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Some astrophysical properties of compact object solutions in the Generalized SU(2) Proca theory

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Introduction

Modified gravity models

- Einstein's theory is very successful.
- General relativity is an effective theory of the gravitational interaction.
- What is dark energy? What is dark matter?
- Metric theories of gravity: additional degrees of freedom.
- This work: additional vectorial degrees of freedom → generalized SU(2) Proca theory (GSU2P).

GSU2P Theory

- Main components: contractions of the Riemann tensor, SU(2) vector field $B^a{}_\mu$, field strength $F^a{}_{\mu\nu} = 2\nabla_{[\mu}B^a{}_{\nu]} + g_c\epsilon^a{}_{bc}B^b{}_\mu B^c{}_\nu$, Antisymmetric $A^a{}_{\mu\nu} = 2\nabla_{[\mu}B^a{}_{\nu]}$, Symmetric $S^a{}_{\mu\nu} = 2\nabla_{(\mu}B^a{}_{\nu)}$.
- Global SU(2) symmetry.
- Equations are maximum of order 2.
- Correct number of propagating degrees of freedom.
- Action principle[1]

$$S = \int \sqrt{-g}d^4x\frac{1}{16\pi}\left(R + F_{a\mu\nu}F^{a\mu\nu} + 2\mu B_{a\mu}B^{a\mu} + \sum_{n=0}^{6}\alpha_n\mathscr{L}^n_{4,2} + \sum_{m=1}^{7}\chi_m\mathscr{L}^m_2\right).$$

[1] A. G. Cadavid, Y. Rodríguez, and L. G. Gómez, Phys. Rev. D 102, 104066 (2020).

GSU2P Lagrangian

- α's Lagrangians

$$\mathscr{L}^{1}_{4,2} = B_{b\alpha}B^{b\alpha}\left(S^{a\mu}{}_{\mu}S_{a\nu}{}^{\nu} - S^{a\mu}{}_{\nu}S_{a\mu}{}^{\nu}\right) + 2B_{a\alpha}B^{\alpha}_{b}\left(S^{\mu a}_{\mu}S^{\nu b}_{\nu} - S^{\mu a}_{\nu}S^{\nu b}_{\mu}\right),$$

$$\mathscr{L}^{2}_{4,2} = A^{a}_{\mu\nu}S^{\nu b}_{\sigma}B^{\nu}_{a}B^{\sigma}_{b} - A^{a}_{\mu\nu}S^{\mu b}_{\sigma}B^{\nu}_{b}B^{\sigma}_{a} + A^{a}_{\mu\nu}S^{\sigma b}_{\sigma}B^{\mu}_{a}B^{\nu}_{b},$$

$$\mathscr{L}^{3}_{4,2} = B^{a\mu}R^{\alpha}{}_{\sigma\rho\mu}B_{a\alpha}B^{c\rho}B_{c}{}^{\sigma} + \frac{3}{4}B^{a}_{\mu}B^{\mu}_{a}B^{\nu}_{b}B^{b}_{\nu}R,$$

$$\mathscr{L}^{4}_{4,2} = \left(B^{a\mu}B_{a\mu}B^{b\nu}B_{b\nu} + 2B^{a}{}_{\mu}B^{b\mu}B_{a\nu}\ B_{b}{}^{\nu}\right)R,$$

$$\mathscr{L}^{5}_{4,2} = G_{\mu\nu}B^{\mu a}B_{a}{}^{\nu}B^{b\alpha}B_{b\alpha},$$

$$\mathscr{L}^{6}_{4,2} = G_{\mu\nu}B^{\mu a}B^{\nu b}B_{a\alpha}B_{b}{}^{\alpha}.$$

GSU2P Lagrangian

- χ's Lagrangians

$$\mathscr{L}_2^1 = B_{a\mu}B^{a\mu}B_{b\nu}B^{b\nu},$$

$$\mathscr{L}_2^2 = B^a{}_\mu B_b{}^\mu B^b{}_\nu B_a{}^\nu,$$

$$\mathscr{L}_2^3 = B^b{}_\mu B_b{}^\rho A^{a\mu\nu}A_{a\rho\nu},$$

$$\mathscr{L}_2^4 = B^b{}_\mu B_a{}^\rho A^{a\mu\nu}A_{b\rho\nu},$$

$$\mathscr{L}_2^5 = B_{\mu a}B^{b\rho}A^{a\mu\nu}A_{\rho\nu b},$$

$$\mathscr{L}_2^6 = B^{b\rho}B^{b\rho}A_{a\mu\nu}A^{a\mu\nu},$$

$$\mathscr{L}_2^7 = B^{b\rho}B_{a\rho}A_{b\mu\nu}A^{a\mu\nu}.$$

Cosmological implications

- Vector fields no isotropy? Cosmic triad, emerges naturally in SU(2)
- Test the theory at cosmological scales[2].
- GW propagation speed is luminal? It is very stringent condition to modified gravity models.
- Only the tensorial perturbations obey the same equations as GR, i.e., luminal speed,

$$\alpha_2 = 2\alpha_3, \alpha_4 = \frac{7\alpha_3}{20} - 2\alpha_1, \alpha_5 = \frac{14\alpha_3}{3} - \frac{20\alpha_1}{3}, \alpha_6 = -8\alpha_3,$$

$$\chi_3 = 0, \chi_7 = 5\alpha_1 + \alpha_3 - \frac{\chi_4}{2} - 3\chi_6$$

- Early inflation: constant roll.
- Late acceleration: dark energy.

[2]J. C. Garnica, L. G. Gomez, A. A. Navarro, and Y. Rodriguez, Annalen der Physik, 534, 2100453 (2021).

Yang-Mills vector fields + Gravity

- Strong field regime, compact objects.
- Breakdown of GR at the singularity of black holes.
- Einstein + Yang-Mills (EYM) , static, localized solutions, asymptotically flat: soliton, particlelike.
- Attractive (gravity) and repulsive (Yang-Mills) interactions.
- EYM Good: particlelike and BHs. Bad: unstable.
- Einstein + Skyrme: Particlelike, BHs. Stable solutions.
- Are there particlelike, BH solutions in GSU2P theory?

Stationary and spherical symmetric solution[3]

- Regular solutions: all algebraic curvature invariants, energy density and pressure are finite.
- Line element,

$$ds^2 = -e^{2\Phi}dt^2 + (1-2m/r)^{-1}dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2),$$

also $e^{2\Phi} \equiv e^{-2\delta}(1-2m/r)$

- Asymptotic flatness: when $r \to \infty$, $m \to M = \text{const.}, \delta \to 1$
- SU(2) spherical configuration,

$$\mathbf{B} = \frac{\tau^i}{\tilde{g}}\left[A_0\frac{x_i}{r}dt + A_1\frac{x_i x_j}{r^2}dx^j + \frac{\phi_1}{r}\left(\delta_{ij} - \frac{x_i x_j}{r^2}\right)dx^j - \epsilon_{ijk}x^j\frac{(1-w)}{r^2}dx^k\right],$$

where τ_i are the Pauli matrices.

3 Martinez, J., Rodriguez, J. F., Rodriguez Y., Gomez, L. G JCAP (2023) 032

Field equations

- Variation with respect to metric,

$$G_{\mu\nu} = 8\pi T^{\text{eff}}_{\mu\nu}\Big|_{\text{GW}},$$

where $T^{\text{eff}}_{\mu\nu} = -\dfrac{1}{8\pi\sqrt{-g}}\dfrac{\delta}{\delta g_{\mu\nu}}\left[F_{a\mu\nu}F^{a\mu\nu} + \sum_{i=0}^{6}\alpha_n \mathscr{L}^n_{4,2} + \sum_{m=1}^{7}\chi_m \mathscr{L}^m_2\right]$

- Variation with respect vector field,

$$\frac{\delta}{\delta B_{a\mu}}\left[F_{a\mu\nu}F^{a\mu\nu} + \sum_{i=0}^{6}\alpha_n \mathscr{L}^n_{4,2} + \sum_{m=1}^{7}\chi_m \mathscr{L}^m_2\right] = 0$$

- Individually the only configuration consistent with asymptotic flatness 't Hooft-Polyakov monopole $A_0 = A_1 = \phi_1 = 0,\ \phi_2 = w.$
- Combined configurations e.g. dyon $A_0 \neq 0,\ \phi_2 \neq 0$, future work.
- Normalized variables by $g_c, r \mapsto r g_c, m \mapsto m g_c.$

