Isotropic expansion and the balance between anisotropic curvature and stresses

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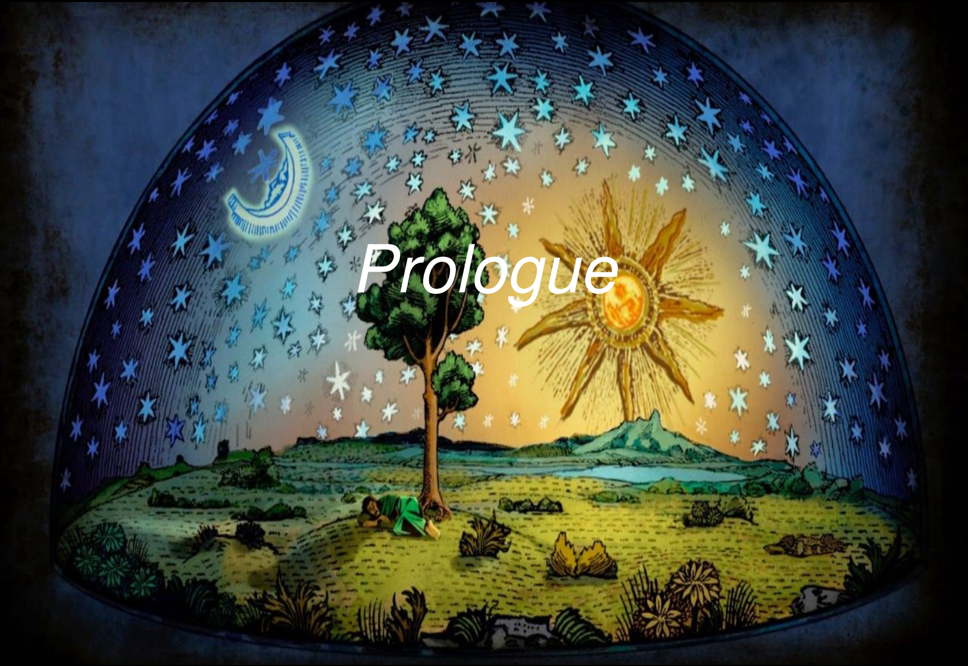

Prologue

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Principle 3 (The strong cosmological principle)

The universe is isotropic around every point.

Why not just study the FLRW universe that best fits the observational evidence?

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[2] Komatsu, E. (2022). *New physics from the polarized light of the cosmic microwave background*. Nature Reviews Physics, 4(7), 452-469.

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- ✓ Anisotropy could have observable consequences on the polarisation and spectrum of the cosmic microwave background radiation[2].

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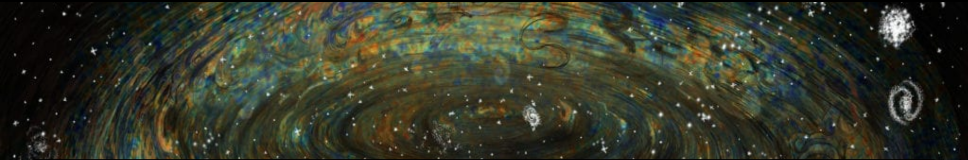
Research question

Can the spatially anisotropic solutions to general relativity be dynamically distinguished from FLRW cosmologies?



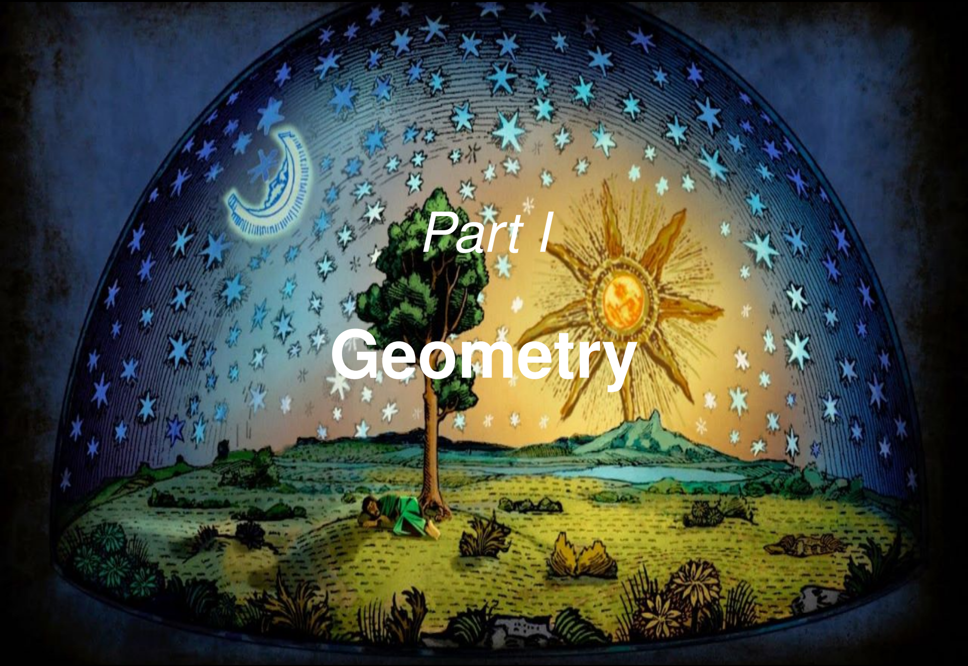
Definition (Extended FLRW model)

It is a cosmological model with underlying anisotropic Bianchi geometry that nevertheless expands isotropically and can be mapped onto a reference FLRW model with the same expansion history[a]*.*

[a] M. Thorsrud, B. D. Normann, and T. S. Pereira, "Extended FLRW Models: dynamical cancellation of cosmological anisotropies," Class. Quant. Grav., vol. 37, no. 6, 2020.

