# Two-loop amplitude reduction in the HELAC framework

## Based on work with Costas Papadopoulos, Giuseppe Bevilacqua, Dhimiter Canko and Nikos Dokmetzoglou

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# Amplitude Construction

- Qgraph $\rightarrow$ symbolic manipulation, dimensionally regularized amplitudes $\rightarrow$ IBP: FIRE Kira or numerical pySecDec
- Numerical unitarity $\rightarrow$ dimensionally regularized amplitudes by gluing tree amplitudes in different integer dimensions $\rightarrow D_s$ (Abreu, Cordero, Ita, Page and Sotnikov, 2021a)
- OpenLoops $\rightarrow$ Feynman graph $\rightarrow$ openning the loops $\rightarrow$ amplitudes in $d = 4 \rightarrow$ coefficients of tensor integrals (Pozzorini, Schär and Zoller, 2022)

# NNLO Results

2-loop amplitude is required for NNLO precision.

$$\hat{\sigma}_{NNLO} \sim |A_{tree}|^2 + \alpha_s \left(2Re[A_{tree}A^*_{1loop}] + |A_{+1p}|^2\right) + \\ + \alpha_s^2 \left(|A_{1loop}|^2 + 2Re[A_{tree}A^*_{1loop+1p} + A_{tree}{\color{red}A^*_{2loop}}] + |A_{+2p}|^2\right)$$

Various results:

- $pp \to W + \gamma + j$ (Badger, Hartanto, Kryś and Zoia, 2022)
- $pp \to H + b\bar{b}$ (Badger, Hartanto, Kryś and Zoia, 2021b)
- $u\bar{d} \to W^+ + b\bar{b}$ (Badger, Hartanto, Kryś and Zoia, 2022)
- $pp \to b\bar{b}$(Hartanto, Poncelet, Popescu and Zoia, 2022)
- $pp \to Zb\bar{b}$ (Mazzitelli, Sotnikov and Wiesemann, 2024)
- $q\bar{q} \to gg\gamma$ and $q\bar{q} \to Q\bar{Q}\gamma$ (Badger, Czakon, Hartanto, Moodie, Peraro, Poncelet and Zoia, 2023)
- $pp \to \gamma\gamma + j(Chawdhry, Czakon, MitovandPoncelet, \ 2021a)$
- $pp \to W + 4partons$ (Abreu, Cordero, Ita, Klinkert, Page and Sotnikov, 2022; Hartanto, Badger, Brønnum-Hansen and Peraro, 2019)
- $gg \to g\gamma\gamma$ (Badger, Brønnum-Hansen, Chicherin, Gehrmann, Hartanto, Henn, Marcoli, Moodie, Peraro and Zoia, 2021a)
- $qg \to q\gamma\gamma$ and $q\bar{q} \to q\gamma\gamma$ (Agarwal, Buccioni, von Manteuffel and Tancredi, 2021)
- $q\bar{q} \to \gamma\gamma\gamma$ (Abreu, Page, Pascual and Sotnikov, 2021b; Chawdhry, Czakon, Mitov and Poncelet, 2021b)
- $pp \to 3j$ (Abreu, Cordero, Ita, Page and Sotnikov, 2021a; Czakon, Mitov and Poncelet, 2021)
- 5 partons (Abreu, Dormans, Febres Cordero, Ita, Page and Sotnikov, 2019b; Abreu, Febres Cordero, Ita, Page and Sotnikov, 2018)
- $gg \to ggg$ (Badger, Frellesvig and Zhang, 2013; Badger, Brønnum-Hansen, Hartanto and Peraro, 2019a; Abreu, Dormans, Cordero, Ita and Page, 2019a; Badger, Chicherin, Gehrmann, Heinrich, Henn, Peraro, Wasser, Zhang and Zoia, 2019b)
- $pp \to \gamma\gamma\gamma$(Kallweit, Sotnikov and Wiesemann, 2021)

# HELAC @ Tree level

Costas Papadopoulos and Aggeliki Kanaki introduced HELAC (Kanaki and Papadopoulos, 2000)
This was an automated way to compute amplitudes using the Dyson-Swinger Equations i.e. recursively expressing n-point Green's Functions in terms of lesser point Green's Functions.

