Finite Feynman Integrals

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work with **Giulio Gambuti**, **Pavel Novichkov**, *and* Lorenzo Tancredi, [2311.16907]

and in progress with **Leonardo de la Cruz** *and* **Pavel Novichkov** [2409.nnnnn];
and **Marc Canay**;
and in progress with Yang Zhang, **Zhihou Wu**, *and* **Rourou Ma**

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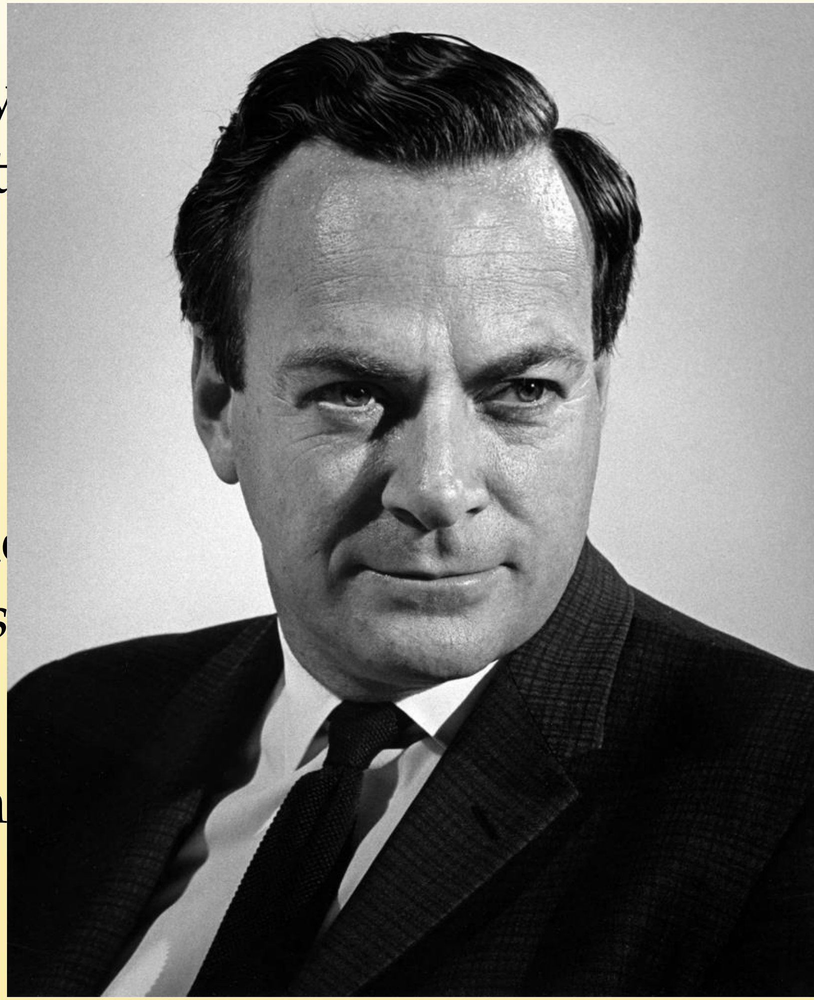
Feynman Integrals



- Key ... op amplitudes, form fact...

$$\ldots \ell_i \, \frac{\mathcal{N}(\ell_i)}{\mathcal{D}_1 \cdots \mathcal{D}_E},$$

- Line ... us to cho se a basis of *mas*...

- Wh...

$\int$ s

Goal

- Select Feynman integrals based on degree of divergence

- Hope: lead to simpler and more transparent representations

- First step: classify and organize **finite** integrals

Henn, Peraro, Stahlhofen, & Wasser; von Manteuffel, Panzer, & Schabinger

- Mostly gloss over fine print
  - look at locally IR-finite integrals (doable strictly in $D = 4$)
  - UV convergent by power counting ("strongly UV convergent")

One-Loop Example

- Canonical basis: box, triangle, bubble
- Box and Triangle have $\frac{1}{\epsilon^2}$ divergence (IR)
- Bubble has $\frac{1}{\epsilon}$ divergence (UV)