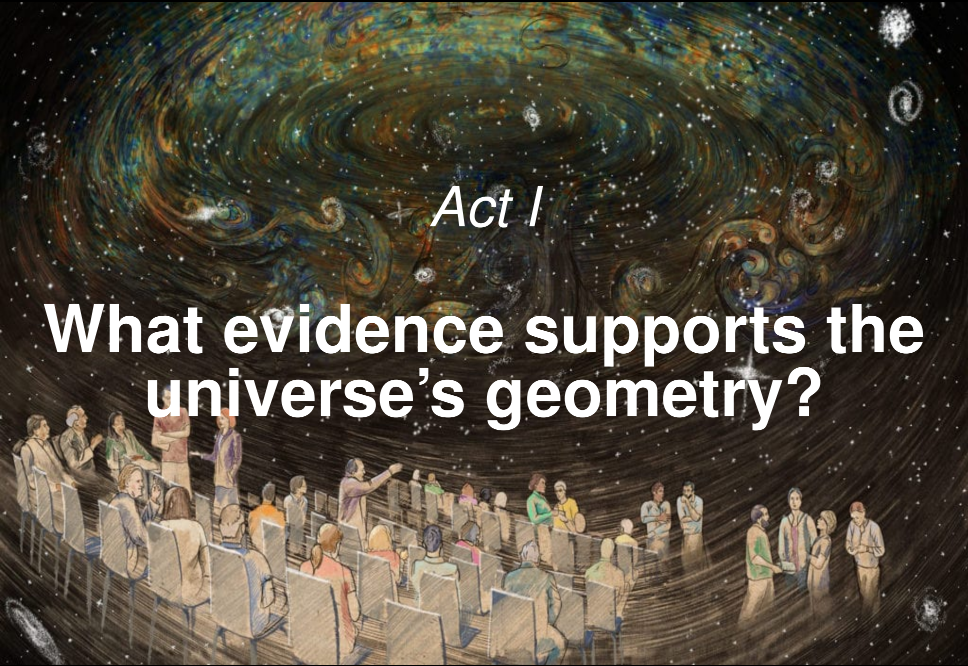
Part I

Geometry



Act I

What evidence supports the universe's geometry?



✓ The universe is evolving, making it a non-homogeneous spacetime[3].

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- ✓ The universe is evolving, making it a non-homogeneous spacetime[3].
- ✓ The universe is spatially homogeneous[4].
- ✓ The universe is isotropic around us[5].

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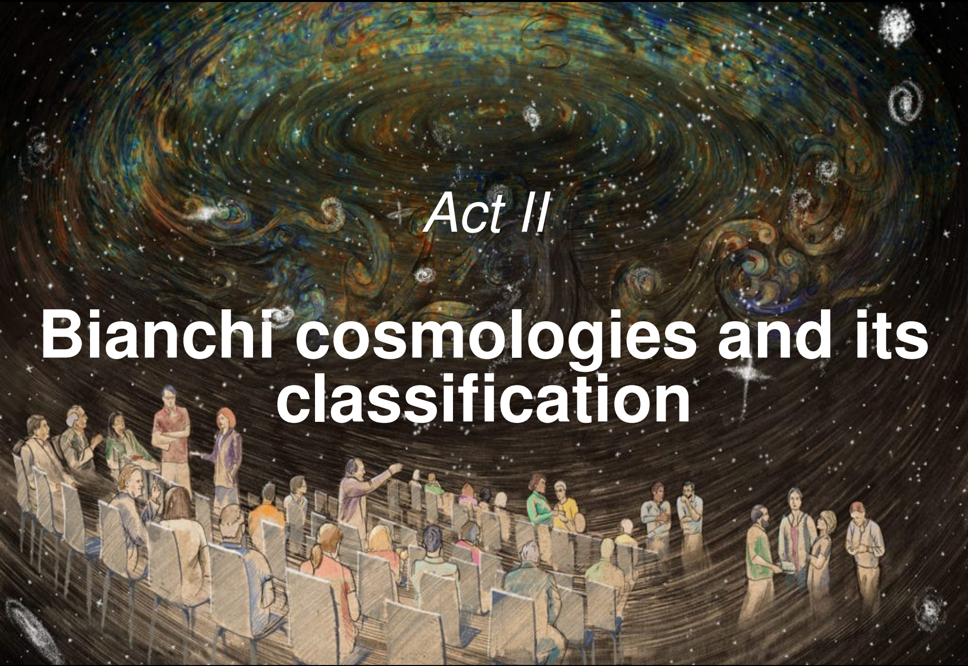

Act II

Bianchi cosmologies and its classification

Definition 5 (Bianchi cosmology)

A Bianchi cosmology $(\mathcal{M}, \mathbf{g}, \mathbf{u})$ *is a model whose metric admits a three-dimensional group of isometries acting simply transitively on spacelike hypersurfaces, which are surfaces of homogeneity in space-time. A Bianchi cosmology thus admits a Lie algebra of KVFs with basis* $\boldsymbol{\xi}_\alpha, \alpha = 1, 2, 3$*, and structure constants* $C^\mu_{\alpha\beta}$ *:*

$$\left[\boldsymbol{\xi}_\alpha, \boldsymbol{\xi}_\beta\right] = C^\mu_{\alpha\beta} \boldsymbol{\xi}_\mu.$$

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Corollary

- ✓ *If* $\mathbf{u} \perp \Sigma$*, we have the family of non-tilted Bianchi cosmologies.*
- ✓ *If not, we have the family of tilted Bianchi cosmologies.*

Classification

The Bianchi cosmologies can be classified by classifying the Lie algebras of KVFs. Thus, one can decompose ${C^\mu}_{\alpha\beta}$ as follows:[6]

$$C^\mu_{\alpha\beta} = \varepsilon_{\alpha\beta\nu}\hat{n}^{\mu\nu} + \hat{a}_\alpha {\delta_\beta}^\mu - \hat{a}_\beta {\delta_\alpha}^\mu,$$

where $\hat{n}^{\mu\nu} = \hat{n}^{\nu\mu}$ and $\hat{a}_\alpha$ are constants, and its trace is $C^\mu_{\alpha\mu} = 2a_\alpha$.