Effective charge and topological charge

- Effective charge is defined by comparing with the RN solution

$$Q^2 \equiv 2r(M-m)$$

- The topological charge is defined from the Bianchi Identity,

$$d(*\mathbf{F}) = 0$$

- The conserved charge topological charge is,

$$Q_M^a = \int \nabla_\mu (*F^{a\mu 0})\sqrt{-g}d^3x = \int \partial_\mu (\epsilon^{\mu 0\alpha\beta}F^a{}_{\alpha\beta})d^3x = \int dS_k \sqrt{-g} * F^{k0}$$

- The topological charge depends on the value of the value of the fields at ∞

$$Q_M^a \propto (1 - w_\infty^2)$$

$$Q = Q_M^a + q$$

Analytic solution for χ_1, χ_2

Schwarzschild and Reissner Nordstrom black hole

- We found an analytic solution for the parameters $\chi_{12} = 2\chi_1 + \chi_2$, with the ansatz $w = w_c = \text{conts}$.
- Schwarzschild black hole, $w_c = -1$, $m = M$, $\Phi = \frac{1}{2}\ln\left(1 - \frac{2M}{r}\right)$
- Reissner-Nordstrom black hole, $w_c = w_{\rm I,II}$, $m = M - \frac{Q^2_{\rm I,II}}{2r}$, $\Phi = \frac{1}{2}\ln\left(1 - \frac{2M}{r} + \frac{Q^2_{\rm I,II}}{r^2}\right)$

  where

$$w_{\rm I,II} = \frac{1 + 2\chi_{12} \pm \sqrt{1 + 8\chi_{12}}}{2 - 2\chi_{12}},$$

  and the effective charge is,

$$Q^2_{\rm I,II} = \frac{1 - 4\chi_{12}(5 + 2\chi_{12}) \mp (1 + 8\chi_{12})\sqrt{1 + 8\chi_{12}}}{2(1 - \chi_{12})^3},$$

  with $\chi_{12} > -1/8$.

- For the solution I the charge can be imaginary in $\chi_{12} \in (0,1)$.

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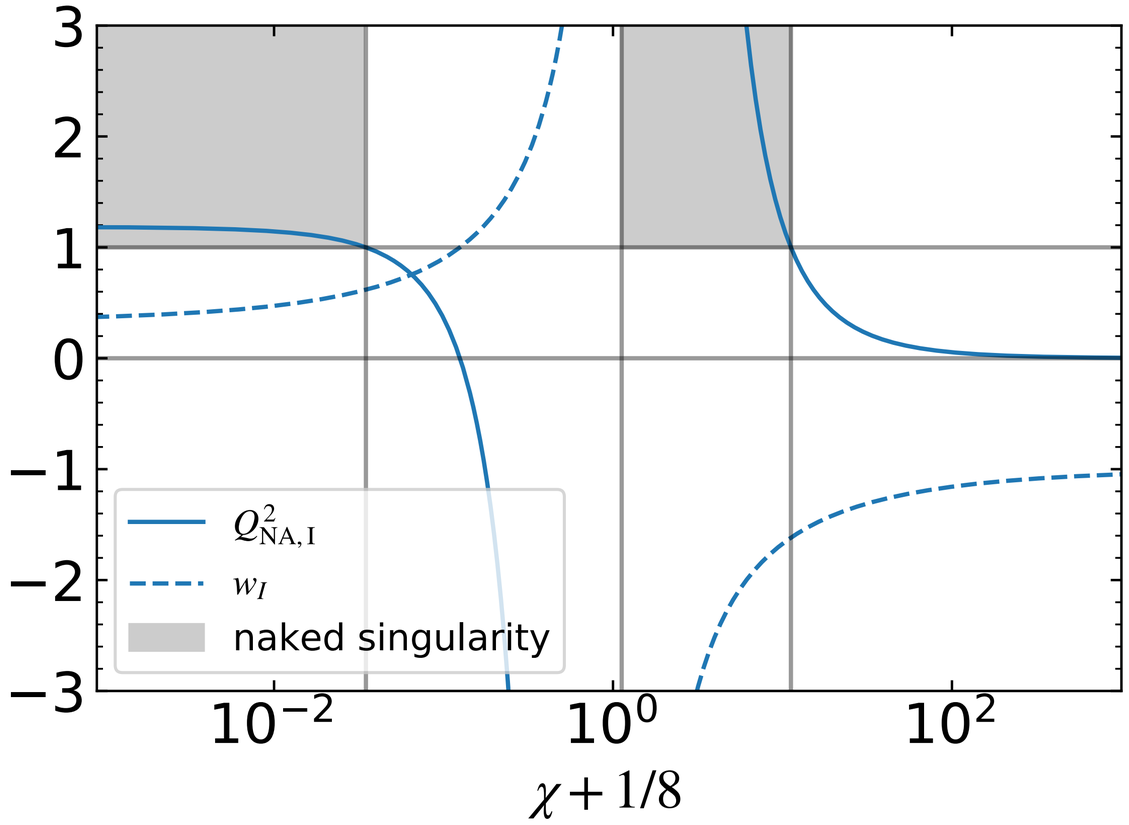
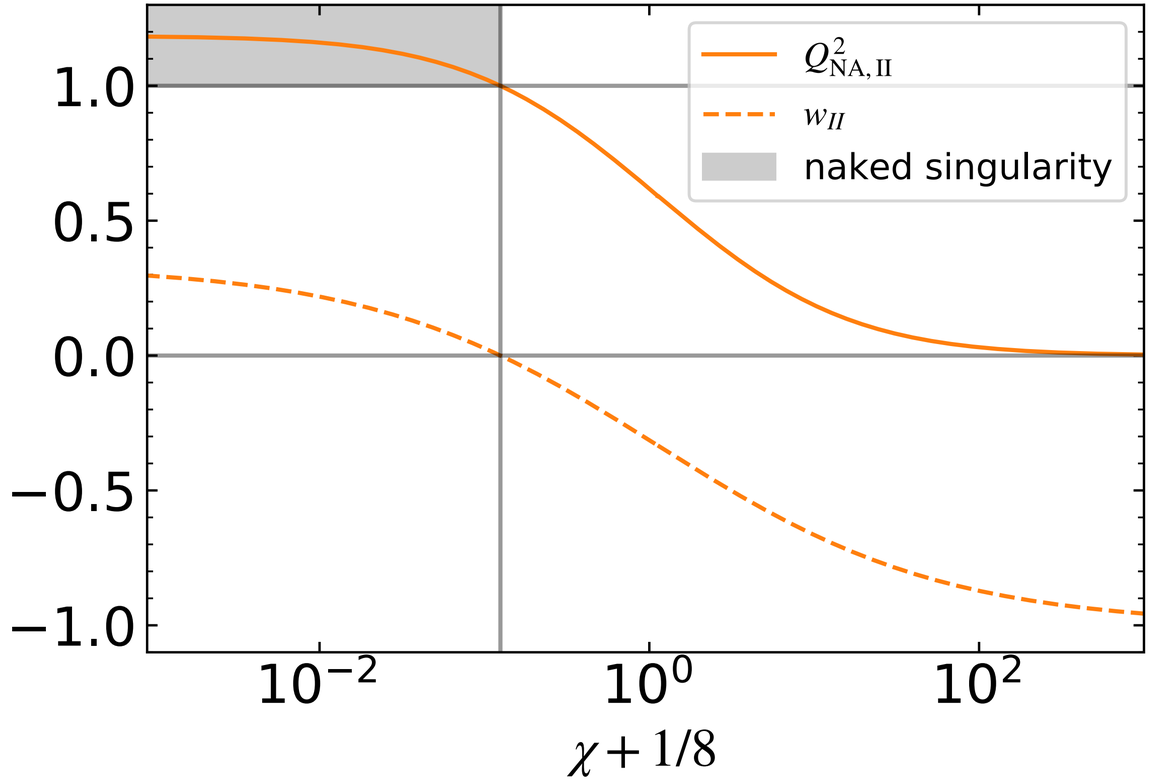
Event Horizon

- Location of the horizons. $r \pm = M \pm \sqrt{M^2 - Q^2_{\rm NA}}$.