# What do we need for HELAC 2-loop?

3 steps:

- 1. Amplitude Construction (*Giuseppe B. talk on Tuesday*)
- **2. Amplitude Reduction at 2-loops (My talk today)**
- 3. Do the integrals (*Nikos D. Talk right after me*)

# Outline

- HELAC amplitude construction
- OPP Amplitude reduction at 1-loop
- 2-loop reduction: 4 vs D-dimensions
- Outlook

# Dyson Schwinger Equations

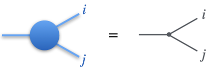



Figura: First Step of the recursion

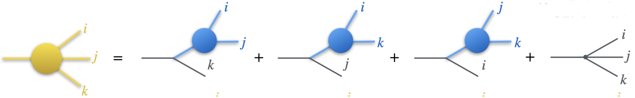



Figura: Second Step of the recursion

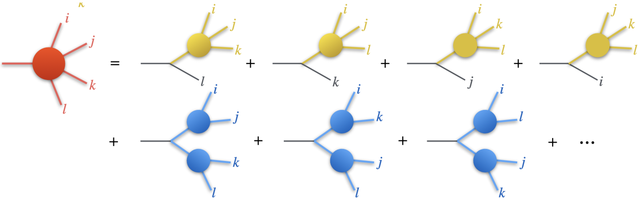



Figura: Third Step of the recursion

# Simple Example $f\bar{f} \to f\bar{f}b$

Binary ID assigned to each particle in powers of 2, so we can write our building blocks as $\psi(1), \bar{\psi}(2), \bar{\psi}(4), \psi(8), A(16)$.
Use this to build all the level 2 sub-amplitudes

- $A_\mu(12) = (ig)\Pi_\mu^\nu \bar{\psi}(4)\gamma_\nu \psi(8)$
- $\bar{\psi}(18) = (ig)\bar{\psi}(2)\not{A}(16)\mathcal{P}$
- $\bar{\psi}(20) = (ig)\bar{\psi}(4)\not{A}(16)\mathcal{P}$
- $\psi(24) = (ig)\mathcal{P}\not{A}(16)\psi(8)$

where $\Pi_{\mu\nu}$ is the boson propagator and $\mathcal{P}$ is the fermion propagator.

# Simple Example $f\bar{f} \to f\bar{f}b$

Continue by building the 3-level subamplitudes

- $A_\mu(12) = \Pi_\mu^\nu \bar{\psi}(4)\gamma_\nu\psi(8)$
- $A_\mu(28) = (ig)\Pi_\mu^\nu \left(\bar{\psi}(20)\gamma_\nu\psi(8) + \bar{\psi}(4)\gamma_\nu\psi(24)\right)$

4-level subamplitudes

- $\bar{\psi}_0(30) = (ig)\left(\bar{\psi}(2)\not{A}(28) + \bar{\psi}(14)\not{A}(16) + \bar{\psi}(18)\not{\psi}(12)\right)$

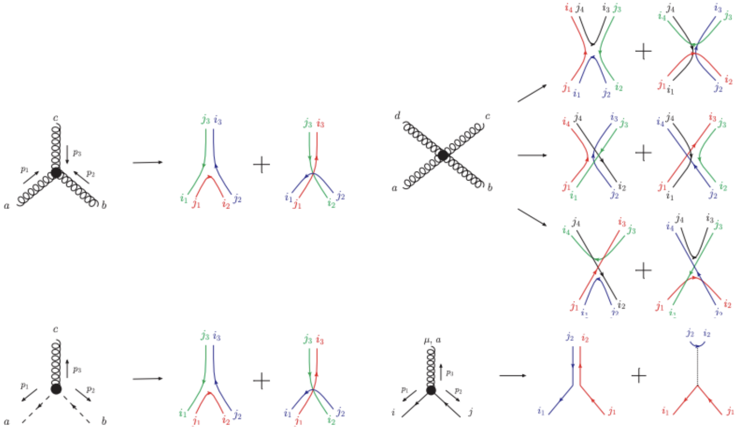
And so the amplitude is finally given by

$$\mathcal{A} = \bar{\psi}_0(30)\psi(1)$$

The recursion is made to always end and the Binary ID 1.
This is how HELAC calculates amplitudes!

