot J^{(n_2)}) + \theta[L^{(n_2)}]\,(J^{(n_1)} \cdot J_{q\varepsilon}^{(n_2)}) \right]$$

$$J_{\varepsilon q}^{(n_1+n_2)\,\alpha} = \theta[L^{(n_1)}] \left[\mathrm{V}^{\alpha}_{\mathrm{ggg}} \left(J_{q\varepsilon}^{(n_1)}, J^{(n_2)}\right) - \left(C_\varepsilon^{(n_1)} + \mu\, C_{q\varepsilon,q}^{(n_1)}\right) J^{(n_2)\,\alpha} \right] + \theta[L^{(n_2)}] \left[\mathrm{V}^{\alpha}_{\mathrm{ggg}} \left(J^{(n_1)}, J_{q\varepsilon}^{(n_2)}\right) + \left(C_\varepsilon^{(n_2)} + \mu\, C_{q\varepsilon,q}^{(n_2)}\right) J^{(n_1)\,\alpha} \right]$$

$$T_{q/\varepsilon}^{(n_1+n_2)\,\alpha} = \theta[L^{(n_1)}] \left[\left(J^{(n_2)} \cdot (2\,p^{(n_1)} + p^{(n_2)})\right) T_{q/\varepsilon}^{(n_1)\,\alpha} \right] + \theta[L^{(n_2)}] \left[-\left(J^{(n_1)} \cdot (p^{(n_1)} + 2\,p^{(n_2)})\right) T_{q\varepsilon}^{(n_2)\,\alpha} \right]$$

$$T^{(n_1+n_2)} = \theta[L^{(n_1)}] \left[\left(J^{(n_2)} \cdot (2\,p^{(n_1)} + p^{(n_2)})\right) T^{(n_1)} \right] + \theta[L^{(n_2)}] \left[-\left(J^{(n_1)} \cdot (p^{(n_1)} + 2\,p^{(n_2)})\right) T^{(n_2)} \right]$$



$$\equiv \left(J^{(n_2)}\cdot(2p^{(n_1)}+p^{(n_2)})\right) J^{(n_1)\,\alpha} - \left(J^{(n_1)}\cdot(p^{(n_1)}+2\,p^{(n_2)})\right) J^{(n_2)\,\alpha} + \left(J^{(n_1)}\cdot J^{(n_2)}\right)\left(p^{(n_2)}-p^{(n_1)}\right)^{\alpha}$$

$$\equiv \mathrm{V}^{\alpha}_{\mathrm{ggg}}\left(J^{(n_1)},\bar{J}^{(n_2)}\right)$$

$$\begin{aligned}
\mathscr{E}[\,J^{(n_1+n_2)\,\alpha}\,] = \; & \left(2\,\theta[L^{(n_1)}]\,(-1)^{L^{(n_1)}+1} + \theta[L^{(n_2)}]\,(-1)^{L^{(n_2)}+1}\right)\left[\theta[L^{(n_2)}]\;\mathscr{E}\left[(J^{(n_2)}\cdot q)\right] J^{(n_1)\,\alpha} + \theta[L^{(n_2)}-1]\left(\mu\,J^{(n_1)}_{q\varepsilon}\right)\right] \\
- & \left(\theta[L^{(n_1)}]\,(-1)^{L^{(n_1)}+1} + 2\,\theta[L^{(n_2)}]\,(-1)^{L^{(n_2)}+1}\right)\left[\theta[L^{(n_1)}]\;\mathscr{E}\left[(J^{(n_1)}\cdot q)\right] J^{(n_2)\,\alpha} + \theta[L^{(n_1)}-1]\left(\mu\,J^{(n_2)}_{q\varepsilon}\right)\right] \\
+ & \left(\theta[L^{(n_1)}]\;\theta[L^{(n_2)}]\right)\;\mathscr{E}\left[(J^{(n_1)}\cdot J^{(n_2)})\right]\;\left(p^{(n_2)}-p^{(n_1)}\right)^{\alpha}
\end{aligned}$$