[6] Krasiński, A., Behr, C., Schücking, E., Estabrook, F., Wahlquist, H., Ellis, G., ... & Kundt, W. (2003). "Golden Oldie": The Bianchi Classification in the Schücking-Behr Approach. General Relativity & Gravitation, 35(3).

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✓ If $C^{\mu}_{\alpha\mu} = 0$, we have the class A Bianchi models.

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- ✓ If $C^\mu_{\alpha\mu} \neq 0$, we have the class B Bianchi models.

[6] Krasiński, A., Behr, C., Schücking, E., Estabrook, F., Wahlquist, H., Ellis, G., ... & Kundt, W. (2003). "Golden Oldie": The Bianchi Classification in the Schücking-Behr Approach. General Relativity & Gravitation, 35(3).

Classification

For the specific Bianchi type, we choose an invariant basis $\{\boldsymbol{\omega}^i\}$ that satisfies

$$\mathbf{d}\boldsymbol{\omega}^\mu = -\frac{1}{2}C^\mu_{\alpha\beta}\boldsymbol{\omega}^\alpha \wedge \boldsymbol{\omega}^\beta.$$

The Bianchi model of the corresponding type can now be written

$$ds^2 = -dt^2 + g_{\mu\nu}(t)\boldsymbol{\omega}^i \otimes \boldsymbol{\omega}^j$$

This metric will in general have the symmetries of the corresponding Bianchi group[7].

[7] Grøn, ∅., & Hervik, S. (2007). Einstein's general theory of relativity: with modern applications in cosmology. Springer Science & Business Media.

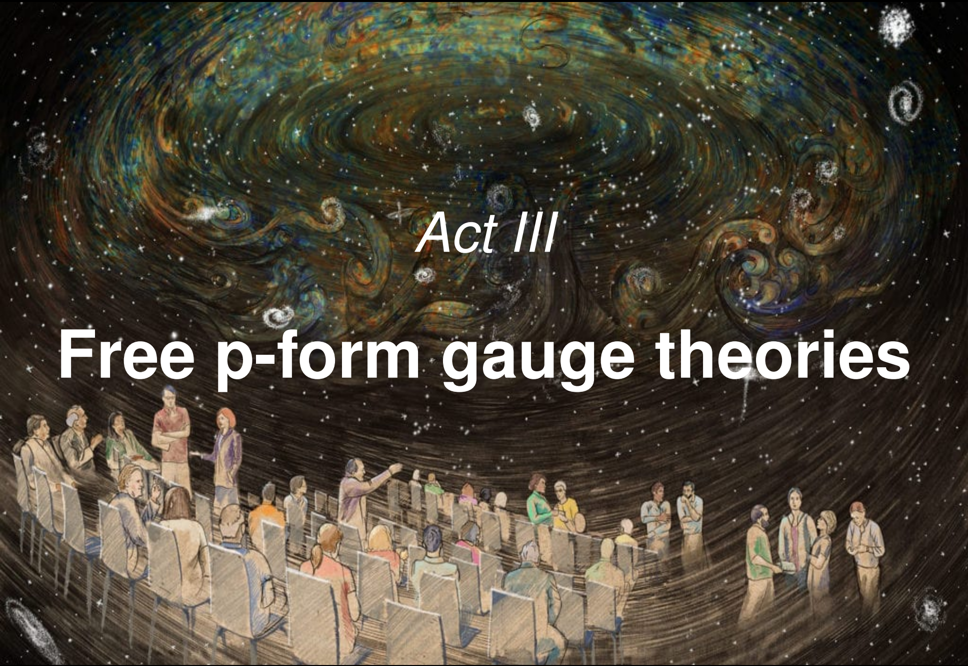
Classification

Class	Type	a	n_1	n_2	n_3
A	I	0	0	0	0
	II	0	+	0	0
	VI_0	0	+	−	0
	VII_0	0	+	+	0
	VIII	0	+	+	−
	IX	0	+	+	+
B	V	+	0	0	0
	IV	+	+	0	0
	VI_h	+	+	−	0
	VII_h	+	+	+	0

Figure: The Bianchi types in terms of the algebraic properties of the structure coefficients.

Part II

Matter



Act III

Free p-form gauge theories

Designing shear-free Bianchi cosmologies require a matter source to balance anisotropic curvature. A natural candidate is the class of free theories defined by the action

$$S = -\frac{1}{2}\int \mathcal{F} \wedge \star\mathcal{F},$$

where $\mathcal{F} = \mathrm{d}\mathcal{A}$ is the field strength of a p-form gauge field

$$\mathcal{A} = \frac{1}{p!}\mathcal{A}_{\mu_1\ldots\mu_p}\ \mathbf{w}^{\mu_1} \wedge \cdots \wedge \mathbf{w}^{\mu_p}.$$

The components of the energy-momentum tensor are

$$T_{\mu\nu} \equiv -\frac{2}{\sqrt{-g}}\frac{\delta\mathcal{L}}{\delta g^{\mu\nu}} = \frac{1}{p!}\mathcal{F}_{\mu}{}^{\alpha_1\ldots\alpha_p}\mathcal{F}_{\nu\alpha_1\ldots\alpha_p} - \frac{1}{2(p+1)!}g_{\mu\nu}\mathcal{F}^{\alpha_1\ldots\alpha_{p+1}}\mathcal{F}_{\alpha_1\ldots\alpha_{p+1}}.$$

We shall consider all non-trivial p-forms in spacetime described by this action, i.e. $p \in \{0, 1, 2, 3\}$.