# Color Flow representation

How do we deal with QCD?



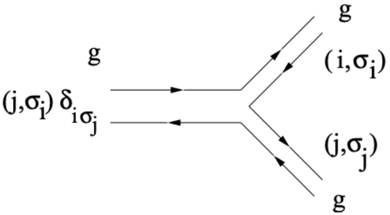
Each colored line corresponds to a Kronecker delta.

# Color Flow Example

How do we deal with QCD? Color connection representation: gluons are represented by a pair of color/anti-color indices $(i, j)$ and the quarks (anti-quarks) by a single color $(i, 0)$ (anti-color $(0, j)$) index, with $i, j \in (1, ..., N_C)$. All the other particles that do not carry color have

$$(0, 0)$$

. First term in the 3-gluon color factor:




or

$$\delta_{1\sigma_2}\delta_{2\sigma_3}\delta_{3\sigma_1}$$

and so on.

# Color Flow representation

Therefore, the color factor can in total be written as

$$\mathcal{F} = \delta_{1\sigma_I(1)}\delta_{2\sigma_I(2)}\dots\delta_{n\sigma_I(n)}$$

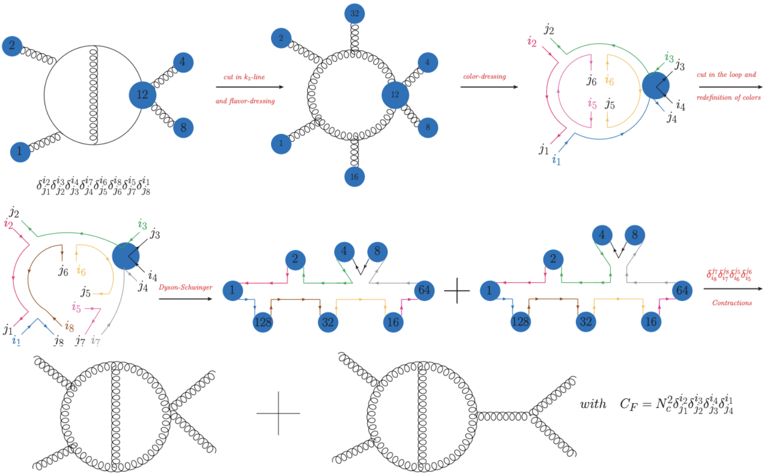
where $\sigma$ is some permutation of $1\dots n$ and $I = 1,\dots n!$
We can therefore write a general amplitude in the form

$$\mathcal{M} = \sum_{\sigma}\mathcal{F}\mathcal{A}_\sigma$$

where the color striped amplitude is properly calculated using appropriate Feynman rules.
**Benefits**:

- Computational complexity is polynomial
- Amplitude squared form is simply a product of Kroenecker deltas leading to a simple form.

# Amplitude Construction at 2-Loops



# Amplitude Construction at 2-Loops: Skeleton

The information for each amplitude is incorporated in the Skeleton

```
          INFO ==================================================
          INFO COLOR             9 out of           24
          INFO number of nums            0
          INFO ==================================================
          INFO COLOR            10 out of           24
          INFO number of nums          332
          INFO NUM             1  of          332             8
 INFO NUM          110  of          332           7
 INFO ======================================
INFO    4  80  35   9   1   1  16  35   5  64  35   7   0   0   0   0   1   2
INFO    4  12  35  10   1   1   4  35   3   8  35   4   0   0   0   0   1   1
INFO    4  92  35  11   1   2  12  35  10  80  35   9   0   0   0   0   1   1
INFO    5  92  35  11   2   2   4  35   3   8  35   4  80  35   9   0   1   5
INFO    4 124  35  12   1   1  32  35   6  92  35  11   0   0   0   0   1   2
INFO    4 126  35  13   1   1   2  35   2 124  35  12   0   0   0   0   1   1
INFO    4 254  35  14   1   1 128  35   8 126  35  13   0   0   0   0   1   2
INFO    6   1  12   1   2  12  35  35  35  35  35  35   0   0   0   0   5   9
```

From the first line to the second-to-last line, there is a sequence of sub-amplitudes accompanied by instructions for their computation.