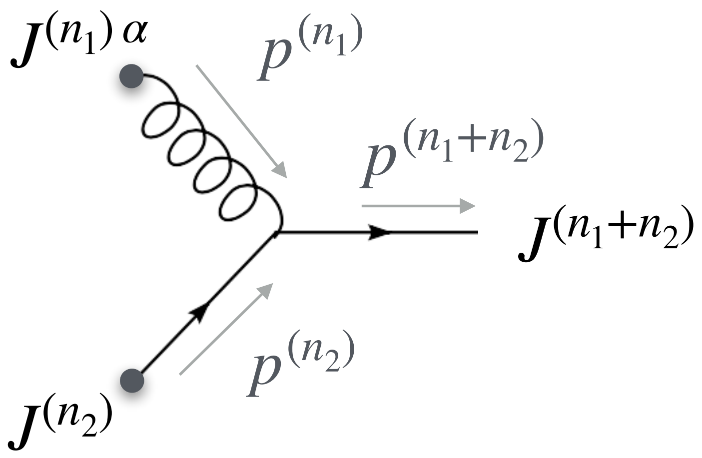
---

where:

$$\mathscr{E}\left[(J^{(n_1)}\cdot q)\right] = \mu\left[C^{(n_1)}_{q} + (\varepsilon_{\lambda}\cdot T^{(n_1)}_{q/\varepsilon})\right] \qquad \mathscr{E}\left[(J^{(n_2)}\cdot q)\right] = \mu\left[C^{(n_2)}_{q} + (\varepsilon_{\lambda}\cdot T^{(n_2)}_{q/\varepsilon})\right]$$

$$\begin{aligned}
\mathscr{E}\left[(J^{(n_1)}\cdot J^{(n_2)})\right] = \; & \theta[L^{(n_1)}]\;\theta[L^{(n_2)}-1]\left[\mu\,C^{(n_1)}_{q\varepsilon,q} + (d-4)\left(C^{(n_1)}_{\varepsilon} - (q\cdot T^{(n_1)}_{q/\varepsilon})\right)\right] \\
+ \; & \theta[L^{(n_1)}-1]\;\theta[L^{(n_2)}]\left[\mu\,C^{(n_2)}_{q\varepsilon,q} + (d-4)\left(C^{(n_2)}_{\varepsilon} - (q\cdot T^{(n_2)}_{q/\varepsilon})\right)\right]
\end{aligned}$$

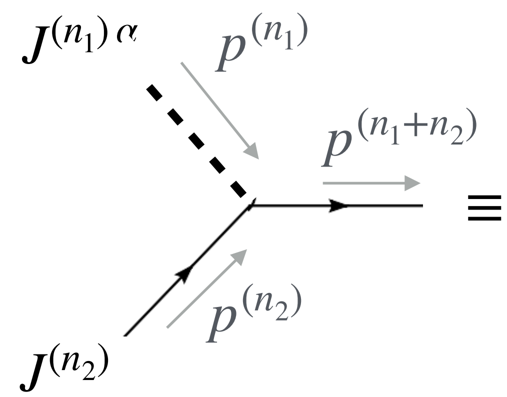
Including fermionic contributions (1-loop)



$$\equiv \not p^{(n_1+n_2)} \not J^{(n_1)} J^{(n_2)}$$

$$p^{(n_1+n_2)} \equiv \hat q^{(n_1+n_2)} + q$$

$$\equiv \mathrm{V}_{\mathrm{qgq}}\left(J^{(n_1)}, J^{(n_2)}, p^{(n_1+n_2)}\right)$$



$$\equiv \not p^{(n_1+n_2)} J^{(n_1)} J^{(n_2)}$$

$$\equiv \mathrm{S}_{\mathrm{qsq}}\left(J^{(n_1)}, J^{(n_2)}, p^{(n_1+n_2)}\right)$$