- Can trade $D = 4$ box ($\frac{1}{\epsilon^2}$ divergence) for $D = 6$ box (finite)

$$\text{Box}^{D=4} = c_0 \text{Box}^{D=6} + \sum c_i^{(3)} \text{Tri}_i$$

This isolates all IR divergences in triangles

One-Loop Example

- Define $I_m^{(1)}[\mathcal{P}(\ell)] = \int d^D\ell \ \frac{\mathcal{P}(\ell)}{\mathrm{Den}_1 \cdots \mathrm{Den}_m}$

- Introduce Gram determinants

$$G\begin{pmatrix} k_1 & \cdots & k_n \\ q_1 & \cdots & q_n \end{pmatrix} = \det(2\, k_i \cdot q_j)$$

vanishes whenever $k_i \parallel k_j$ or $q_i \parallel q_j$

$$I_4\left[G\begin{pmatrix} \ell k_1 k_2 k_4 \\ \ell k_1 k_2 k_4 \end{pmatrix}\right] \text{ is finite (and in fact } \propto \mathrm{Box}^{D=6}\text{)}$$

One-Loop Example

- Can use similar relation for pentagon

$$\text{Pent}^{D=4} = d_0 \text{Pent}^{D=6} + \sum c_i^{(4)} \text{ Box}_i$$

- $D = 6$ integral is finite, and d_0 is $\mathcal{O}(\epsilon)$: drop pentagon if we truncate to $\mathcal{O}(\epsilon^0)$
  - "evanescent relation"

- Relation is generated by $I_5 \left[G \begin{pmatrix} \ell k_1 k_2 k_3 k_4 \\ \ell k_1 k_2 k_3 k_4 \end{pmatrix} \right] = \mathcal{O}(\epsilon)$

- Generalize these ideas to higher loops

How Do IR Singularities Arise?

- Look at

$$\int d^D \ell \frac{1}{\cdots (\ell - K_1)^2 (\ell - K_2)^2 (\ell - K_3)^2 \cdots}$$

Singularities arise from regions where the denominator vanishes

- One denominator vanishing is integrable
- Two denominators vanishing in an invariant-independent way gives a $\frac{1}{\epsilon}$ singularity
- Three denominators vanishing in an invariant-independent way gives a $\frac{1}{\epsilon^2}$ singularity

How Do IR Singularities Arise?

- Generically,
  - two denominators vanishing must be adjacent propagators separated by a massless leg:
    $K_2 - K_1$ massless, singularity arises from $\ell \sim K_2 - K_1$
  - three denominators vanishing must be adjacent propagators separated by a pair of massless legs:
    $K_2 - K_1$and $K_3 - K_2$ massless, singularity arises when middle momentum is soft $\ell \sim K_2$

- Generalize this to higher loops

- Find numerators that vanish in those regions

Analytic Strategy

Gambuti, Novichkov, Tancredi, DAK

- Derivable
- Proceed topology by topology

- Solve Landau equations in mixed representation:
  - for all loops i, $\sum_{d=1}^{N} \alpha_d \frac{\partial}{\partial \ell_i} \mathrm{Den}_d = 0$
  - for all denominators d, $\alpha_d \mathrm{Den}_d = 0$
  - at least one α_d strictly positive, all nonnegative
  - subtleties for nonplanar integrals

- Each solution is a singular surface

Analytic Strategy

- Classify degree of divergence following Anastasiou & Sterman (based on Libby & Sterman)
  - planar: logarithmic soft & collinear singularities
  - nonplanar: soft singularities can collide to give power divergences
- Build finite numerators
  - start with all factors: $\ell_1^2, \ell_1 \cdot \ell_2, \ell_2^2; \ell_i \cdot k_{1,2,4}$
  - build all numerators of fixed degree, *e.g.*

$$c_1\ \ell_1 \cdot \ell_2 + c_2\ \ell_1 \cdot k_4\, \ell_2 \cdot k_1 + c_3\ (\ell_1 \cdot k_2)^2 + \cdots$$

  - for each singular surface, require coefficients of singular scaling terms to vanish

→ Linear equation(s) for the c_i

Independent Numerators

- How many are there (cumulative)?