# Amplitude Construction

So far, everything is done in 4 dimensions. But, this is not necessary! The skeleton doesn't know about the number of dimensions! **See the talks by Giuseppe/Costas past Tuesday!**

# Amplitude reduction OPP @ 1-loop

Reduction is done at the *integrand* level. OPP (Ossola, Papadopoulos and Pittau, 2007) master formula a numerator at 1 loop:

$$
\begin{aligned}
N(q) = & \sum_{i_0<i_1<i_2<i_3}^{m-1} d\left(i_0 i_1 i_2 i_3\right) + \tilde{d}\left(q; i_0 i_1 i_2 i_3\right) \prod_{i \neq i_0, i_1, i_2, i_3}^{m-1} D_i \\
& + \sum_{i_0<i_1<i_2}^{m-1} c\left(i_0 i_1 i_2\right) + \tilde{c}\left(q; i_0 i_1 i_2\right) \prod_{i \neq i_0, i_1, i_2}^{m-1} D_i \\
& + \sum_{i_0<i_1<i_2}^{m-1} b\left(i_0 i_1\right) + \tilde{b}\left(q; i_0 i_1\right) \prod_{i \neq i_0, i_1}^{m-1} D_i \\
& + \sum_{i_0}^{m-1} a\left(i_0\right) + \tilde{a}\left(q; i_0\right) \prod_{i \neq i_0}^{m-1} D_i
\end{aligned}
$$

where $\tilde{d}$, $\tilde{c}$, $\tilde{b}$, $\tilde{a}$ are terms which vanish upon integration.

# Amplitude reduction OPP: Fitting

The system is solved by iteratively:

- Evaluate the numerator on values of $k$ for which $D_i(k) = 0$, starting from the first line of the previous equation, where 4 propagators are put on shell
- This condition by default sets to zero all the rest of the terms, allowing us to calculate $d$ and $\tilde{d}$
- Repeat for $c$, $\tilde{c}$ and so on until we fit all the coefficients

# Rational Terms

We need to account for the mismatch between 4 and $D = 4 - 2\epsilon \longrightarrow$ Rational Terms.

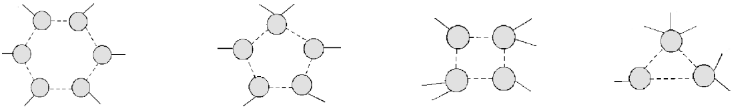
Categorized into two terms (Ossola, Papadopoulos and Pittau, 2008b):

- R1, arising from the mismatch between thh of D- and 4- dimensional propagators.
- R2, arising from the extra dimensional parts of the loop momentum, metric tensor and gamma matrices.

# 1 Loop Example

Example of a 6 point amplitude

$$\mathcal{A}(q) = \sum \underbrace{\frac{N_i^{(6)}(q)}{\bar{D}_{i_0}\bar{D}_{i_1}\cdots\bar{D}_{i_5}}} + \underbrace{\frac{N_i^{(5)}(q)}{\bar{D}_{i_0}\bar{D}_{i_1}\cdots\bar{D}_{i_4}}} + \underbrace{\frac{N_i^{(4)}(q)}{\bar{D}_{i_0}\bar{D}_{i_1}\cdots\bar{D}_{i_3}}} + \underbrace{\frac{N_i^{(3)}(q)}{\bar{D}_{i_0}\bar{D}_{i_1}\bar{D}_{i_2}}} + \cdots$$