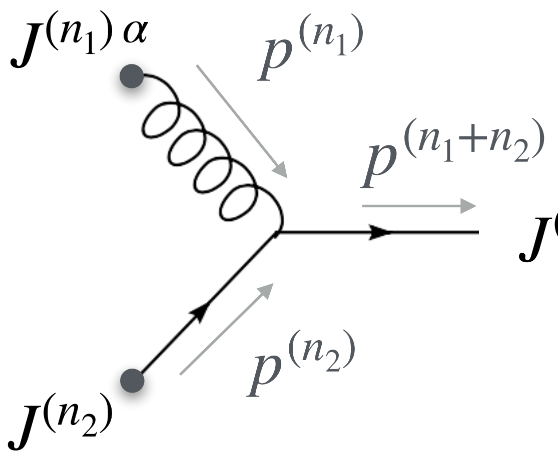
$$\psi_q^{(n_1+n_2)} = \theta[L^{(n_1)}]\left[-\mathrm{S}_{\mathrm{qsq}}\left(C_q^{(n_1)}, J^{(n_2)}, \hat q^{(n_1+n_2)}\right)\right] + \theta[L^{(n_2)}]\left[\mathrm{V}_{\mathrm{qgq}}\left(J^{(n_1)}, \bar\psi_q^{(n_2)}, \hat q^{(n_1+n_2)}\right)\right] + \left(1-\delta_{(n_1+n_2)N_{end}}\right) \not J^{(n_1)} J^{(n_2)}$$

$$\psi_\varepsilon^{(n_1+n_2)} = \theta[L^{(n_1)}]\left[-\mathrm{S}_{\mathrm{qsq}}\left(C_\varepsilon^{(n_1)}, J^{(n_2)}, \hat q^{(n_1+n_2)}\right)\right] + \theta[L^{(n_2)}]\left[\mathrm{V}_{\mathrm{qgq}}\left(J^{(n_1)}, \bar\psi_\varepsilon^{(n_2)}, \hat q^{(n_1+n_2)}\right)\right]$$

$$\psi_{q/\varepsilon}^{(n_1+n_2)} = \theta[L^{(n_2)}]\left[\mathrm{V}_{\mathrm{qgq}}\left(J^{(n_1)}, \bar\psi_{q/\varepsilon}^{(n_2)}, \hat q^{(n_1+n_2)}\right)\right] + \theta[L^{(n_1)}]\left(1-\theta[L^{(n_2)}]\right)\left[C_\varepsilon^{(n_1)} J^{(n_2)}\right] + \theta[L^{(n_2)}]\left(1-\theta[L^{(n_1)}]\right)\left[-\not J^{(n_1)} \psi_\varepsilon^{(n_2)}\right]$$

$$\psi_{\varepsilon q}^{(n_1+n_2)} = \theta[L^{(n_1)}]\left[\mathrm{V}_{\mathrm{qgq}}\left(J_{\varepsilon q}^{(n_1)}, J^{(n_2)}\right) + \mu\, C_{\varepsilon q,q}^{(n_1)} J^{(n_2)}\right] + \theta[L^{(n_2)}]\left[\mathrm{V}_{\mathrm{qgq}}\left(J^{(n_1)}, J_{\varepsilon q}^{(n_2)}\right) - \mu \not J^{(n_1)} \psi_{\varepsilon q,q}^{(n_1)}\right]$$

$$\psi_{\varepsilon q,q}^{(n_1+n_2)} = \theta[L^{(n_1)}]\left[-\mathrm{S}_{\mathrm{qsq}}\left(C_{\varepsilon q,q}^{(n_1)}, J^{(n_2)}, \hat q^{(n_1+n_2)}\right)\right] + \theta[L^{(n_2)}]\left[\mathrm{V}_{\mathrm{qgq}}\left(J^{(n_1)}, \bar\psi_{\varepsilon q,q}^{(n_2)}, \hat q^{(n_1+n_2)}\right)\right]$$
$$+ \left(\theta[L^{(n_1)}]\left(1-\theta[L^{(n_2)}]\right)\right)\left[\not J_{\varepsilon q}^{(n_1)} J^{(n_2)}\right] + \left(\theta[L^{(n_2)}]\left(1-\theta[L^{(n_1)}]\right)\right)\left[\not J^{(n_1)} J_{\varepsilon q}^{(n_2)}\right]$$