- 31 solutions to the Landau equations for planar double box

Max Order in ℓ	1	2	3	4	5
Finite	**0**	**2**	**18**	**89**	**247**

- Not all truly independent

$$\text{Poly}(\ell_i)\ \text{(Finite numerator)}$$

(subject to UV power-counting)

- Mathematical structure: ideal (before UV power-counting)
- "truncated ideal" (linear space) after

A Foreign-Language Lesson

Don't learn animal behavior from DuoLingo

A Foreign-Language Lesson

- Study systems of polynomial equations in variables x_i

- We start with some basis polynomials $P_j(x_i)$, or *generators*
  - coefficients are numbers, or rational functions of other variables

- Consider all polynomials built out of them

$$Q(x_i) = f_j(x_i)\, P_j(x_i)$$

This is the *ideal* generated by the $P_j(x_i)$: $\langle P_j \rangle$

- Functions that vanish when all $P_j(x_i)$ do

A Foreign-Language Lesson

- Questions we want to ask:
  - What is the "simplest" set of polynomials P_j that generate the ideal?
  - Can a polynomial $Q_2(x_i)$ be written in terms of the P_j?
  - Can we recover the coefficients f_j?
  - Are the equations $P_j(x_i) = 0$ consistent?

- To answer them
  - We need to choose an ordering of monomials $x_1^{i_1} x_2^{i_2} \cdots x_n^{i_n}$
  - Build a *Gröbner* basis, a special set of generators that allows us to answer these questions

A Foreign-Language Lesson

- A Gröbner basis and ordering
  - Makes the result of polynomial division independent of ordering
  - Makes the membership question equivalent to zero remainder
  - Gives us an equivalent set of equations to $P_j = 0$
  - In particular, the equations have *no* solution iff the GB is $\{1\}$

- Any set of generators can be written in terms of another

- The Gröbner basis can be written in terms of our original generators: gives us the *cofactor* matrix

$$g_i = A_{ij} P_j$$

Independent Numerators

- Appropriate technology: Gröbner bases

- Compute Gröbner basis of order 2, retain independent remainders after dividing over the basis; iterate
- Or, just compute overall Gröbner basis all at once

Max Order in ℓ	1	2	3	4	5
Finite	0	2	18	89	247
Independent New	0	2	4	4	0

- Define (van Neerven & Vermaseren)

$$v_i^\mu \equiv \frac{G\begin{pmatrix} k_1 & \cdots & \mu & \cdots & k_R \\ k_1 & \cdots & k_i & \cdots & k_R \end{pmatrix}}{G\left(k_1 \quad \cdots \quad k_R\right)}, \quad \nu_{ij} \equiv G\begin{pmatrix} l_i & k_1 & \cdots & k_R \\ l_j & k_1 & \cdots & k_R \end{pmatrix} \Big/ G\left(k_1 \quad \cdots \quad k_R\right)$$

to get nice forms for generators

Nicer Packaging

- Gram determinants: $G\begin{pmatrix} p_1 & \cdots & p_R \\ q_1 & \cdots & q_R \end{pmatrix} \equiv \det\left(2p_i \cdot q_j\right)$

- Order 2 planar double box

$$G\begin{pmatrix} \ell_1 & k_1 & k_2 \\ \ell_2 & k_3 & k_4 \end{pmatrix}; \quad G\begin{pmatrix} \ell_1 & k_1 & k_2 \\ k_1 & k_2 & k_4 \end{pmatrix} G\begin{pmatrix} \ell_2 & k_2 & k_4 \\ k_1 & k_2 & k_4 \end{pmatrix}$$

- Basis elements can be written as
  - product of Grams
  - propagator denominator(s) times Grams

Planar Double Box

- Basis of finite numerators

rank-2 finite $\quad G\begin{pmatrix}\ell_1 & 1 & 2\\ 1 & 2 & 4\end{pmatrix} G\begin{pmatrix}\ell_2 & 3 & 4\\ 1 & 2 & 4\end{pmatrix} \qquad G\begin{pmatrix}\ell_1 & 1 & 2\\ \ell_2 & 3 & 4\end{pmatrix}$

rank-3 finite $\quad G\left(\ell_1 \;\; 1 \;\; 2\right) G\begin{pmatrix}\ell_2 & 3 & 4\\ 1 & 2 & 4\end{pmatrix} \qquad G\left(\ell_2 \;\; 3 \;\; 4\right) G\begin{pmatrix}\ell_1 & 1 & 2\\ 1 & 2 & 4\end{pmatrix}$