1+3 decomposition of differential forms

Case 1: 1-form

Let $\mathcal{J}$ a one-form. Given a unit timelike vector $\vec{u}$ (in practice, it will be the 4 -velocity of some observer), there exists a one-form $\boldsymbol{v}$ and a unique vector scalar field φ such that [a]

$$\mathcal{J} = -\phi\,\underline{\mathbf{u}} + \underline{\mathbf{v}} \quad \Longrightarrow \quad \mathcal{J}_\mu = -\varphi u_\mu + v_\mu.$$

[a] Gourgoulhon, É. (2016). Special relativity in general frames. Springer-Verlag Berlin An.

1+3 decomposition of differential forms

Case 2: 2-form

Let $\mathcal{A}$ be a 2-form:

$$\forall(\vec{v},\vec{w}) \in E^2, \quad \mathcal{A}(\vec{v},\vec{w}) = -\mathcal{A}(\vec{w},\vec{v}).$$

Given a unit timelike vector $\overrightarrow{\boldsymbol{u}}$ (in practice, it will be the 4 -velocity of some observer), there exists a unique linear form $\boldsymbol{q} \in E^*$ and a unique vector $\overrightarrow{\boldsymbol{b}} \in E$ such that

$$\mathcal{A} = \underline{u} \otimes q - q \otimes \underline{u} + \epsilon(\vec{u},\vec{b},\ldots,), \quad \langle \boldsymbol{q}, \overrightarrow{\boldsymbol{u}} \rangle = 0 \quad \textit{and} \quad \vec{u}\cdot\vec{b} = 0.$$

1+3 decomposition of differential forms

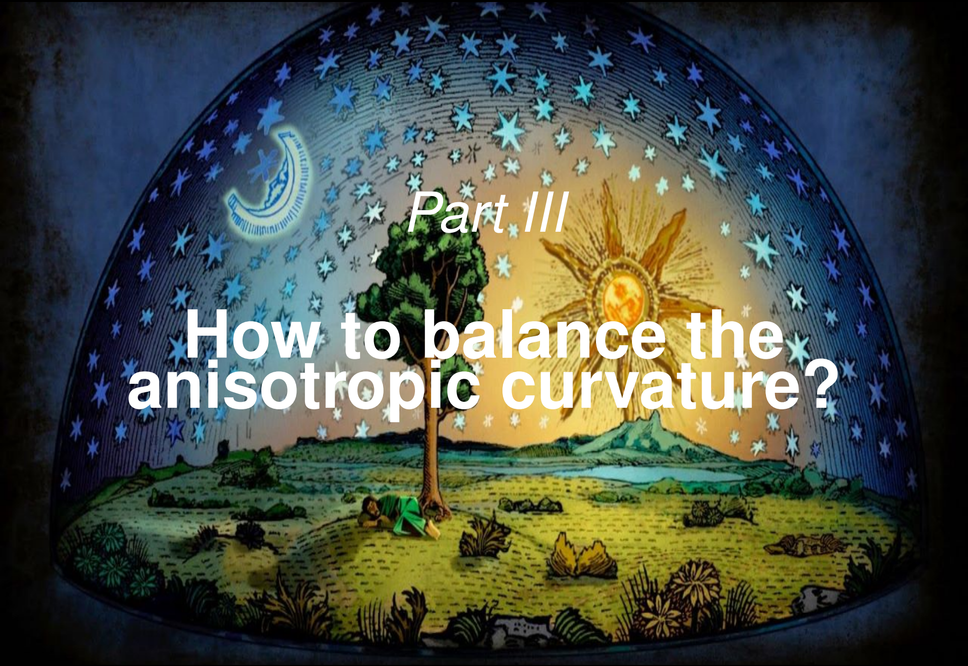
Case 3: 3-form

Let $\mathcal{B}$ be a 3-form:

$$\forall(\vec{v},\vec{w},\vec{f}) \in E^3, \quad \mathcal{B}(\vec{v},\vec{w},\vec{f}) = -\mathcal{B}(\vec{w},\vec{v},\vec{f}) = -\mathcal{B}(\vec{v},\vec{f},\vec{w}).$$

Given a unit timelike vector $\overrightarrow{u}$ (in practice, it will be the 4 -velocity of some observer), there exists a unique scalar field φ and a unique vector $\overrightarrow{\mathbf{b}} \in E$ such that

$$\mathcal{B} = \underline{\mathbf{u}} \wedge \epsilon(\vec{\mathbf{u}},\vec{\mathbf{b}},\ldots,) + \varphi\,\epsilon(\vec{\mathbf{u}},\vec{\mathbf{b}},\ldots,), \quad \vec{\mathbf{u}}\cdot\vec{\mathbf{b}} = 0.$$



Part III

How to balance the anisotropic curvature?

Act IV

Shear-free solutions

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3. That expand isotropically.
4. A collection of matter fields modelled as comoving perfect fluids, accounting for standard Λ CDM constituents.
5. A collection of n independent p-form gauge fields with vanishing energy flux and action specified by $S = -\frac{1}{2}\int \mathcal{F} \wedge \star\mathcal{F}$.

[a] Thorsrud, M., Normann, B. D., Pereira, T. S. (2020). Extended FLRW models: dynamical cancellation of cosmological anisotropies. Classical and Quantum Gravity, 37(6), 065015.