$$J^{(n_1+n_2)} \equiv \not{p}^{(n_1+n_2)} \not{J}^{(n_1)} J^{(n_2)}$$

$$p^{(n_1+n_2)} \equiv \hat{q}^{(n_1+n_2)} + q$$

$$\equiv \mathrm{V}_{\mathrm{qgq}}\left(J^{(n_1)}, J^{(n_2)}, p^{(n_1+n_2)}\right)$$

$$\begin{aligned}\mathcal{E}[\,J^{(n_1+n_2)}\,] = {} & \theta[L^{(n_1)}]\,\left[\hat{\not{q}}^{(n_1+n_2)}\,\mathcal{E}[\not{J}^{(n_1)}]\,J^{(n_2)}\right] + \theta[L^{(n_1)}]\left(1-\theta[L^{(n_2)}]\right)\left[\mathcal{E}[\not{q}\not{J}^{(n_1)}]\,J^{(n_2)}\right] \\ & + \left(1-\theta[L^{(n_1)}]\right)\theta[L^{(n_2)}]\left[-\not{J}^{(n_1)}\,\mathcal{E}[\not{q}J^{(n_2)}]\right] \\ & + \theta[L^{(n_1)}-1]\,\theta[L^{(n_2)}]\left[\hat{\not{q}}^{(n_1+n_2)}\,\mathcal{E}[\not{J}^{(n_1)}J^{(n_2)}]\right]\end{aligned}$$

where:

$$\mathcal{E}[\not{J}^{(n_1)}] = -(d-4)\,T^{(n_1)}$$

$$\mathcal{E}[\not{q}\not{J}^{(n_1)}] = \mu\,C_q^{(n_1)} - (d-4)\,T^{(n_1)}\not{q}$$

$$\mathcal{E}[\not{q}J^{(n_2)}] = \mu\left(\psi_q^{(n_2)} + \not{\varepsilon}_\lambda\,\psi_{q/\varepsilon}^{(n_2)}\right)$$

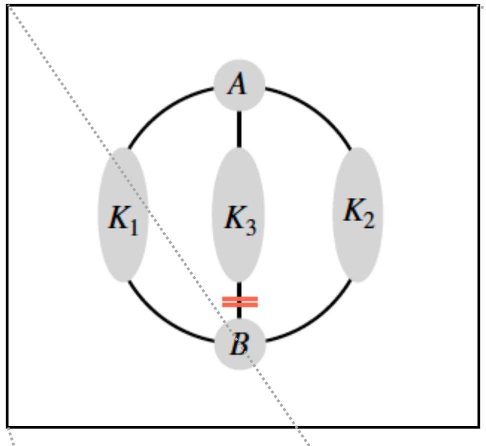
$$\mathcal{E}\left[\not{J}^{(n_1)}J^{(n_2)}\right] = \mu\,\psi_{\varepsilon q,q}^{(n_2)} + (d-4)\left[\psi_\varepsilon^{(n_2)} - \not{q}\,\psi_{q/\varepsilon}^{(n_2)}\right]$$