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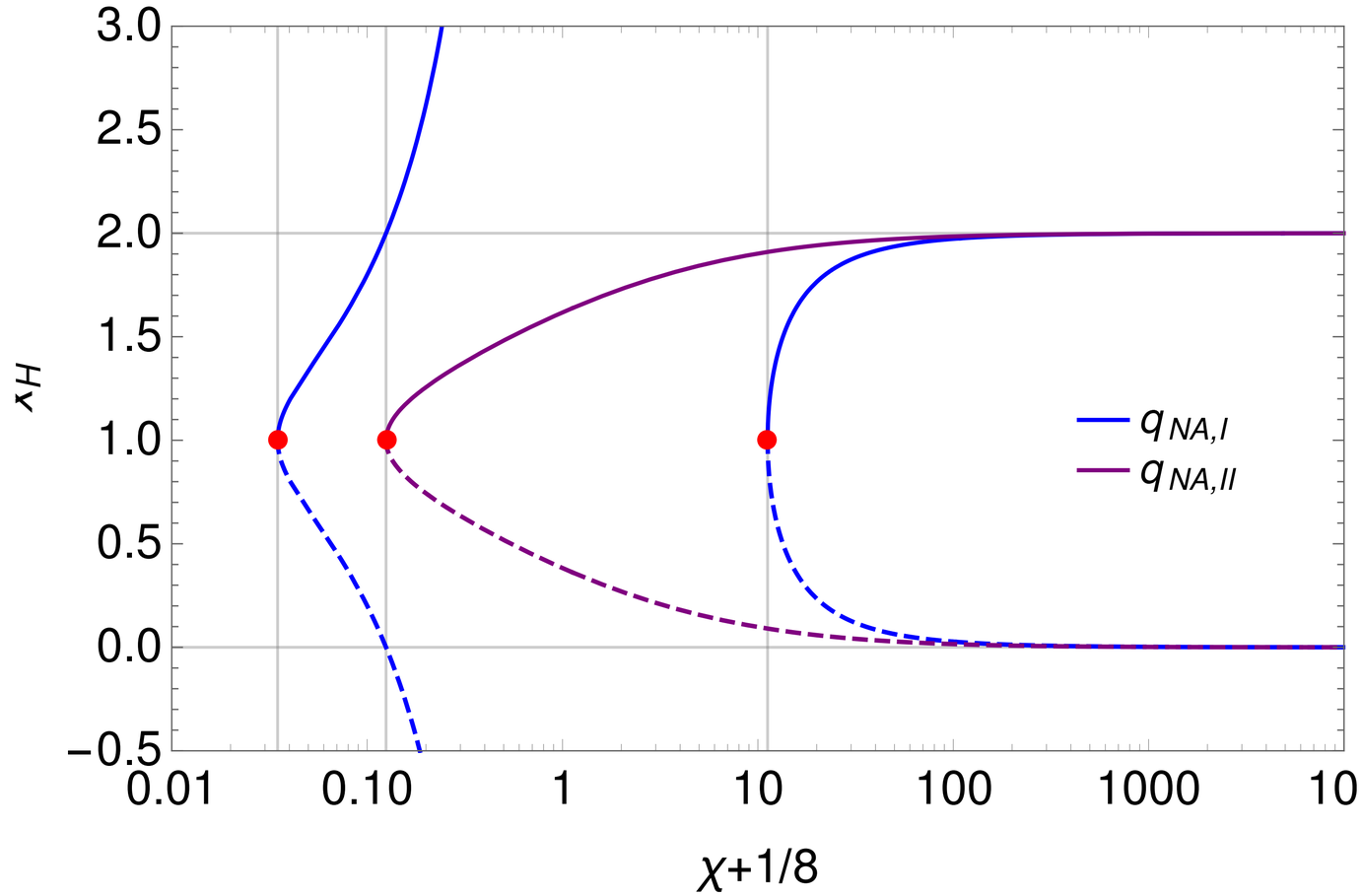
Event Horizon

- $x = r/M, q = Q/M$



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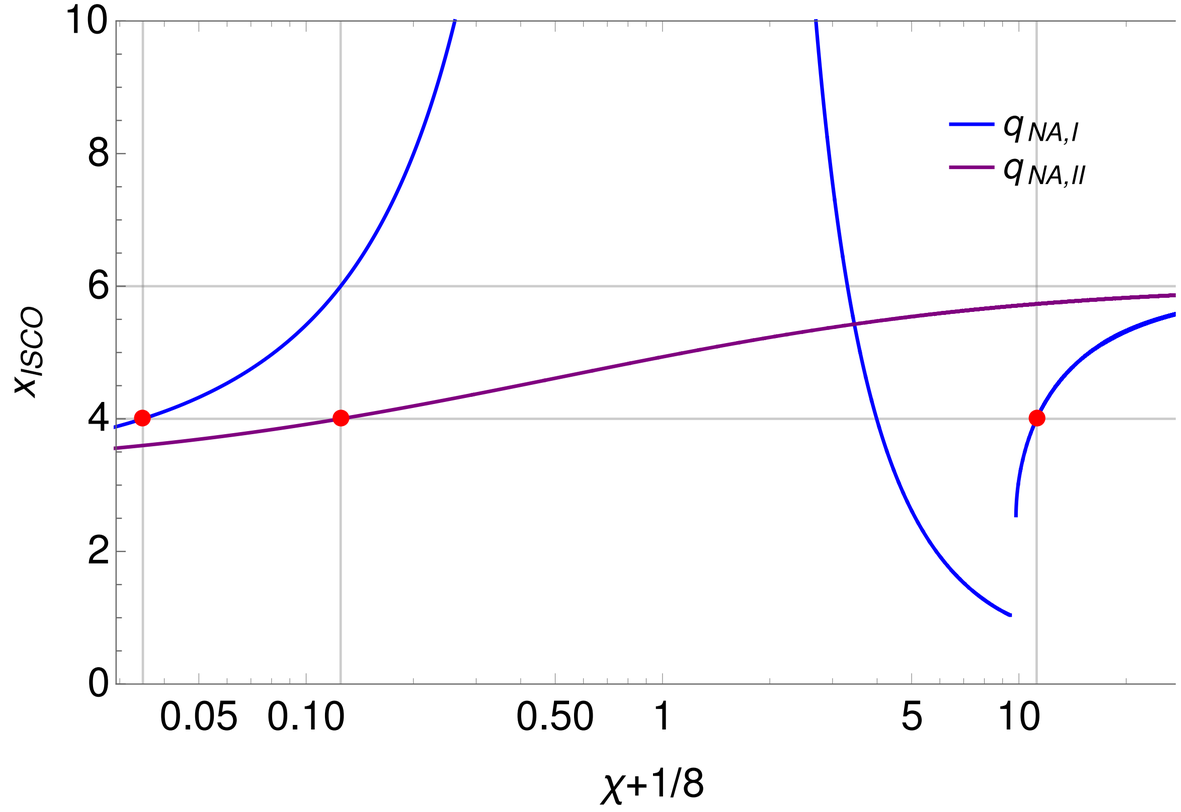
Effective potential for massive test particles

- Geodesic equation $\left(\frac{dr}{ds}\right)^2 = \left[E^2 - N\left(1 + \frac{L^2}{r^2}\right)\right]$
- Effective Potential $V_{\text{eff}} = N\left(1 + \frac{L^2}{r^2}\right).$
- Angular momentum. $L^2 = \frac{N'(r)r^3}{2N(r) - rN'(r)}.$

ISCO

- Location of the inner-most stable circular orbit,

$$r_{\rm ISCO} = 2M + \frac{4M^4 - 3M^2Q_{\rm NA}^2 + \left(8M^6 - 9M^4Q_{\rm NA}^2 + 2M^2Q_{\rm NA}^4 + \sqrt{5M^8Q_{\rm NA}^4 - 9M^6Q_{\rm NA}^6 + 4M^4Q_{\rm NA}^8}\right)^{2/3}}{M\left(8M^6 - 9M^4Q_{\rm NA}^2 + 2M^2Q_{\rm NA}^4 + \sqrt{5M^8Q_{\rm NA}^4 - 9M^6Q_{\rm NA}^6 + 4M^4Q_{\rm NA}^8}\right)^{1/3}}.$$