$\quad (\ell_1 - k_1)^2\, G\begin{pmatrix}\ell_2 & 3 & 4\\ 1 & 2 & 4\end{pmatrix} \qquad (\ell_2 - k_4)^2\, G\begin{pmatrix}\ell_1 & 1 & 2\\ 1 & 2 & 4\end{pmatrix}$

rank-4 finite $\quad (\ell_1 - k_1)^2\, G\left(\ell_2 \;\; 3 \;\; 4\right) \qquad (\ell_2 - k_4)^2\, G\left(\ell_1 \;\; 1 \;\; 2\right)$

$\quad (\ell_1 - k_1)^2 (\ell_2 - k_4)^2$

- Evanescent basis (rank-4)

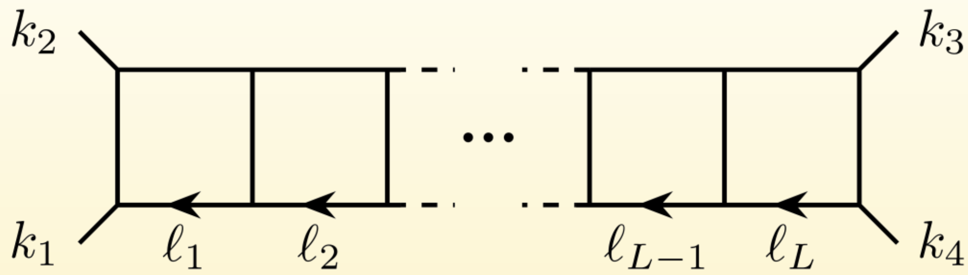
$$G(\ell_1, \ell_2, 1,2,3)$$

Three-Loop Ladder

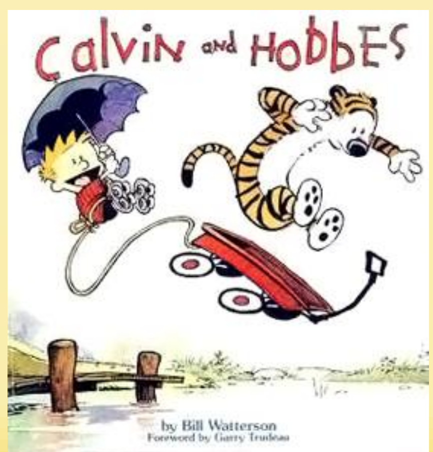
Max Order in ℓ	1	2	3	4	5	6	7
Finite	0	2	26	184	850	2807	6044
Independent New	0	2	6	9	0	0	0
$\mathcal{O}(\epsilon)$	0	0	0	4	42	229	678
$\mathcal{O}(\epsilon)$ **independent**	0	0	0	4	2	2	0

- $\mathcal{O}(\epsilon)$
  - Finite
  - Vanish when any loop momentum is four-dimensional
  - Generalizes one-loop pentagon Gram

All-Loop Ladder Conjecture



Max Order in ℓ	1	2	3	4	5	6	7	8
Independent	0	2	$2L$	L^2	0	0	0	0
$\mathcal{O}(\epsilon)$ independent	0	0	0	$(3L^2-9L+8)/2$	$(L-2)(L^2-4L+5)$	$(L-2)\times$ $(L^3-9L+16)/8$	$(L-2)(L-3)\times$ $(L^2-5L+8)/4$	$(L-2)(L-3)\times$ $(L^2-5L+8)/8$



Geometric Strategy

De la Cruz, Novichkov, DAK

- Not yet derivable — lots of conjecture
- Parametric representation
- Focus on exponents of monomials

$$\alpha_1^{e_1}\alpha_2^{e_2}\cdots\alpha_n^{e_n}$$
$$\boldsymbol{e} \equiv (e_1, e_2, \ldots, e_n)$$