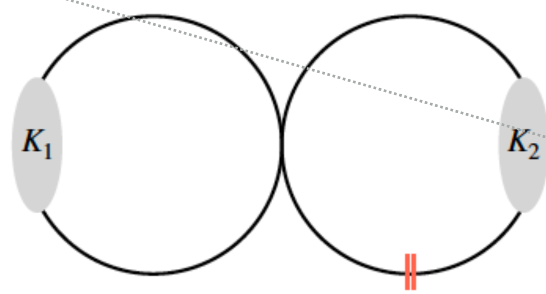
III. Beyond 1-loop

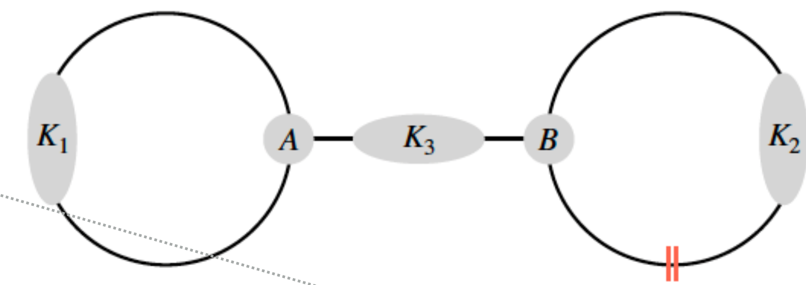
Steps to 2-loop construction

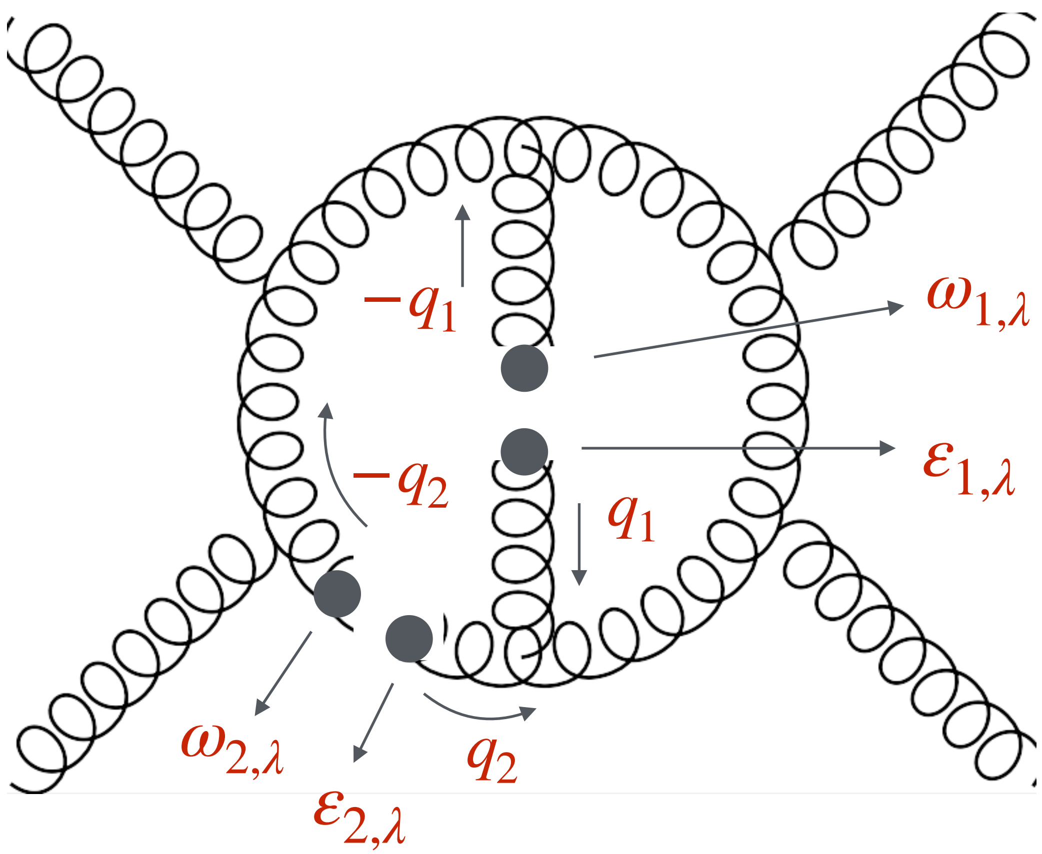
[work in progress]