We consider a collection of comoving perfect matter fields with total energy-momentum tensor

$$T_{\mu\nu} = \rho u_\mu u_\nu + P h_{\mu\nu}, \quad \rho = \sum_\ell \rho_\ell, \quad P = \sum_\ell P_\ell,$$

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We also consider a p-form gauge field $\mathcal{A}_{\mu_1 \ldots \mu_p}$ with zero energy flux in Σ_t, and energy-momentum tensor

$$T^{(\mathcal{A})}_{\mu\nu} = \rho_{\mathcal{A}} u_\mu u_\nu + P_{\mathcal{A}} h_{\mu\nu} + \pi_{\mu\nu}.$$

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The matter fields satisfy the evolution equations

$$\begin{aligned} &\dot{\rho} + 3H(\rho + P) = 0, \\ &\dot{\rho}_{\mathcal{A}} + 3H\left(\rho_{\mathcal{A}} + P_{\mathcal{A}}\right) = -\pi^{\mu\nu}\sigma_{\mu\nu}. \end{aligned}$$

The congruence evolves according to the Raychudhuri equation and the shear propagation equation

$$\dot{H} + H^2 = -\frac{1}{6}(\rho + 3P) - \frac{1}{6}\left(\rho_{\mathcal{A}} + 3P_{\mathcal{A}}\right) - \frac{2}{3}\sigma^2,$$
$$\dot{\sigma}_{\mu\nu} + 3H\sigma_{\mu\nu} = \pi_{\mu\nu} - {}^3S_{\mu\nu}.$$

Here, ${}^3S_{\mu\nu}$, is the trace-free three-dimensional Ricci tensor on the hypersurfaces Σ_t, given by

$${}^3S_{\mu\nu} = {}^3R_{\mu\nu} - \frac{{}^3R}{3}h_{\mu\nu}.$$

Additionally, we have the constraint between variables

$$3H^2 - \sigma^2 + \frac{{}^3R}{2} = \rho + \rho_{\mathcal{A}}.$$

To satisfy the conditions behind the Bianchi cosmologies, we shall consider the case in which we have a shear-free scenario. Thus, the shear propagation equation takes the form

$$\pi_{\mu\nu} = {}^{3}S_{\mu\nu}.$$

In this project, we shall investigate systematically under which conditions gauge fields can balance the anisotropic curvature in this way.

Act V

Known facts: Abelian case

✓ The components of ${}^{3}S_{\mu\nu}$, relative to an orthonormal basis, decay as $1/a^2$ in the shear-free limit, where $a(t)$ is the scale factor controlling distances in Σ_t[b].

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- ✓ According to the shear-free condition, the anisotropic stress $\pi_{\mu\nu}$ also must decay as $1/a^2$ [b].

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- ✓ The fact that the energy-momentum tensor (of the gauge field) is homogeneous quadratic in the field strength, suggests the implication[b]

$$\pi_{\mu\nu} \propto 1/a^2 \Longrightarrow \rho_{\mathcal{A}} \propto 1/a^2.$$

[b] Thorsrud, M., Normann, B. D., & Pereira, T. S. (2020). Extended FLRW models: dynamical cancellation of cosmological anisotropies. Classical and Quantum Gravity, 37(6), 065015.

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- ✓ The 1-form case has an equation of state $+1/3$ and can thus be ruled out as a candidate to balance anisotropic curvature ${}^3S_{\mu\nu}$ [b].

- ✓ We can already rule out the case $p = 3$ which is equivalent to the cosmological constant Λ [c].

[b] Thorsrud, M., Normann, B. D., & Pereira, T. S. (2020). Extended FLRW models: dynamical cancellation of cosmological anisotropies. Classical and Quantum Gravity, 37(6), 065015.

[c] Hawking, S. W. (1984). The cosmological constant is probably zero. Physics Letters B, 134(6), 403-404.

Spacetime (subtype)	sign K_{eff}	sign ${}^3R$	n_{min}	Hamiltonian bounded from below?
Bianchi type II	$+$	$-$	2	No
Bianchi type VI_0 ($N^a{}_a = 0$)	0	$-$	1	No
Bianchi type III ($N^a{}_a = 0$, LRS)	$-$	$-$	1	Yes
Kantowski–Sachs	$+$	$+$	1	No
FLRW (closed/flat/open)	$+/0/-$	$+/0/-$		

Figure: Thorsrud, M., Normann, B. D., & Pereira, T. S. (2020). Extended FLRW models: dynamical cancellation of cosmological anisotropies. Classical and Quantum Gravity, 37(6), 065015.

Act VI

Results

Attempt one: 1-form non-abelian case

We start with the action for the gauge field sector, defined by

$$S = -\frac{1}{4}\int \mathrm{d}x^4 \sqrt{-|g|}\, \mathcal{F}^2.$$

In the above, $\mathcal{F}$ is the $\mathrm{SO}(3)$-gauge field-strength tensor $\mathcal{F} = D\mathcal{A}$. Thus, using a general basis, the above quantities may be expressed as

$$\mathcal{F} = \frac{1}{2}\mathcal{F}^a{}_{\mu\nu}\mathrm{T}_a \boldsymbol{\omega}^\mu \wedge \boldsymbol{\omega}^\nu \quad \rightarrow \quad \mathcal{F}^a{}_{\mu\nu} = 2\nabla_{[\mu}\mathcal{A}^a{}_{\nu]} + g\varepsilon^a_b{}_{bc}\mathcal{A}^b{}_\mu \mathcal{A}^c{}_\nu,$$

or using a 1+3 decomposition, we arrive at

$$\mathcal{F}^a = \left(u_{[\mu}\mathcal{E}^a{}_{\nu]} + \frac{1}{2}\eta_{\lambda\mu\nu}\mathcal{B}^{a\lambda}\right)\omega^\mu \wedge \omega^\nu \rightarrow \mathcal{F}^a{}_{\mu\nu} = 2u_{[\mu}\mathcal{E}^a{}_{\nu]} + \eta_{\lambda\mu\nu}\mathcal{B}^{a\lambda},$$

where

$$\mathcal{B}^a \equiv \star_3 \mathcal{F}^a = \frac{1}{2}\eta_{ijk}\mathcal{F}^{aij}\omega^k \to \mathcal{B}^a_k = \frac{1}{2}\eta_{ijk}\mathcal{F}^{aij} \quad \text{and} \quad \mathcal{E}^a_\mu \equiv \mathcal{F}^a_{\mu\nu}u^\nu .$$