- Build on theorem of Berkesch, Forsgård, & Passare on convergence of Euler–Mellin integrals

Geometric Strategy

- Newton polytope: convex hull of all positive-weight linear combinations of all exponent vectors in a given polynomial
- *H*-representation: region bounded by set of inequalities
- Relation to tropical geometry to be explored

- BFP instructs us to look at Newton polytope of Symanzik polynomials

$$\text{Newton}\left(\left[\mathcal{U}^{E-\frac{D}{2}(L+1)-r}\mathcal{F}^{\frac{DL}{2}-E}\right]^{-1}\right)$$

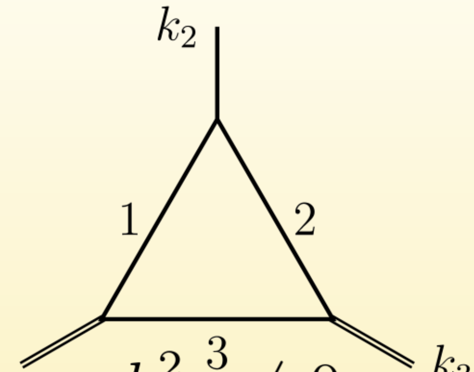
E propagators, D dimensions, L loops, rank r

Geometric Strategy

- Reexpress polytope as weighted Minowski sum
$$\left(r - E + \frac{D}{2}(L+1)\right)\text{Newton}(\mathcal{U}) + (E - DL/2)\text{Newton}(\mathcal{F})$$
- BFP: integral of a Feynman-parameter monomial converges if
  - $\mathcal{U}$ and $\mathcal{F}$ have no zeros on faces of polytope (true for planar integrals)
  - the vector $\boldsymbol{e} + \mathbf{1}$ lies in the \`relative interior' of the polytope
- Find interior with tools like NConvex, or via conjecture on generating function
- Conjecture: integral is finite iff each Feynman-parameter monomial is in the relative interior

Geometric Strategy: Toy Example

- Consider the two-mass triangle



$$I_\triangle[\mathcal{N}(\ell)] = \int \mathrm{d}^D\ell \, \frac{\mathcal{N}(\ell)}{\ell^2\,(\ell-k_2)^2\,(\ell+k_1)^2}, \qquad k_{1,3}^2 \neq 0$$

- In parametric form

$$I_{\triangle,r}[\mathcal{N}(\ell)] = \sum_{\mathbf{m}\in B} c_{\mathbf{m}} \int \frac{\mathrm{d}^3\alpha}{\alpha_1\alpha_2\alpha_3}\delta(1-\alpha_1-\alpha_2-\alpha_3)\,\alpha^{\mathbf{m}+1}\mathcal{U}^{3-D-r}\mathcal{F}^{D/2-3}$$

$$\mathcal{U} = \alpha_1 + \alpha_2 + \alpha_3, \quad \mathcal{F} = -(k_1^2\alpha_1\alpha_3 + k_3^2\alpha_2\alpha_3)$$