Basic notation at 2-loop









$$\sum_\lambda \varepsilon^\alpha_{1,\lambda}\, \omega^\beta_{1,\lambda} = g^{\alpha\beta}$$

$$\sum_\lambda \varepsilon^\alpha_{2,\lambda}\, \omega^\beta_{2,\lambda} = g^{\alpha\beta}$$

Origin of evanescent terms at 2 loops

*QCD, only gluons

$$\mathscr{E}[\,(q_i \cdot q_j)\, X\,] = \mu_{ij}\, X \qquad [i,j=1,2]$$

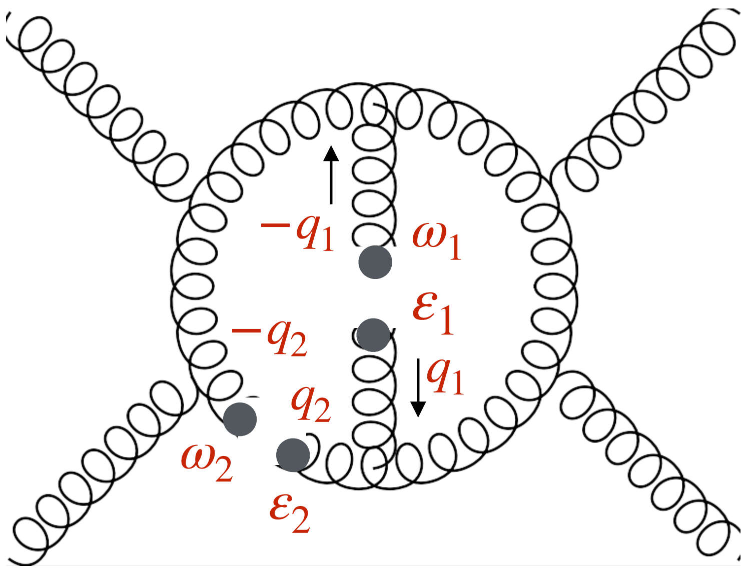
$$\mathscr{E}[\,\sum_{\lambda} (q_i \cdot \varepsilon_{k,\lambda})\,(q_j \cdot \omega_{k,\lambda})\; X\,] = \mu_{ij}\, X \qquad [i,j=1,2]$$

$$\mathscr{E}[\,\sum_{\lambda} (\varepsilon_{i,\lambda} \cdot \omega_{i,\lambda})\; X\,] = (d-4)\, X \qquad [i,j=1,2]$$

$$\mathscr{E}[\,\sum_{\lambda_1,\lambda_2} (\varepsilon_{1,\lambda_1} \cdot \omega_{1,\lambda_1})\,(\varepsilon_{2,\lambda_2} \cdot \omega_{2,\lambda_2})\; X\,] = (d-4)\, X$$

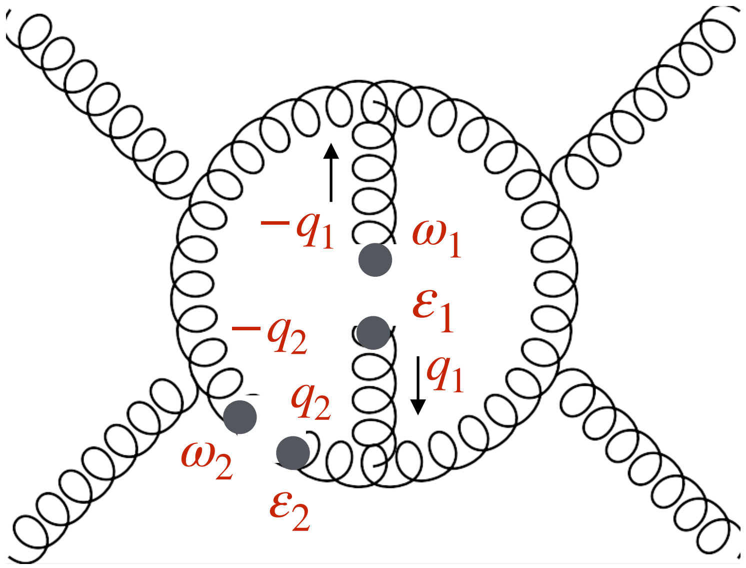
$$\mathscr{E}[\,\sum_{\lambda_1,\lambda_2} (\varepsilon_{1,\lambda_1} \cdot \omega_{2,\lambda_2})\,(\varepsilon_{2,\lambda_2} \cdot \omega_{1,\lambda_1})\; X\,] = (d-4)\, X$$