From this one finds that the components of the energy-momentum tensor in the standard irreducible decomposition are such that

$$\rho^{\mathcal{F}} = \frac{1}{2}\left(\mathcal{E}^2 + \mathcal{B}^2\right), \qquad p^{\mathcal{F}} = \frac{1}{6}\left(\mathcal{E}^2 + \mathcal{B}^2\right), \qquad q^{\mathcal{F}}_\lambda = \eta_{\lambda\gamma\beta}\mathcal{E}_a{}^\gamma\mathcal{B}^{a\beta}$$

$$\pi^{\mathcal{F}}_{\mu\nu} = -\left(\mathcal{B}^a{}_\mu\mathcal{B}_{a\nu} + \mathcal{E}^a{}_\mu\mathcal{E}_{a\nu}\right) + \frac{1}{3}h_{\mu\nu}\left(\mathcal{E}^2 + \mathcal{B}^2\right).$$

In the above $\mathcal{E}^2 = \mathcal{E}^a\mathcal{E}_a = \mathcal{E}^a\mathcal{E}^b g_{ab} = -2\delta_{ab}\mathcal{E}^a\mathcal{E}^b$, and similarly for $\mathcal{B}^2$, and

$$\mathcal{E}^a_i = 2\nabla_{[i}\mathcal{A}^a{}_{0]} + g\varepsilon^a_{bc}\mathcal{A}^b_0\mathcal{A}^c_i,$$

$$\mathcal{B}^a_i = \eta_{ijk}\nabla^{[j}\mathcal{A}^{ak]} + \frac{1}{2}g\eta_{ijk}\varepsilon^a{}_{bc}\mathcal{A}^{bj}\mathcal{A}^{ck}.$$

Attempt two: 0-form non-abelian case

We start with the action for the gauge field sector, defined by

$$S = -\frac{1}{2}\int \mathrm{d}x^4 \sqrt{-|g|}\, \mathcal{J}^2,$$

where $\mathcal{J}^2 \equiv \mathcal{J}^a \mathcal{J}_a$, and $\mathcal{J}^a = D_\mu \mathcal{X}^a$ with $\mathcal{X}^a$ being a scalar field. Furthermore, the covariant derivative D of a vector $\mathcal{X}$ is in this context defined to be

$$D_\mu \mathcal{X}^a \equiv \nabla_\mu \mathcal{X}^a + g\varepsilon^a{}_{bc} A^b_\mu \mathcal{X}^c.$$

Thus, $\mathcal{J}^a$ can be decomposed as

$$\mathcal{J}^a_\mu = -\,\omega^a\, u_\mu + v^a_\mu.$$

Attempt three: 0-form plus 1-form non-abelian case

We start with the action for the gauge field sector, defined by

$$S = -\int \mathrm{d}x^4 \sqrt{-|g|} \left(\frac{f}{4}\, \mathcal{F}^2 + \frac{1}{2}\, \mathcal{J}^2 + V \right),$$

where f is a scalar function that considers the interaction between the non-abelian gauge fields.

Attempt three: 0-form plus 1-form non-abelian case

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$$S = -\int \mathrm{d}x^4 \sqrt{-|g|}\left(\frac{f}{4}\,\mathcal{F}^2 + \frac{1}{2}\,\mathcal{J}^2 + V\right),$$

where f is a scalar function that considers the interaction between the non-abelian gauge fields.

Another possibility consists of imposing that

$$q_\mu^{\mathcal{J}} + q_\mu^{\mathcal{F}} = 0.$$

Attempt four: 3-form non-abelian case

We start with the action for the gauge field sector, defined by

$$S = -\frac{1}{12}\int \mathrm{d}x^4 \sqrt{-|g|}\,\mathcal{H}^2,$$

where $\mathcal{H}^2 \equiv \mathcal{H}^a\mathcal{H}_a$, and $\mathcal{H}^a = D\,\mathcal{B}^a$ with $\mathcal{B}^a$ being a two-form, which, in the 1+3 decomposition can be written as

$$\mathcal{B} = \underline{\mathbf{u}} \wedge \epsilon(\vec{\mathbf{u}}, \vec{\mathbf{b}}, \ldots,) + \varphi\,\epsilon(\vec{\mathbf{u}}, \vec{\mathbf{b}}, \ldots,), \quad \vec{\mathbf{u}} \cdot \vec{\mathbf{b}} = 0.$$