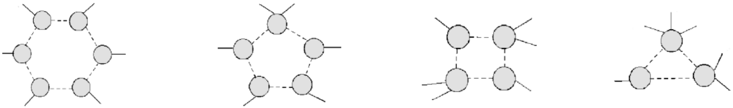


- HELAC numerically computes the numerators taking into account all flavors and colors that are consistent with the given process
- The value of the loop momentum on the cut is given by CutTools (Ossola, Papadopoulos and Pittau, 2008a)

# 1 Loop Example

Example of a 6 point amplitude

$$\mathcal{A}(q) = \sum \underbrace{\frac{N_i^{(6)}(q)}{\bar{D}_{i_0}\bar{D}_{i_1}\cdots\bar{D}_{i_5}}} + \underbrace{\frac{N_i^{(5)}(q)}{\bar{D}_{i_0}\bar{D}_{i_1}\cdots\bar{D}_{i_4}}} + \underbrace{\frac{N_i^{(4)}(q)}{\bar{D}_{i_0}\bar{D}_{i_1}\cdots\bar{D}_{i_3}}} + \underbrace{\frac{N_i^{(3)}(q)}{\bar{D}_{i_0}\bar{D}_{i_1}\bar{D}_{i_2}}} + \cdots$$



Now we can do the reduction:

- Calculate the coefficients of the OPP formula for each numerator
- Use Integration by Parts relations to get a Master Integral Basis
- Compute the rational terms
- **This has been implemented!** $\longrightarrow$**HELAC One Loop!**(Bevilacqua, Czakon, Garzelli, van Hameren, Kardos, Papadopoulos, Pittau and Worek, 2013)

# 2-Loop Amplitude: Reduction

Reduction is done at the *integrand* level. **The fit will be more complex!**
A generic 2-loop integrand can be written using the following scalar product set:

$$\{p_i \cdot p_j, k_i \cdot k_j, k_i \cdot p_j, k_i \cdot \eta_j\}$$

as well as any masses inside the loops. For the rest of the discussion we will ignore the case of massive loops.
The integrand can be written in the general form

$$\mathcal{R} = \frac{\mathcal{N}}{\mathcal{D}} = \frac{\sum_a c_a (z_1^{(a)})^{\beta_1} \ldots (z_{n_a}^{(a)})^{\beta_N}}{D_1 \ldots D_{N_p}} \quad (1)$$

where the $z_i$ are any of the scalar products in the set.

# 2-Loop Amplitude reduction

The above can we written in a more reduced form:

$$\mathcal{R} = \frac{\mathcal{N}}{\mathcal{D}} = \sum_{m=0}^{N_p} \sum_{\sigma} \frac{\sum_a \bar{c}_a (\bar{z}_1^{(a)})^{\alpha_1} \dots (\bar{z}_{N_a}^{(a)})^{\alpha_N}}{D_{\sigma_1} \dots D_{\sigma_m}} \quad (2)$$

where now the $\bar{z}_i$ are only the scalar products which cannot be eliminated by being written as linear combinations of $D_i$, know as irreducible scalar products (ISPs) or the transverse $k_i \cdot \eta_j$ and $\sigma$ is any subset of $\{1, \dots, N_p\}$ with m elements.

Write the numerator of the integrand level amplitude as follows:

**Numerator Formula**

$$\mathcal{N} = \sum_{m=0}^{N_p} \sum_{\sigma} \sum_{a} \bar{c}_a \prod_{i}^{N_{T+ISP}} (\bar{z}_i)^{\alpha_i} \prod_{j} D_j \quad (3)$$

where $N_{T+ISP}$ is the number of ISP and transverse scalar products and $j \neq \sigma_i$ for all $i$ (Bevilacqua, Canko and Papadopoulos, 2024)

# 2-Loop Amplitude reduction

Our goal now is the calculate the coefficients $\bar{c}_a$ which generically depend on the set of scalar products. For each topology, we

- Identify the maximal set of loop propagators we can set to zero, i.e. the Maximal Cut and solve the equations that put all of them on shell simultaneously, AKA *cut solutions*
- Write a linear system for the coefficients $\mathbf{M} \cdot \vec{\bar{c}} = \vec{\mathcal{N}}$ where $\mathbf{M}$ is a matrix of all monomials $\bar{z}_i$ evaluated on different values of cut-solutions, $\vec{c}$ is all the $\bar{c}_a$ and $\vec{\mathcal{N}}$ is a vector of equal length with values of the numerator evaluated on the cut-solutions
- Solve the system of equations
- Subtract the result from the numerator