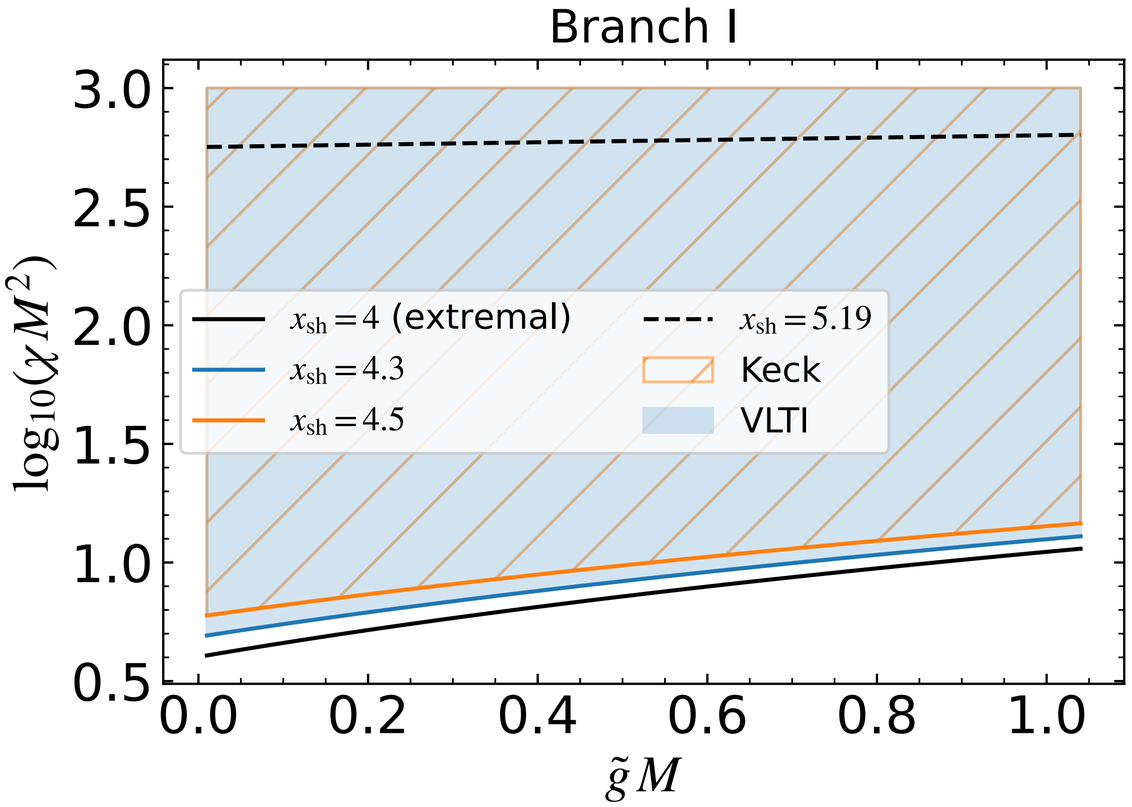


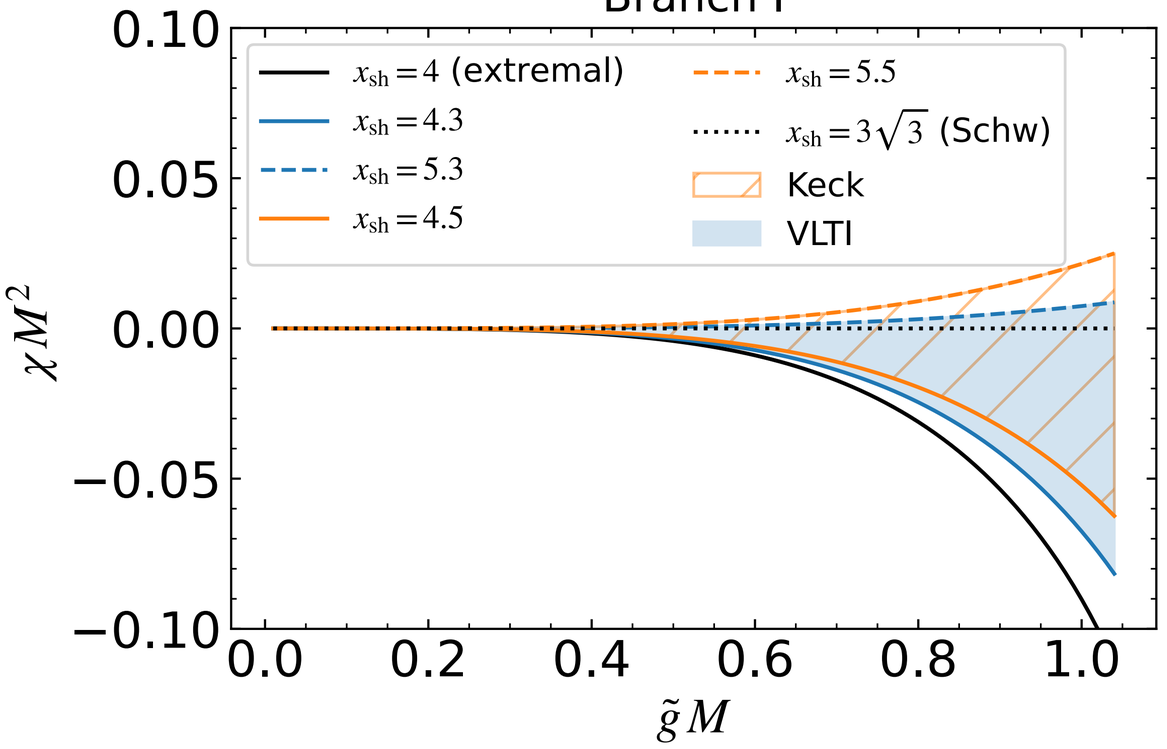
Effective potential for massless particles

- Photon sphere $r_{\rm ph} g'_{tt}(r_{\rm ph}) - 2 g_{tt}(r_{\rm ph}) = 0.$
- Location of the shadow $r_{\rm sh} = \dfrac{r_{\rm ph}}{\sqrt{g_{tt}(r_{\rm ph})}}\,.$
- For this non Abelian charged black holes $\dfrac{r_{\rm sh}}{M} = x_{\rm sh} = \dfrac{\sqrt{2}\left(\sqrt{9 - 8q_{\rm NA}^2} + 3\right)}{\sqrt{\dfrac{4q_{\rm NA}^2 + \sqrt{9 - 8q_{\rm NA}^2} - 3}{q_{\rm NA}^2}}}\,.$

EHT and charged non-abelian Black Holes

- Shadow $4.5 \lesssim x_{\rm sh} \lesssim 5.5$, with Keck and $4.3 \lesssim x_{\rm sh} \lesssim 5.3$, with VLTI





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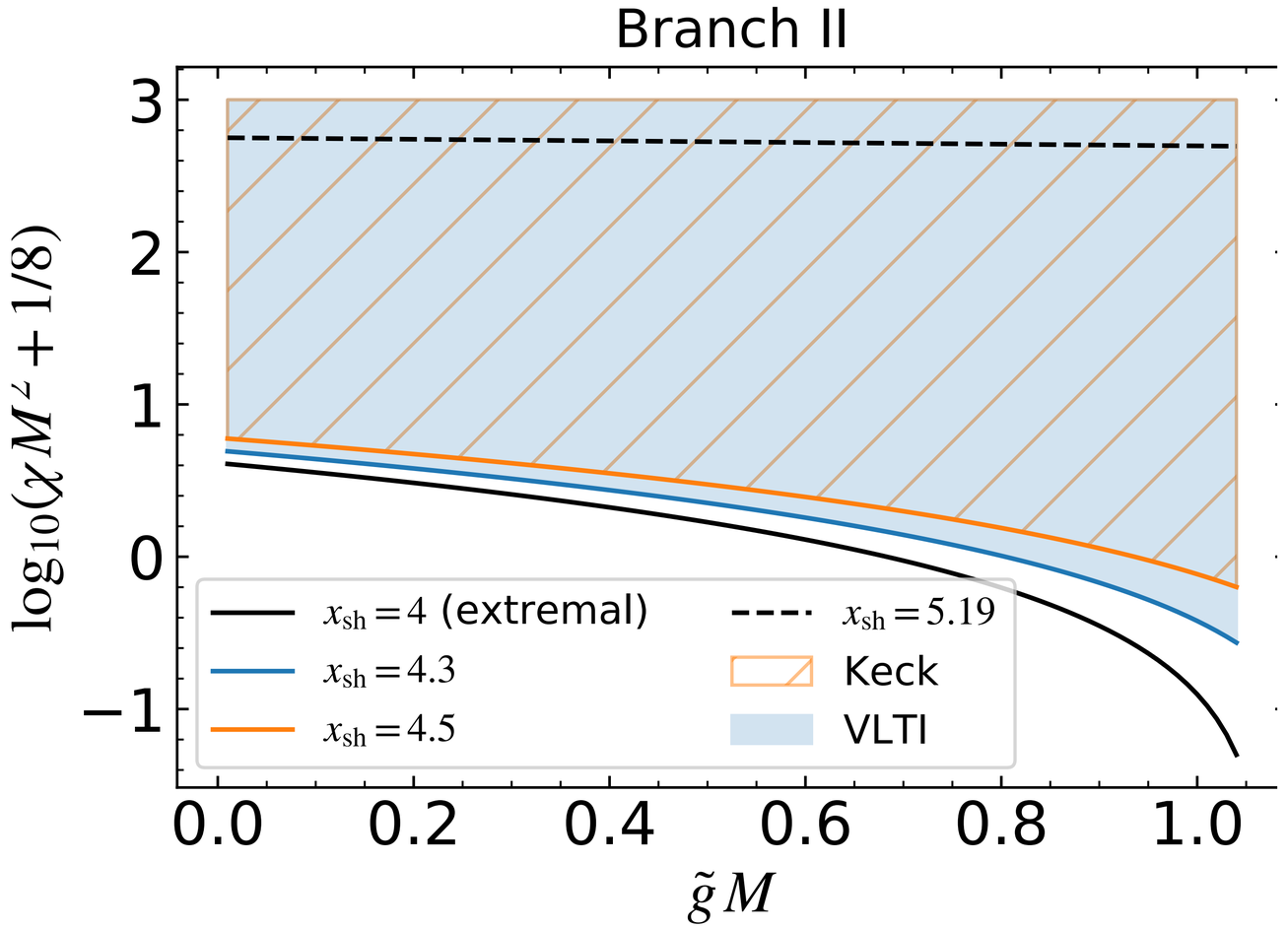
EHT and charged non-abelian Black Holes