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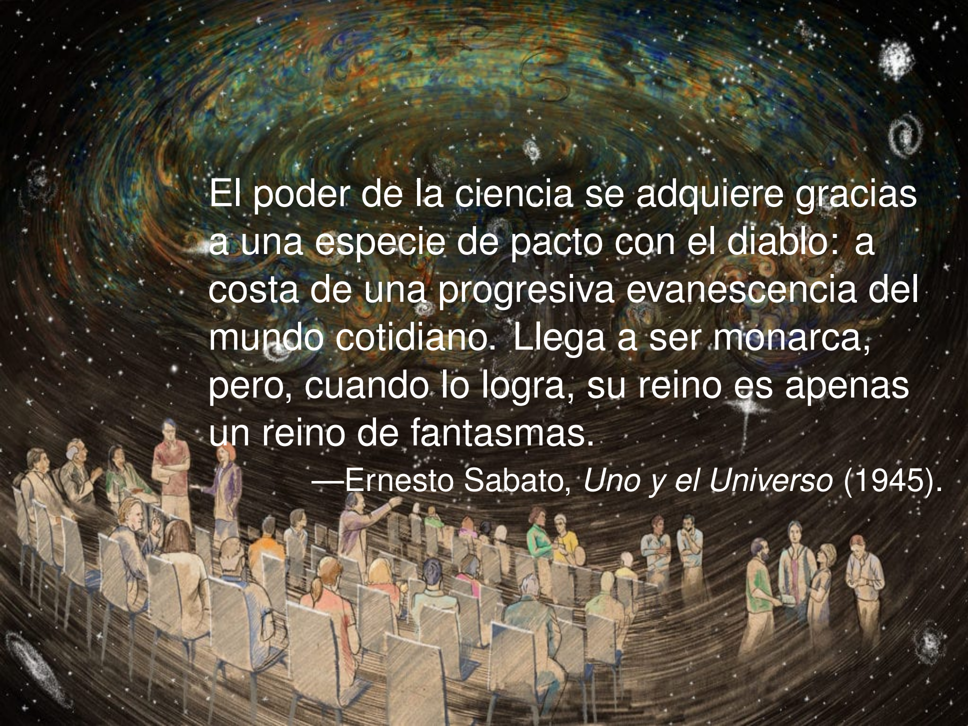
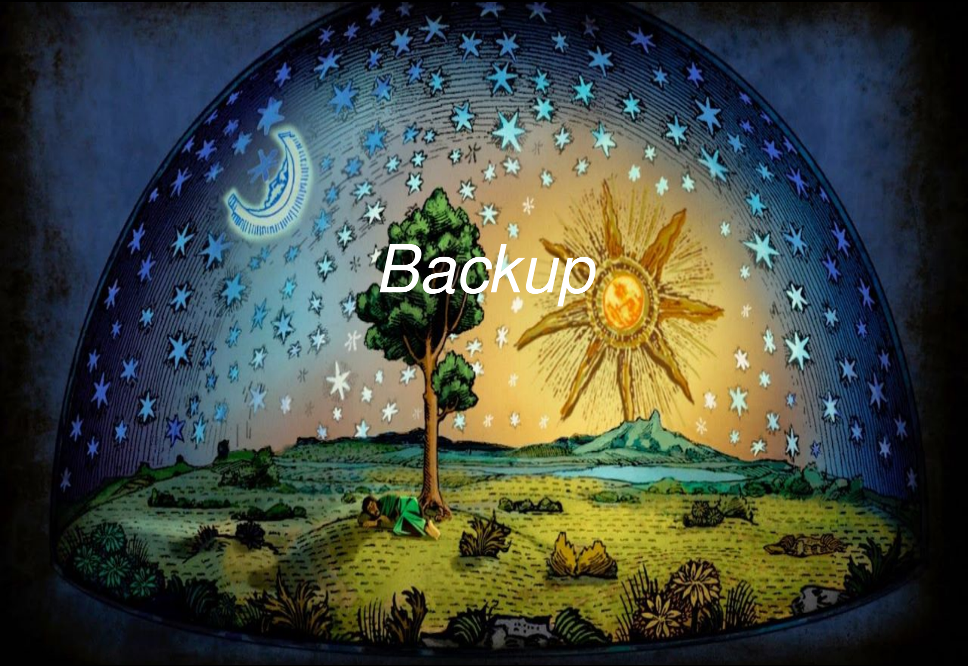
Main result

The anisotropic curvature can only be balanced in the free 0-form or 2-form cases through Hodge duality.


El poder de la ciencia se adquiere gracias a una especie de pacto con el diablo: a costa de una progresiva evanescencia del mundo cotidiano. Llega a ser monarca, pero, cuando lo logra, su reino es apenas un reino de fantasmas.
—Ernesto Sabato, *Uno y el Universo* (1945).

Backup



Homogeneity

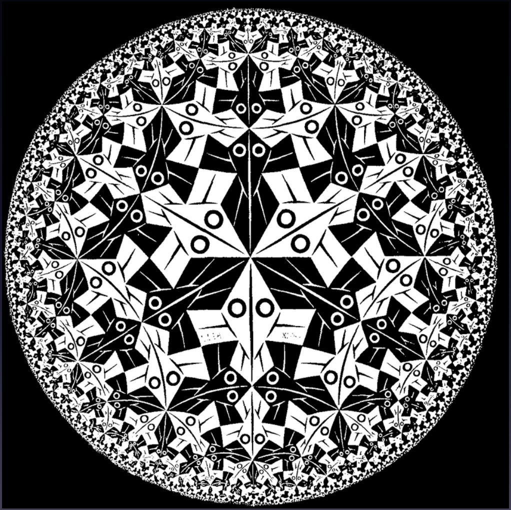


Figure: Circle limit I, by M.C. Escher.

Spatial homogeneity

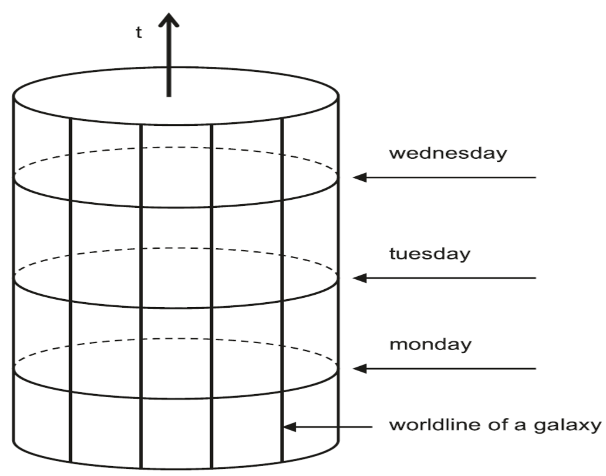


Figure: Taken of Perlov, D., & Vilenkin, A. (2017). *Cosmology for the Curious*. Springer International Publishing.