# 2-Loop Amplitude reduction

- Move to the next cut, where one less propagator is put on shell, AKA a *subtopology*.
- Do this till this for all subtopologies (up to a 2-cut for massless legs)
- Algebraic test that the reduced amplitude is equal to the original amplitude, known as the **N=N test**.
- The reduction is complete!

# 2-Loop Amplitude: Final Form

$$\mathcal{N} = P_{maxcut} + \sum_i P_{maxcut-1} D_i + \sum_{ij} P_{maxcut-2} D_i D_j + \ldots \tag{4}$$

where the Ps are polynomials in the transverse elements and ISPs, with the values of $\bar{c}_a$ such that this equation is satisfied.

**Final Form**:

$$\mathcal{A} = \sum_i \bar{c}_i F_i \tag{5}$$

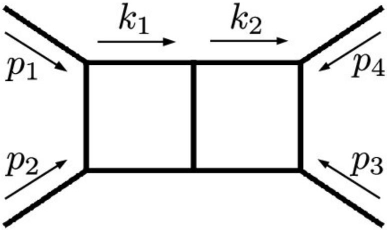
where:

$$F_i \equiv F_{a_1 \ldots a_N} = \int d^d k \frac{\overbrace{(D_{m+1})^{a_{m+1}} \cdots (D_N)^{a_N}}^{ISP}}{\underbrace{(D_1)^{a_1} \cdots (D_m)^{a_m}}_{RSP}}$$

Use Integration by Parts (IBP) and/or other methods to further simplify the integral basis.

# 2-Loop reduction example

Let's look at a specific $2 \to 2$ gluon-topology example, part of current work:




Maximal cut: 7 propagators on shell. Question arises: Can/should we fit in 4 dimensions or $D = 4 - 2\epsilon$ dimension?

# 2-Loop reduction example: 4-dimensions

Begin with 8 free parameters (the components of the loop momenta). Construct solutions to the cut equations. After imposing the on-shell conditions in the maximal cut only 1 parameter is left independent. Can use relations between the various monomials to reduce the number of coefficients we need to fit, implemented through BasisDet (Zhang, 2012). This is the current way we are building the *Ansatz*. The main input we can vary is the max exponent of each monomial.

**Recent result**

Completed a *numerical* Mathematica simulation for all subtopologies of this example, including the **N=N test**.

# 2-Loop reduction example: 4-dimensions

- Some monomials which form the matrix **M** have the same values for different cut solution sets. Danger that we could get $Rank[\mathbf{M}] < Lenght[\vec{c}]$
  $\longrightarrow$ **Need to use full set of solutions, treating each of them independently.**
- How can we derive all the gram determinant relations that hold at each step of the recursion?
  $\longrightarrow$ Use BasisDet for the Ansatz
  $\longrightarrow$ **Write directly in terms of 4d $k_i$ components**.
  *Not unique, different basis lead to a different form of the cut-solutions!*

# 2 loop reduction interesting questions

- What goes into to constructing the polynomial Ansatz? One option is BasisDet (Zhang, 2012). Is there some apriori way of determining it for each topology/subtopology apriori? What is the "correct" power for each monomial at each subtopology?
- Structure of the solution space for the cut solutions (Algebraic Geometry question) (Frellesvig, 2014)
- Are there any integrand level symmetries on the cuts which we can take into account to further simplify?

# More specific 4-d questions

But Frellesvig (2014) mention issues with lower topology cuts:

- A) Some sub-topologies having divergences that require careful subtraction
- B) Other sub-topologies where $\mathrm{Rank}[\mathbf{M}] < \mathrm{Length}[\vec{c}]$, i.e not enough independent equations to fit the polynomial Ansatz.