$$\mathscr{E}[\,\sum_{\lambda_1,\lambda_2} (\varepsilon_{1,\lambda_1} \cdot \varepsilon_{2,\lambda_2})\,(\omega_{1,\lambda_1} \cdot \omega_{2,\lambda_2})\; X\,] = (d-4)\, X$$

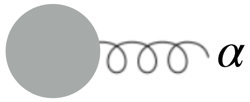


Origin of evanescent terms at 2 loops

- Case study: *planar* configurations



$$
\begin{aligned}
J^{(N)\,\alpha} \equiv\; & C_{q_1}{}^{(N)} q_1^\alpha + C_{q_2}{}^{(N)} q_2^\alpha + C_{\varepsilon_1}{}^{(N)} \varepsilon_1^\alpha + C_{\varepsilon_2}{}^{(N)} \varepsilon_2^\alpha + C_{\omega_1}{}^{(N)} \omega_1^\alpha \\
& + (\varepsilon_1 \cdot \varepsilon_2) \, \underbrace{\left[C^{(N)\,\alpha}_{\varepsilon_1\varepsilon_2,q_1} q_1^\alpha + C^{(N)\,\alpha}_{\varepsilon_1\varepsilon_2,q_2} q_2^\alpha + C^{(N)\,\alpha}_{\varepsilon_1\varepsilon_2,\omega_1} \omega_1^\alpha + X^{(N)\,\alpha}_{\varepsilon_1\varepsilon_2} \right]}_{\equiv\, J^{(N)\,\alpha}_{\varepsilon_1\varepsilon_2}} \\
& + \sum_{i,j=1}^{2} (\varepsilon_i \cdot q_j) \, \underbrace{\left[C^{(N)\,\alpha}_{\varepsilon_i q_j,q_1} q_1^\alpha + C^{(N)\,\alpha}_{\varepsilon_i q_j,q_2} q_2^\alpha + C^{(N)\,\alpha}_{\varepsilon_i q_j,\omega_1} \omega_1^\alpha + X^{(N)\,\alpha}_{\varepsilon_i q_j} \right]}_{\equiv\, J^{(N)\,\alpha}_{\varepsilon_i q_j}} \\
& + \sum_{i,j=1}^{2} (\varepsilon_1 \cdot q_i)\,(\varepsilon_2 \cdot q_j) \, \underbrace{\left[C^{(N)\,\alpha}_{\varepsilon_1 q_i \varepsilon_2 q_j,q_1} q_1^\alpha + C^{(N)\,\alpha}_{\varepsilon_1 q_i \varepsilon_2 q_j,q_2} q_2^\alpha + X^{(N)\,\alpha}_{\varepsilon_1 q_i \varepsilon_2 q_j} \right]}_{\equiv\, J^{(N)\,\alpha}_{\varepsilon_1 q_i \varepsilon_2 q_j}}
\end{aligned}
$$



Summary and outlook

- Work is underway to enable HELAC to to perform numerical computations of loop numerators in $d = 4 - 2\varepsilon$ dimensions
- Interesting applications in the context of the two-loop reduction problem
- We are formulating an alternative method to compute explicit dependence of the numerators upon ε and μ_{ij} terms via modified recursion relations
- Selected examples of modified recursion relations have been discussed for the case of massless QCD at 1-loop
- Extension to the 2-loop case in progress