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Constraint in the parameters

- For instance, when $\tilde{g}M = 0.9$ the minimum values of χM^2 are -0.0326136 and -0.0414562, for mass-to-distance ratio with Keck and VLTI, respectively.

- if $\tilde{g}M = 1.01$, the constraints are $(-0.0545704 \lesssim \chi M^2 \lesssim 0.0223263) \cup 14.3271 \lesssim \chi M^2$, for Keck and $(-0.0708403 \lesssim \chi M^2 \lesssim 0.00774184) \cup 12.6237 \lesssim \chi M^2$, for VLTI.

Neutral Black Hole Solutions

- Line element $ds^2 = -fdt^2 + h^{-1}dr^2 + r^2 d\Omega^2,$
- With $h \equiv 1 - \dfrac{2m}{r}$
- Horizon $f(r_H) = 0$
- Asymptotic series near the horizon,

$$m = r_H/2 + a_1 x + a_2 x^2 + \ldots,$$

$$w = w_H + b_1 x + b_2 x^2 \ldots,$$

$$f = c_1 x + c_2 x^2 + \ldots,$$

With $x = r - r_H$

Asymptotic solution near the horizon α_1

- $$a_1 = -\Big[(1+w_H)^2\Big(27g_c^8 r_H^8(w_H-1)^2 + 22208\alpha_1^3(1+w_H)^8$$
$$+9\alpha_1 g_c^4 r_H^4(1+w_H)^2\Big\{g_c^2 r_H^2\big[53+(22-19w_H)w_H\big]$$
$$+8(w_H^2-1)^2\Big\} - 24\alpha_1^2\Big\{69g_c^4 r_H^4(1+w_H)^4$$
$$-g_c^2 r_H^2(1+w_H)^6\big[133+w_H(-138+61w_H)\big]\Big\}\Big)\Big] \Big/ \Big[2\Big\{-27g_c^{10}r_H^{10} + 7360\alpha_1^3(1+w_H)^{10}$$
$$+9\alpha_1 g_c^6 r_H^6(1+w_H)^2\big[3g_c^2 r_H^2 - 16(1+w_H)^2\big] - 48\alpha_1^2 g_c^2 r_H^2(1+w_H)^6\big[53g_c^2 r_H^2 + 4(1+w_H)^2\big]\Big\}\Big]$$

Asymptotic solution near the horizon α_1

- $$b_1 = -\Big[(1+w_H)\Big\{9g_c^{10}r_H^{10}(1-w_H)w_H + 12288\alpha_1^3(1+w_H)^{10}$$
$$-24\alpha_1 g_c^6 r_H^6(1+w_H)^2\left[3g_c^2r_H^2 + 2(w_H-1)(1+w_H)^2(-1+2w_H)\right]$$
$$-64\alpha_1^2 g_c^2 r_H^2(1+w_H)^6\left[29g_c^2r_H^2 - (-1+w_H)(1+w_H)^2(-12+11w_H)\right]\Big\}\Big]/$$
$$\Big[r_H\Big(-9856\alpha_1^3(1+w_H)^{10} + 9g_c^8r_H^8\left[g_c^2r_H^2 - (w_H^2-1)^2\right] - 3\alpha_1\Big\{3g_c^8r_H^8(1+w_H)^2$$
$$+8g_c^4r_H^4(-1+w_H)^2(1+w_H)^6 - g_c^6r_H^6(1+w_H)^4\left[-37+w_H(-22+19w_H)\right]\Big\} + 8\alpha_1^2\Big\{175g_c^4r_H^4(1+w_H)^6$$
$$-g_c^2r_H^2(1+w_H)^8\left[125+w_H(-138+61w_H)\right]\Big\}\Big)\Big],$$

Asymptotic solution when $r \to \infty$

- SU(2) global symmetry breaks $w \mapsto -w$.
- There are two solutions, $w \to -1$

$$m = M_\infty - \frac{b_{1,\infty}}{g_c^2 r^3} + \mathcal{O}(r^{-4})$$

$$w = -1 + \frac{b_{1,\infty}}{r} + \frac{3(2M_\infty + b_{1,\infty})b_{1,\infty}}{4r^2} + \frac{b_{1,\infty}(48M_\infty^2 - 42M_\infty b_{1,\infty} + 11b_{1,\infty}^2)}{20r^3} + \mathcal{O}(r^{-4}),$$