Both currently under investigation with our approach

The polynomial we use needs to properly capture the functional behaviour of the numerator

Can have

- Full rank $\mathrm{Rank}[\mathbf{M}] = \mathrm{Length}[\vec{c}]$ and no solution (i.e. incomplete Ansatz)

Depends on the relation between the Ansatz and the numerator $\rightarrow$ **Not just a matter of the rank of the matrix!**

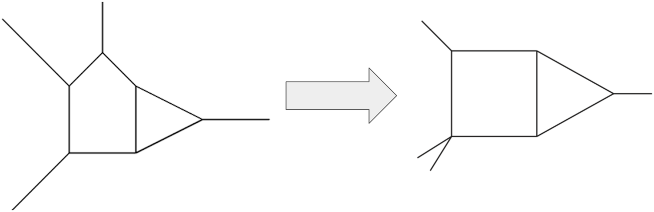
**Open Question:** Can we know the Ansatz apriori from the topological family? Seems difficult...

# Too much Freedom?

# Too much Freedom?

*"Freedom's just another word for nothing left to lose..."*
"Me and Bobby McGee", Kris Kristofferson 1936-2024

# Penta-triangle and subtologies (Preliminary)



Examining the penta-triangle and its 6-cut subtologies

- Do not get a full rank for **M** for *any* of the 6-cut subtopologies in 4-d ...
- We can still fit all of them!

# More specific answers

- A) Don't seem to appear (so far) in the way we construct the amplitude
- B) Some sub-topologies do have $\mathrm{Rank}[\mathbf{M}] < \mathrm{Length}[\vec{c}]$. This does not lead to an unsolvable system!

**Need to have an Ansatz that properly describes the Numerator in question!**

# More specific (partial) Answers: Ansatz

Can have

- $\mathrm{Rank}[\mathbf{M}] < \mathrm{Length}[\vec{c}]$ and get a correct solution!

Depends on the relation between the Ansatz and the numerator.

*The polynomial we use needs to properly capture the functional behaviour of the numerator.*

**Open Question:** Can we know the Ansatz apriori from the topological family?

We don't need to be exact, *just know enough to get a solution!*

**Work is ongoing for this more difficult topology**

Still need to do the lower subtopology cuts numerically

# 2-Loop reduction example: D-dimensions

Begin with 11 free parameters: 8 from the components of the two 4-momenta, and 3 $\mu_{11}, \mu_{22}, \mu_{12}$ the $\epsilon$ components of $k_1^2$, $k_2^2$ and $k_1 \cdot k_2$ respectively.
The maximal cut has 7 cut equations, we have a remainder of 4 free parameters.
Construct solutions with the remaining 4 free parameters use them to solve the system.

## Result

We have completed an *analytic* Mathematica simulation of this fit.
Agreement with known results: Caravel (Abreu, Cordero, Ita, Page and Sotnikov, 2021a)

## Upcoming

Complete cut+fit for all subtologies, and then implement numerically.

# D-dimensional amplitude construction

Currently underway in our group, including a method of numerically evaluating the amplitude in D-dimensions by keeping track of $\epsilon$ and $\mu$ terms.

$\rightarrow$ Is an option since **the skeleton is dimensionally agnostic!**

Note that in principle construction and reduction of the amplitude are completely modular, i.e **independent of each other!**

## None of the above issues appear if all is done in D-dimensions!

# A bit of optimism

*"...feeling good was good enough for me!"*
"Me and Bobby McGee" Kris Kristofferson 1936-2024, "cited" by Janis Joplin, 1943-1970

# Next Steps/Outlook

Immediate steps

- Numerically implement the 4-d reduction within the HELAC framework @ 2-Loops for all $2 \to 2$ topologies
- Implement the fully D-dimensional fit @ 1 and 2 loops

Outlook-Near Future

- D-dimensional fit at 2-loops for all $2 \to 2$ topologies
- $2 \to 3$ topology reduction

Outlook-Further Future Generalize for further topologies!