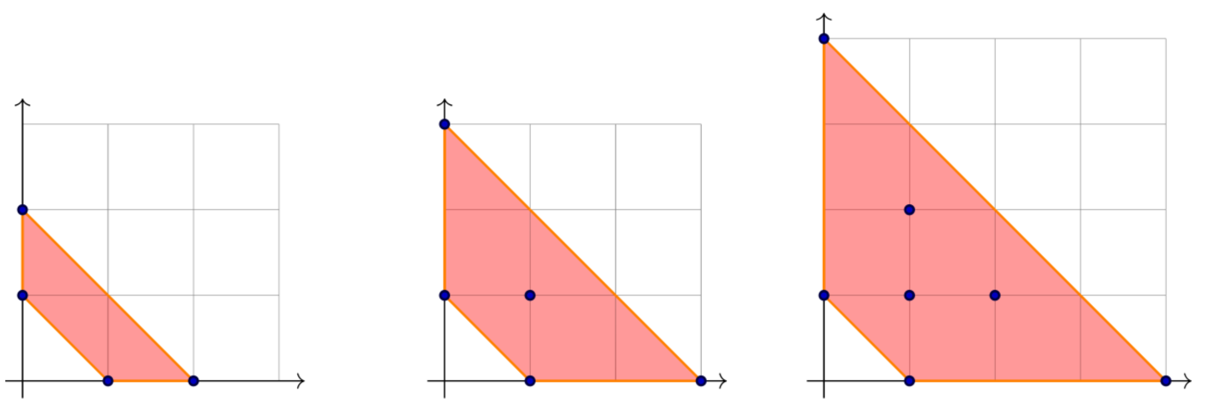
Fix $\alpha_3 = 1$

Feynman Polytope

- All possible exponent vectors:

(0,0); (1,0); (0,1)

$$P_3 = (r + D - 3)\mathrm{Newt}(\mathcal{U}) + \left(3 - \frac{D}{2}\right)\mathrm{Newt}(\mathcal{F})$$

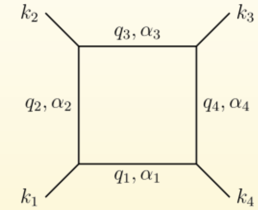


(a) $r = 0$ (b) $r = 1$ (c) $r = 2$

$\Downarrow$ $\Downarrow$

divergent $\ell \cdot k_2$

Geometric Strategy: Example



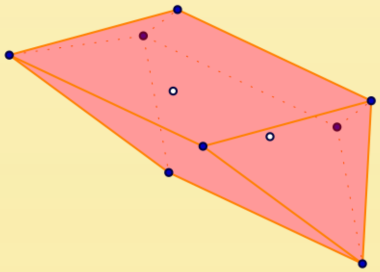
- Massless box

$$\mathcal{U} = \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4, \quad \mathcal{F} = -s\alpha_1\alpha_3 - t\alpha_2\alpha_4$$

fix $\alpha_4 = 1$

- Look at rank two: exponents w/lattice points in polytope

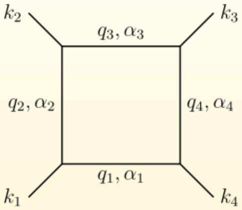
(0, 0, 0), (0, 0, 1), (0, 0, 2), (0, 1, 0), (0, 1, 1), (0, 2, 0), (1, 0, 0), (1, 0, 1), (1, 1, 0), (2, 0, 0) + 18 others



- Relative-interior exponents

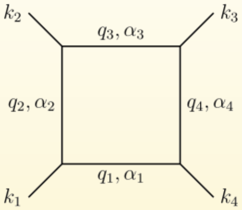
(1,0,1),(0,1,0)

Geometric Strategy: Example



- Require general loop-momentum numerator to yield only these
- Write down general numerator up to desired degree
$$c_1\,\ell\cdot k_1 + c_2\,\ell\cdot k_2 + c_3\ell^2 + c_4\,\ell\cdot k_1\,\ell\cdot k_2 + c_5(\ell\cdot k_1)^2 + \cdots$$
- Convert to parametric form
- Set coefficients of non-interior monomials to vanish
- Require coefficients to be *D*-independent

Geometric Strategy: Example



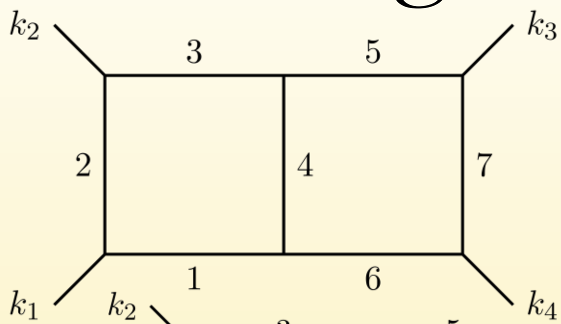
N_1: $(s+t)\ell\cdot k_1 + t\ell\cdot k_2 - s\ell\cdot k_4 - (s+t)\,\ell^2$