$$f = f_\infty\left(1 - \frac{2M_\infty}{r}\right) + \mathcal{O}(r^{-4}),$$

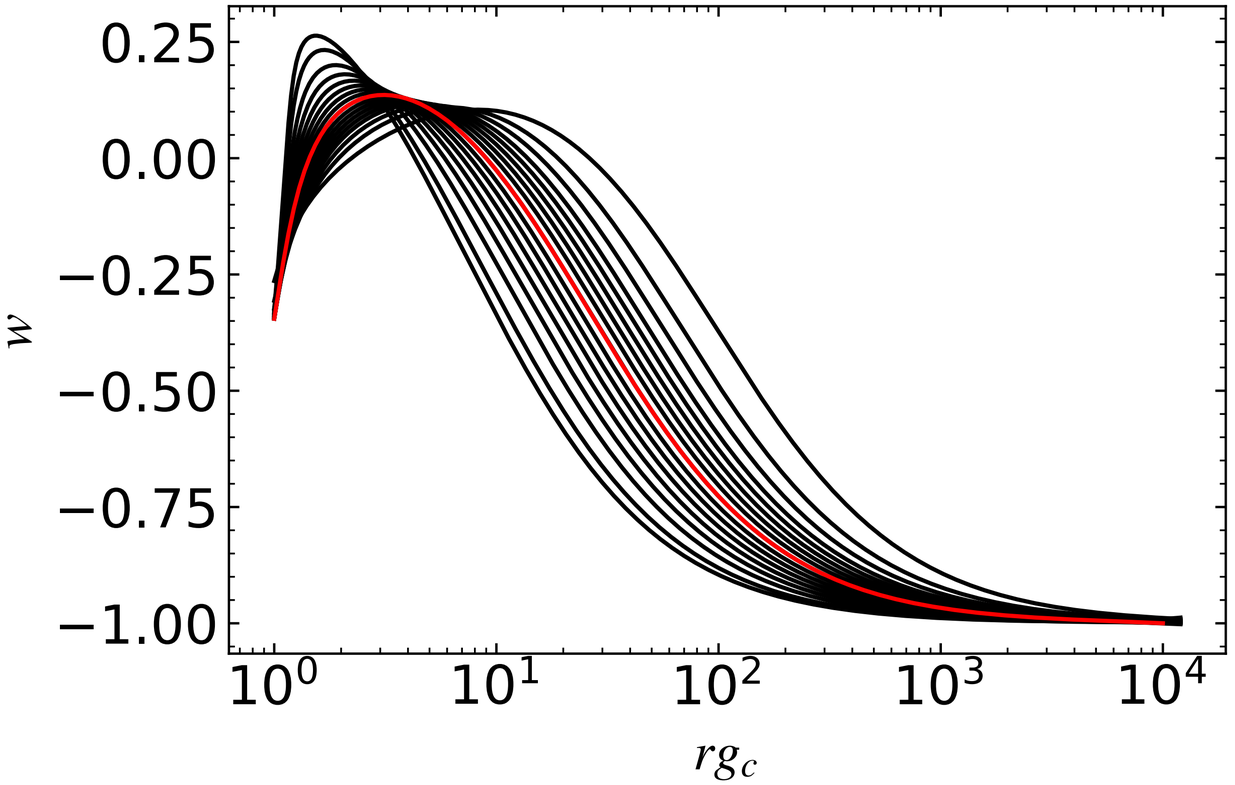
- $w \to 1$

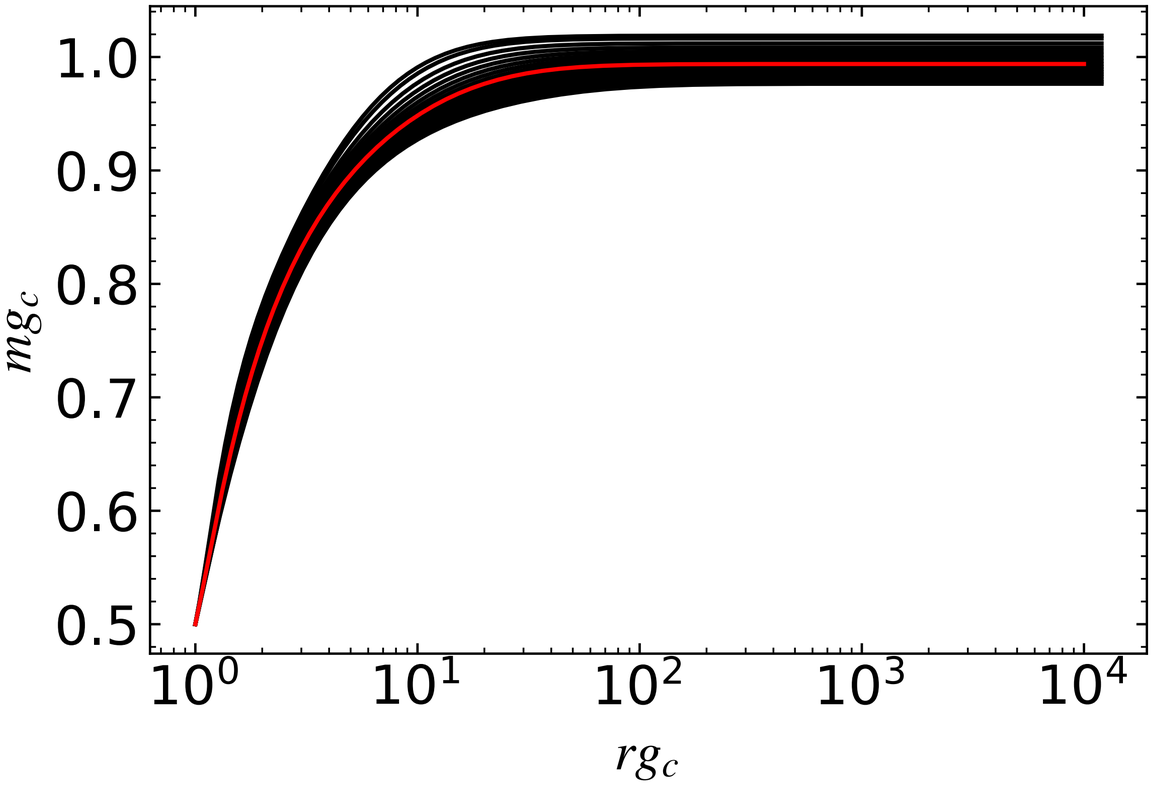
$$m = M_\infty + \frac{1}{r^3 g_c^3}(320\alpha_1/g_c - b_1^2 g_c) + \mathcal{O}(r^{-4}),$$

$$w = 1 + \frac{b_1}{r} + \frac{1}{r^2 g_c^2}\left[16\alpha_1 + 3b_1 g_c^2(2M_\infty + b_1)\right] + \mathcal{O}(r^{-3}),$$

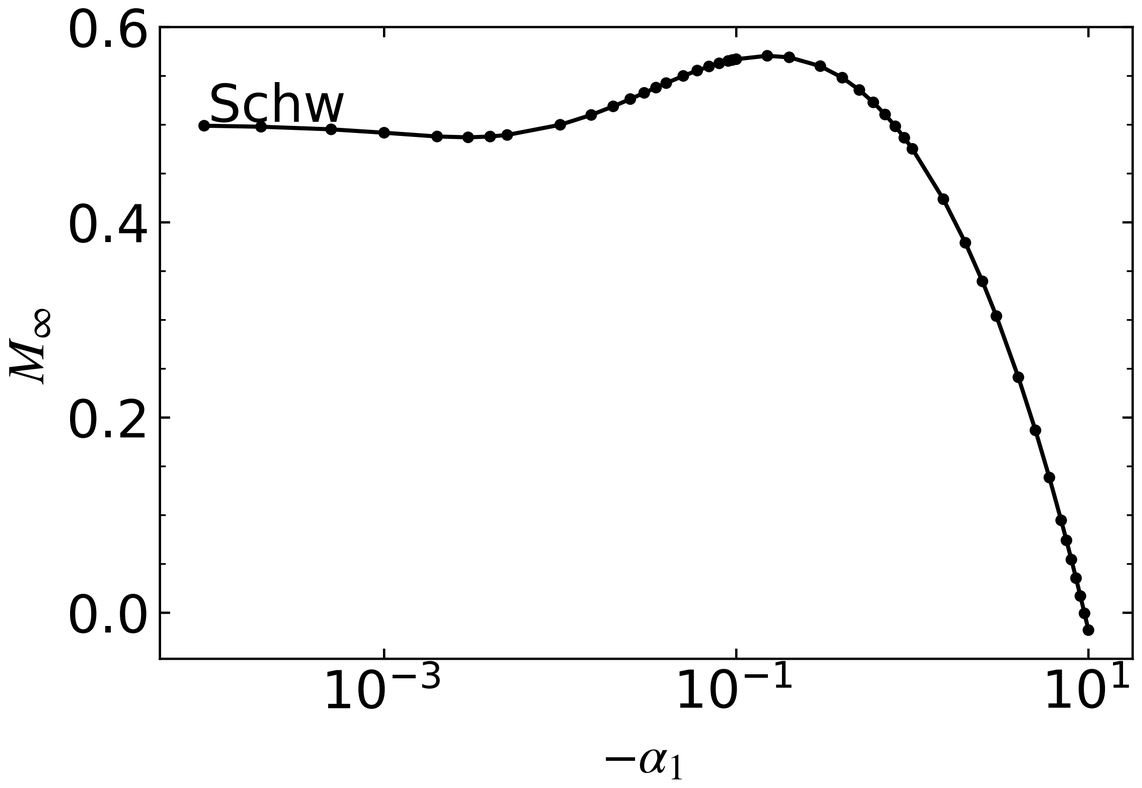
$$f = f_\infty\left[1 - \frac{2M_\infty}{r} + \frac{1}{3r^4 g_c^4}(512\alpha_1 + 3b_1^2 g_c^2)\right],$$