# Thank you!

# Thank you for listening!

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H.F.R.I.
Hellenic Foundation for Research & Innovation

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# Simple Example $f\bar{f} \to f\bar{f}b$

Binary ID assigned to each particle in powers of 2, so we can write our building blocks as $\psi(1), \bar{\psi}(2), \bar{\psi}(4), \psi(8), A(16)$.
Use this to build all the level 2 sub-amplitudes

- $A_\mu(12) = (ig)\Pi_\mu^\nu \bar{\psi}(4)\gamma_\nu \psi(8)$
- $\bar{\psi}(18) = (ig)\bar{\psi}(2)\not{A}(16)\mathcal{P}$
- $\bar{\psi}(20) = (ig)\bar{\psi}(4)\not{A}(16)\mathcal{P}$
- $\psi(24) = (ig)\mathcal{P}\not{A}(16)\psi(8)$

where $\Pi_{\mu\nu}$ is the boson propagator and $\mathcal{P}$ is the fermion propagator.

# Simple Example $f\bar{f} \to f\bar{f}b$

Continue by building the 3-level subamplitudes

- $A_\mu(12) = \Pi_\mu^\nu \bar{\psi}(4)\gamma_\nu\psi(8)$
- $A_\mu(28) = (ig)\Pi_\mu^\nu \left(\bar{\psi}(20)\gamma_\nu\psi(8) + \bar{\psi}(4)\gamma_\nu\psi(24)\right)$

4-level subamplitudes

- $\bar{\psi}_0(30) = (ig)\left(\bar{\psi}(2)\not{A}(28) + \bar{\psi}(14)\not{A}(16) + \bar{\psi}(18)\not{\psi}(12)\right)$

And so the amplitude is finally given by

$$\mathcal{A} = \bar{\psi}_0(30)\psi(1)$$

The recursion is made to always end and the Binary ID 1.
This is how HELAC calculates amplitudes!

# Amplitude Construction in $D = 4 - 2\epsilon$ dimensions

Clarification: Going from 4 to $D$ dimensions is simple to do analytically for example in Mathematica.

| Structure in $d = 4$ | | Extra term |
|---|---|---|
| $q^2\,X$ | $\longrightarrow$ | $\mu\,X$ |
| $\sum_\lambda (\varepsilon_{\ell_1,\lambda} \cdot \varepsilon_{\ell_2,\lambda})\,X$ | $\longrightarrow$ | $(d-4)\,X$ |
| $\sum_\lambda (q \cdot \varepsilon_{\ell_1,\lambda})(q \cdot \varepsilon_{\ell_2,\lambda})\,X$ | $\longrightarrow$ | $(q^2 + \mu)\,X$ |

How can we do this numerically?

Work is ongoing, led by Giuseppe B., optimistic progress so far.

# Amplitude Construction: Performance

Performace of amplitude generation for various processes

| Process | Loops | Loop-Flavors | Color | Skeleton Size | Timing | Numerators |
|---|---|---|---|---|---|---|
| $gg \to gg$ | 2 | $\{g, c, \bar{c}\}$ | Leading | 8.9 MB | 15.017s | 4560 |
| $gg \to gg$ | 2 | $\{g, q, \bar{q}, c, \bar{c}\}$ | Full | 110.6 MB | 6m 54.574s | 89392 |
| $gg \to q\bar{q}$ | 2 | $\{g, q, \bar{q}, c, \bar{c}\}$ | Full | 16.1 MB | 3m 14.509s | 13856 |
| $gg \to ggg$ | 2 | $\{g, c, \bar{c}\}$ | Leading | 300.0 MB | 21m 42.609s | 81480 |
| $gg \to gg$ | 1 | $\{g, q, \bar{q}, c, \bar{c}\}$ | Full | 537.8 kB | 2.386s | 768 |
| $gg \to ggg$ | 1 | $\{g, q, \bar{q}, c, \bar{c}\}$ | Full | 15.1 MB | 8m 53.349s | 11496 |
| $gg \to gggg$ | 1 | $\{g, c, \bar{c}\}$ | Leading | 394.0 MB | 104m 14.95s | 19680 |

Results obtained running 1-core in a personal laptop (i7 processor, 8-core, 25GB RAM).