Isotropy

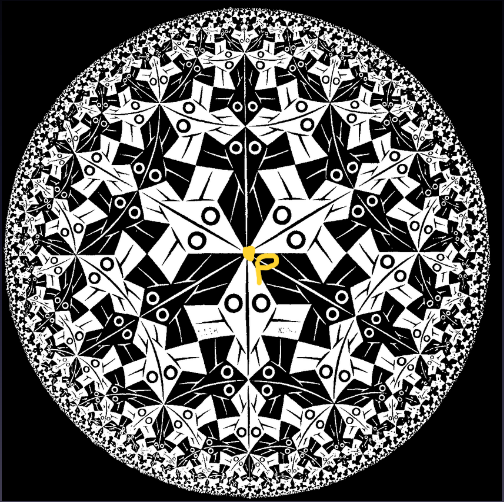


Figure: Circle limit I, by M.C. Escher.

Homogeneity: formal definition

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Definition 1 (Isometry)

A diffeomorphism $\varphi : \mathcal{M} \mapsto \mathcal{M}$ *will be said to be an isometry if it carries the metric into itself, that is, if the mapped metric* $\varphi^* \boldsymbol{g}$ *by the pull-back map* φ^**, is equal to* $\boldsymbol{g}$ *at every point. Then, the map* φ^* *preserves the scalar products, as*

$$\boldsymbol{g}(\boldsymbol{X}, \boldsymbol{Y})\big|_p = \varphi^* \boldsymbol{g}\left(\varphi_* \boldsymbol{X}, \varphi_* \boldsymbol{Y}\right)\big|_{\varphi(p)} ,$$

with $\boldsymbol{X}$ *and* $\boldsymbol{Y}$ *being vector fields on* $\mathcal{T}_p(\mathcal{M})$*.*

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with $\boldsymbol{X}$ and $\boldsymbol{Y}$ being vector fields on $\mathcal{T}_p(\mathcal{M})$.

Definition 2 (Isometry group)

The isometry group is denoted by Isom $(\mathcal{M})$ and defined by

$$Isom(\mathcal{M}) := \{\varphi : \mathcal{M} \longmapsto \mathcal{M} \mid \varphi \text{ is an isometry } \}.$$

Homogeneity: formal definition

Definition 3 (Homogeneous spaces)

If for each pair of points $p, q \in \mathcal{M}$ *there exists a* $\varphi \in Isom(\mathcal{M})$ *so that* $\varphi(p) = q$*, then we say that* $\mathcal{M}$ *is a homogeneous space. In other words, a homogeneous space is a space where you can get from one point to any other point using an isometry.*

Homogeneity: formal definition

Definition 3 (Homogeneous spaces)

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Definition 4 (Killing vector field)

If a local one-parameter group of diffeomorphisms φ_t generated by a vector field $\boldsymbol{X}$ is a group of isometries, we call the vector field $\boldsymbol{X}$ a Killing vector field. Thus, the Lie derivative of the metric tensor $\boldsymbol{g}$ respect to $\boldsymbol{X}$ is

$$\mathcal{L}_{\boldsymbol{X}}\boldsymbol{g} = \lim_{t\to 0} \frac{1}{t}\left(\boldsymbol{g} - \varphi_t^{*}\boldsymbol{g}\right) = 0.$$

Isotropy: formal definition

Homogeneity condition

For a manifold $\mathcal{M}$ of dimension n to be homogeneous, the number m of Killing-vectors $\{\boldsymbol{\xi}_\alpha\}$ must be equal to or larger than n. Hence we require

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$$m \geq n.$$

In the case where $m > n$, not all Killing-vectors can be linearly independent. We denote the difference as

$$d = m - n.$$

This corresponds to the measure of what we shall call *isotropy*.

Isotropy: formal definition

As an example, consider the maximally symmetric three-spaces, i.e., $n = 3$. Furthemore, its number of killings vector are equal to

$$m = \frac{n(n+1)}{2} = 6 \Longrightarrow d = 3.$$

The remaining transformations are the three rotations around a point p.

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The remaining transformations are the three rotations around a point p.

Isotropy subgroup

Take a point $p \in \mathcal{M}$. Then the isotropy subgroup of p is defined as

$$\mathrm{Iso}_p(\mathcal{M}) = \{\varphi \in I\,\mathrm{som}(\mathcal{M}) \mid \varphi(p) = p\}.$$

Hence, the isotropy group is a subgroup of $Isom(\mathcal{M})$ that leaves the point p fixed.

Isotropy: formal definition

The Lie algebra of a Lie group

Let $\mathcal{G}$ be a Lie group. Then the tangent space of $\mathcal{G}$ at the identity element $\mathcal{T}_e(\mathcal{G})$ is a Lie algebra, i.e.,

$$\mathfrak{g} = \mathcal{T}_e(\mathcal{G}).$$

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Corollary

The Killing vector field $\{\boldsymbol{\xi}_\mu\}$ of the whole manifold $\mathcal{M}$ forms a finite-dimensional vector space, such that its algebra is isomorphic to the Lie algebra $\{\boldsymbol{e}_\mu\}$ of $Isom(\mathcal{M})$.

Isotropic and homogeneous space properties

- ✓ The dimension d of $I\,\mathrm{sop}_p(\mathcal{M})$ of the manifold $(\mathcal{M}, \boldsymbol{g})$ determines the *isotropic* properties of the manifold.

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Our interest

We shall study expanding cosmological models, so that we shall impose that the dimension of the orbit at $p \in \mathcal{M}$ under $\mathrm{Isom}(\mathcal{M})$ is equal to the dimension of the spatial hypersurface Σ, i.e. we impose $n = 3$.

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✓ $\boldsymbol{d = 0}$ (*Anisotropic*): They are called the **Bianchi models**.

Summary

In this research work, we will focus only on the Bianchi models. So, we shall restrict our study to consider a manifold that has

$$m = 3 \quad \textit{and} \quad d = 0.$$