N_2: $(t^2-s^2)(\ell\cdot k_1)^2 + 2t^2\ell\cdot k_1\,\ell\cdot k_2 + t^2(\ell\cdot k_2)^2 - 2s^2\ell\cdot k_1\,\ell\cdot k_4 - s^2(\ell\cdot k_4)^2 + st^2\ell^2$

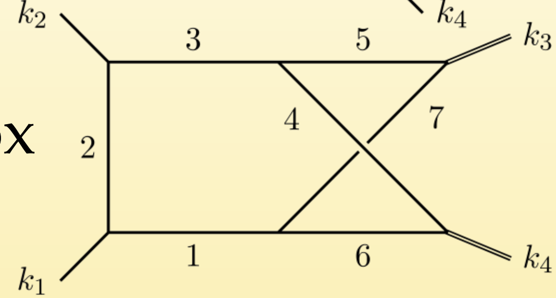
N_3: $-(s+t)(\ell\cdot k_1)^2 - t\ell\cdot k_1\,\ell\cdot k_2 - (2s+t)\ell\cdot k_1\,\ell\cdot k_4 + t\ell\cdot k_2\,\ell\cdot k_4 - s(\ell\cdot k_4)^2 - \frac{1}{2}st\ell^2$

Other Integrals

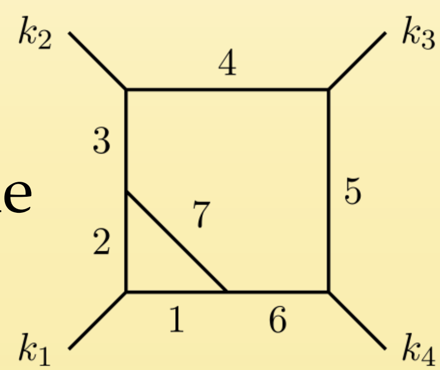
- Planar double box



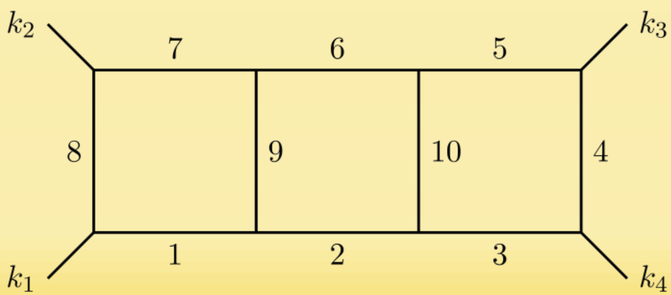
- Nonplanar double box



- Beetle



- Three-loop ladder



Comparison

- Do the results of the two approaches agree?

- Yes…

…but the comparison is subtle

Comparisons

- Strongly UV convergent (by strict power counting)
  vs [simply] weakly UV convergent (coefficient vanishes)

- Nontrivial numerators can vanish in parameters
  - Special total derivatives

- General total derivatives caught in
  - Not locally finite but still scooped up by polytopes

- Compared
  - Planar & nonplanar double box
  - Beetle
  - Three-loop ladder

Integration by Parts

Canay, Novichkov, Ma, Wu, Zhang, DAK

- Can avoid doubled propagators using generating vectors *aka* "syzygy method"

- Choose v in

$$\int d^D \ell_j \;\; \frac{\partial}{\partial \ell_i^\mu}\left[\frac{v^\mu \, N}{D_1 D_2 \cdots D_E}\right]$$

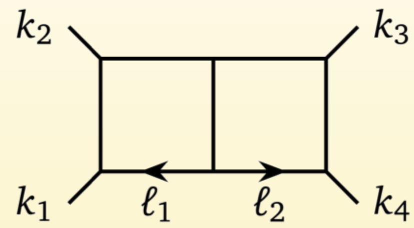
Such that $v \cdot \frac{\partial}{\partial \ell_i} D_j \propto D_j$ for all D_j