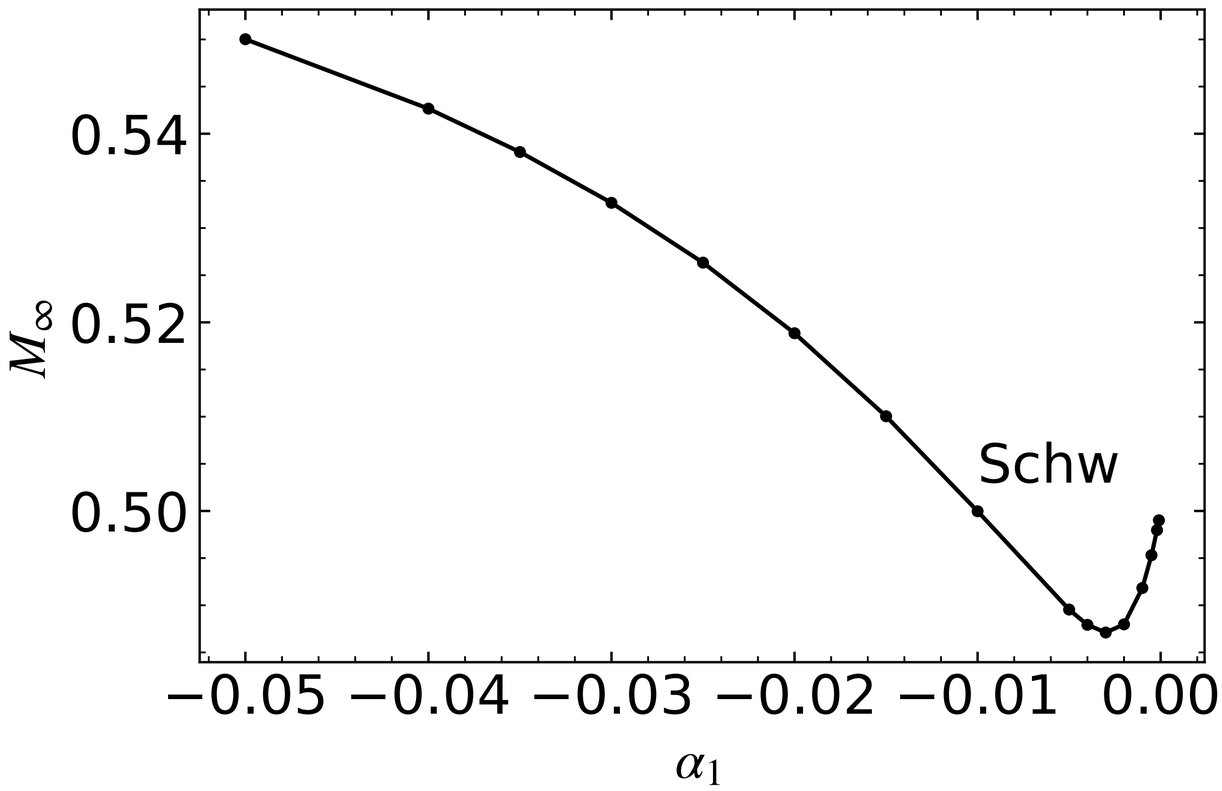
Exterior BH solutions α_1



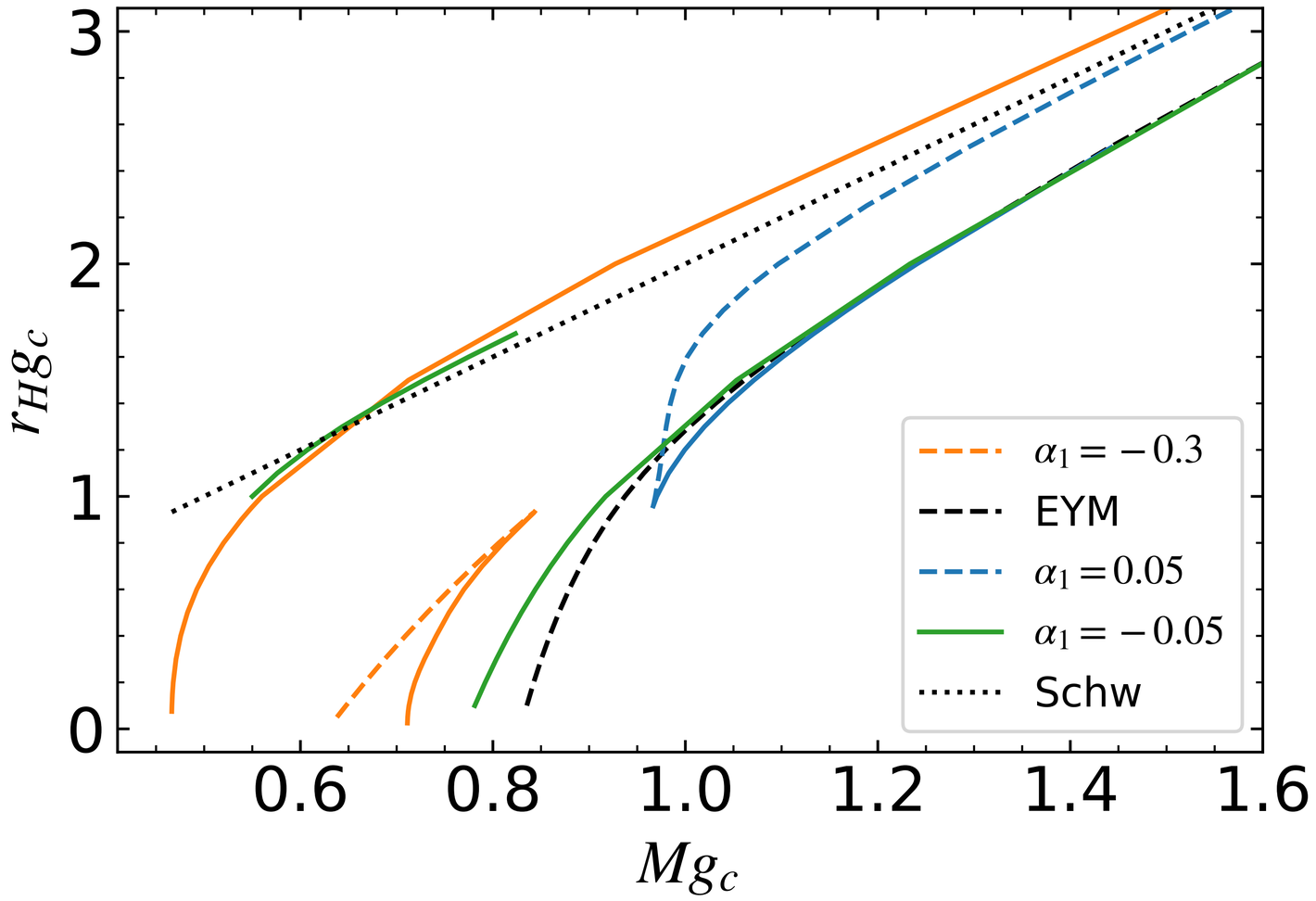


Equilibrium sequences α_1

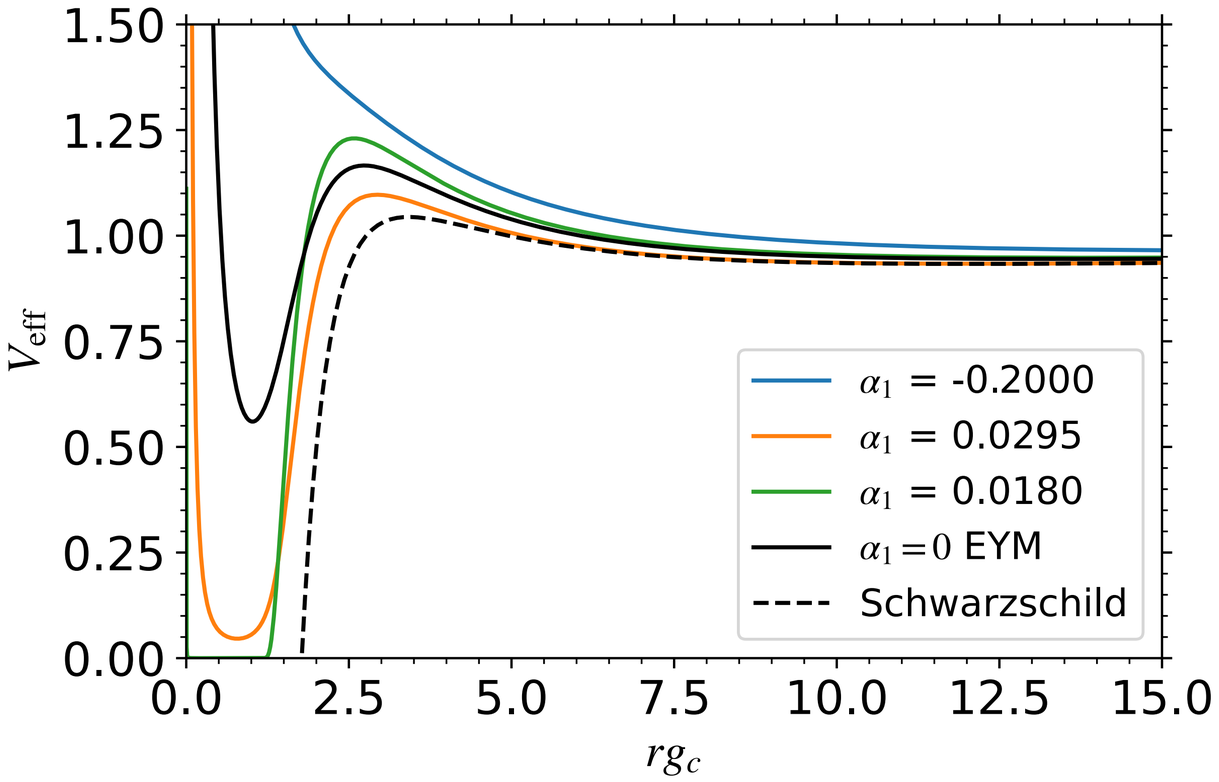


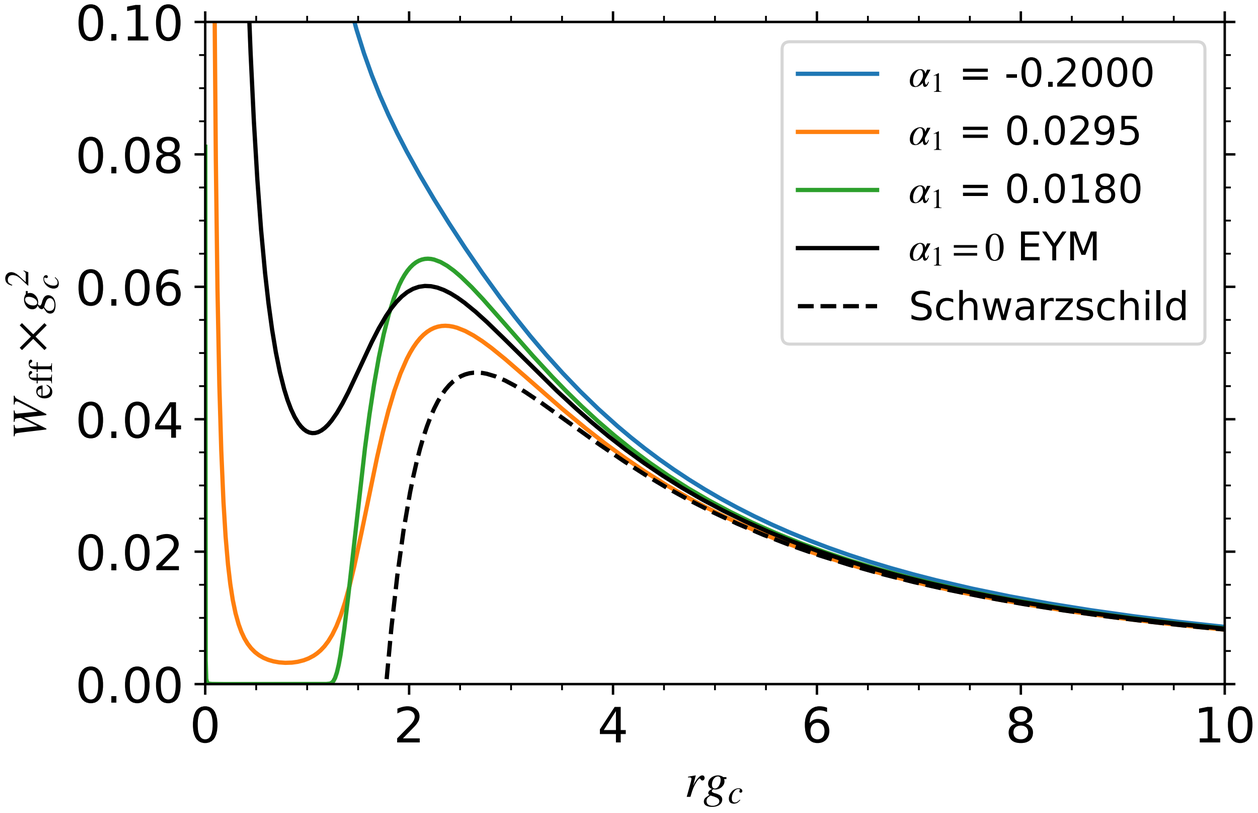


Equilibrium sequences α_1



Effective potential case α_1

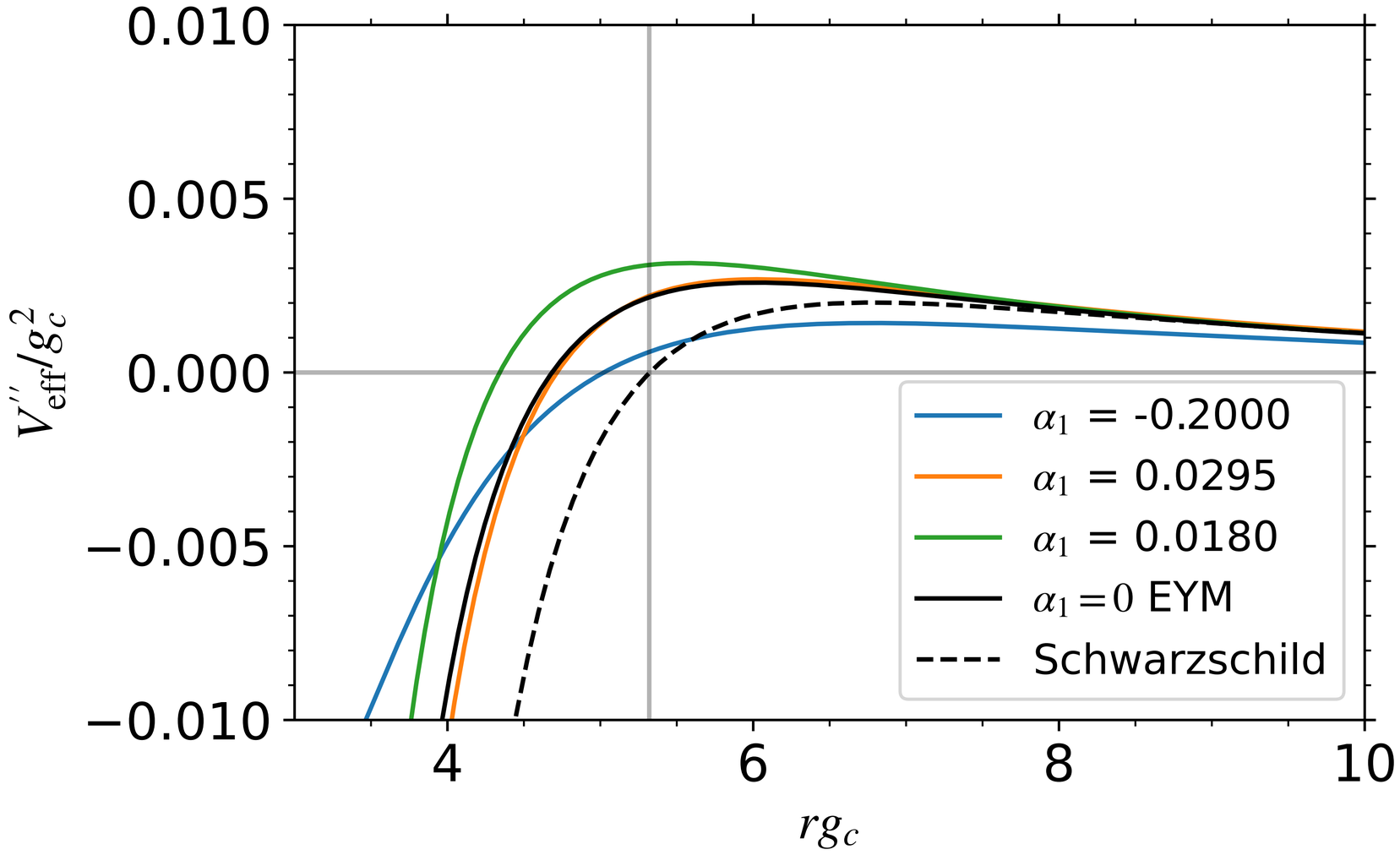




- $\frac{f}{h}\dot{r}^2 + V_{\text{eff}} = E^2, \quad V_{\text{eff}} = f\left(1 + \frac{L^2}{r^2}\right),$

$$\frac{f}{L^2 h} r'^2 + W_{\text{eff}} = \frac{E^2}{L^2} \equiv \frac{1}{b^2}, \quad W_{\text{eff}} = \frac{f}{r^2},$$

ISCO


0.010
0.005
0.000
−0.005
−0.010
4
6
8
10
$V''_{\rm eff}/g_c^2$
rg_c
α_1 = -0.2000
α_1 = 0.0295
α_1 = 0.0180
$\alpha_1 = 0$ EYM
Schwarzschild

Results

- We have found BH solutions in the GSU2P theory.
- Generalization of the EYM case. New charged solutions. EHT results can be used to constraint χ_1 and χ_2
- We have constructed equilibrium sequences and found bifurcations points for BH solutions case α_1, which hints towards the existence of stable solutions.
- Highly compact objects with photon sphere

Future work

- Stability, perturbations: radial and angular.
- GW near these compact objects
- Rotating solutions.
- Static black holes with axial symmetry.
- Accretion disks.