Find only IBPs for finite numerators by also requiring

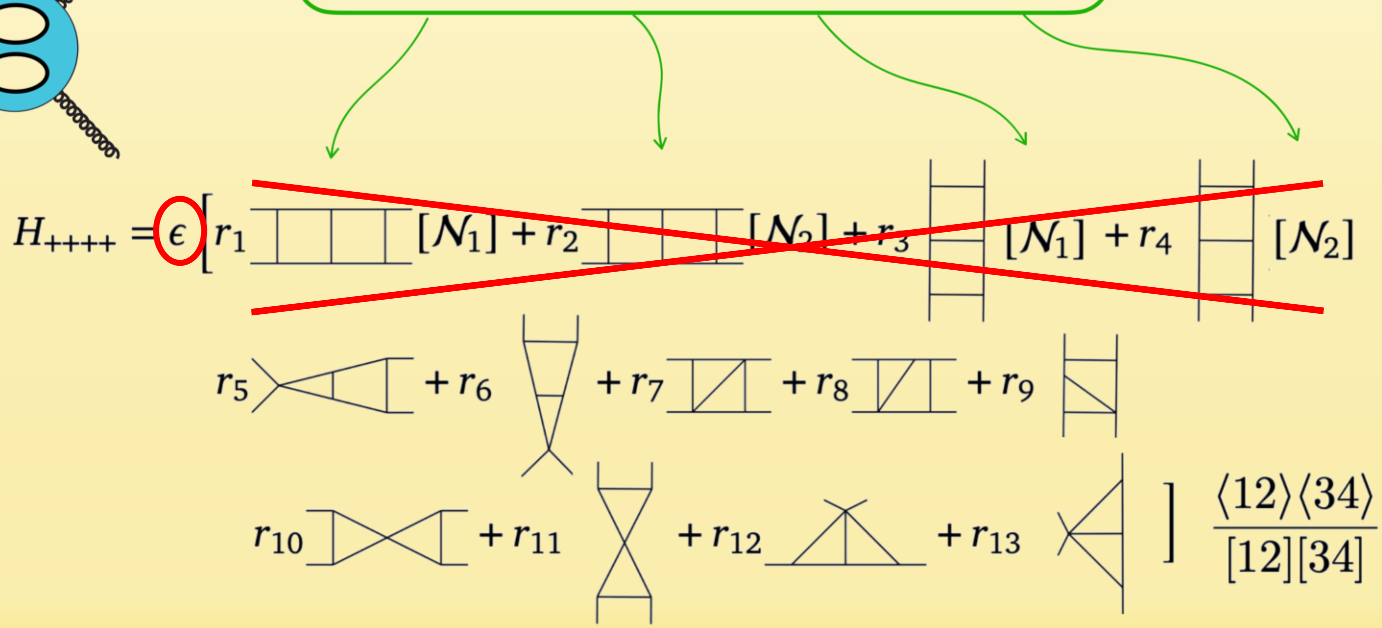
$$v \cdot \frac{\partial}{\partial \ell_i} N_j = c_m N_m$$

Use of New Integrals

Look at two-loop $A_4(++++)$



$$\mathcal{N}_1 = G\begin{pmatrix}\ell_1 & 1 & 2\\ 1 & 2 & 4\end{pmatrix} G\begin{pmatrix}\ell_2 & 3 & 4\\ 1 & 2 & 4\end{pmatrix}, \quad \mathcal{N}_2 = G\begin{pmatrix}\ell_1 & 1 & 2\\ \ell_2 & 3 & 4\end{pmatrix}$$



Coefficients simpler too

Integrating

Canay, Novichkov, DAK

- New opportunity to use existing tool: HyperInt

Panzer

- Top-level topology has complicated functions
- Rank-two finite pentabox
- All integrations but one are linear
- Separate $\sqrt{\Delta}$ by hand: $\alpha_6^2 - \Delta$

Beyond Scope

- Weakly UV convergent: coefficients vanish
- Cancellation of IR or UV singularities with ϵ from evanescence
  - Key role in rational terms
- In Landau approach: relax some constraints, obtain integrals with controlled degree of divergence

Summary

With numerators
Feynman integrals settle
happily bounded

- Finite integrals are a first step to exploring a new organization of scattering amplitudes
- Two approaches
  - Analytic approach: Landau equations + Anastasiou–Sterman scaling
  - Geometric approach: Newton polytopes + BFP theorem
- Applications
  - Amplitude structure
  - Integrating