Analyzing Running Vacuum Energy and Viscous Dark Matter Models

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Third Workshop on Current Problems in Cosmology

Topics

1. Motivation
   - The ΛCDM model: successes and tensions
   - Alleviating the tensions of the ΛCDM model
2. Viscous Cosmologies
   - A new parametrization for the bulk viscosity
3. The model
   - Dynamical system analysis
   - Cosmological constraints
4. Conclusions

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The ΛCDM model: successes and tensions

Features of the Standard cosmological model

- It describes an homogeneous isotropic universe.
- For late times the universe is described by cold dark matter and a cosmological constant (Λ).

The Hubble parameter in the ΛCDM model

$$H(z) = H_0 \sqrt{\Omega_{m,0}(1+z)^3 + \Omega_{\Lambda,0}},$$

$$\Omega_{m,0} \equiv \frac{\rho_{m,0}}{3H_0^2},$$

$$\Omega_{\Lambda,0} \equiv \frac{\Lambda}{3H_0^2}.$$

Constraint

$\Omega_{m,0}$ y $\Omega_{\Lambda,0}$ se are constrained by Friedmann's equation: $\Omega_{\Lambda,0} = 1 - \Omega_{m,0}$.

Successes of the ΛCDM model

- It describes the Ia-type of supernovaes observations (SNe Ia)[a].
- It describes consistently the Hubble parameter observations (OHD)[b].
- It describes the acustic barionic oscillations (BAO)[c].
- It describes the cosmic radiation background observations (CMB)[d].

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[a] A. Riess et al. [High-z Supernova Search Team], Astron. J. **116**, 1009 (1998).
[b] M. Moresco, Mon. Not. Roy. Astron. Soc. **450**, L16 (2015).
[c] D. Eisenstein et al. [SDSS Collaboration], Astrophys. J. **633**, 560 (2005).
[d] N. Aghanim et al. [PLanck Collaboration], Astron. Astrophys. **641**, A6 (2020).

Tensions of the ΛCDM model

- The comological constant problem (at least on fundamental level)[a].
- The coincidence problem[b].
- Tensions of measurements in the $\sigma_8 - \Omega_m$ among Planck CMB and the large scale structure (LSS)[c].
- Excess of radiation found by the EDGES experiment compared to the ΛCDM model at $z \approx 17$[d].

[a] S. Weinberg, Rev. Mod. Phys. **61**, 1 (1989).
[b] H. E. S. Velten et al., Eur. Phys. J. C **74**, 3160 (2014).
[c] S. Anand et al., J Cosmol. Astropart. Phys. **11**, 005 (2017).
[d] J. D. Bowman et al., Nature **555**, 67 (2018).

Tension in the measurement of H_0

The measurements of H_0 differs in 5σ among the value obtained by cefeidas observations and the one extrapolated from Planck CMB[a].

[a] A. G. Riess et al., Astrophys. J. Lett. **934**, L7 (2022).

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Alleviating the tensions of the ΛCDM model

A possible solution

- A bulk viscosity fluid representing the DM can explain the recent accelerated expansion of the universe without the inclusion of Λ[a].
- The excess of radiation detected by EDGES can be explained by the ΛCDM model including a viscous DM component[b].
- The $\sigma_8 - \Omega_m$ tensions can be alleviated as well by considering a bulk viscoues fluid, shear or both effects combined [c].
- The tensions of the H_0 mesurements can also be alleviated by introducing viscous fluids[d].

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[a] A. Avelino et al., J. Cosmol. Astropart. Phys. **04**, 006 (2009).
[b] J. R. Bhatt et al., Phys. Rev. D **100**, 063539 (2019).
[c] S. Anand et al., J. Cosmol. Astropart. Phys. **11**, 005 (2017).
[d] I. H. Brevik et al., Mod. Phys. Lett. A **36**, 2150198 (2021).

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Viscous Cosmologies

Friedmann's equations

$$H^2 \equiv \left(\frac{\dot{a}}{a}\right)^2 = \frac{\rho + \Lambda}{3},$$

$$2\dot{H} + 3H^2 = -p + \Lambda.$$

Conservation equations

$$\dot{\rho} + 3H(\rho + p) = 0.$$

Degrees of freedom

Only two of the above three equations are independent which compels to propose a barotroic equation of state (EoS) among ρ and p of the form $p = \omega\rho$.

Coupled equations including viscosity

$$3H^2 \equiv 3\left(\dot{a}/a\right)^2 = \rho_r + \rho_m + \rho_\Lambda,$$

$$2\dot{H} + 3H^2 = -P_r - P_m^{eff} + P_\Lambda,$$

$$\dot{\rho}_r + 4H\left(\rho_r\right) = 0, \quad \dot{\rho_m} + 3H\left(\rho_m + \Pi\right) = 0$$

$$P_r = \rho_r/3, \quad P_\Lambda = -\Lambda, \quad P_m^{eff} = p_m + \Pi = -3H\xi$$

Dissipative pressure in Eckart's frame

$$\Pi = -3H\xi^{a}.$$

[a] C. Eckart, Phys. Rev. **58**, 919 (1940).

Israel-Stewart's frame of dissipative pressure

$$\tau\dot{\Pi} + \Pi = -3H\xi - \frac{1}{2}\tau\Pi\left(3H + \frac{\dot{\tau}}{\tau} - \frac{\dot{\xi}}{\xi} - \frac{\dot{T}}{T}\right)^{a}.$$

[a] W. Israel et al., Phys. Lett. A **58**, 213 (1976).

Some observations to be considered

- Althought Eckart's theroy in not causal, due to its mathematical simplicity it is widely used in the literature to study the behavior of viscous cosmologies.
- Up to now, there is no functional dependence for ξ coming from first principles.
- The Ansatz for the viscosity $\xi = \xi_0 H$ is one of the more popular in the literature.
- A more generalized paramtrization like $\xi = \xi_0 \rho^s$ leads to autonomous dynamical system of equation only for particular s-values (i.e. $s = 1/2$).

Observation

Because of the homogenity an isotropy of the FLRW unverse there is no shear but only a bulk viscosity.

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A new parametrization for the bulk viscosity

A new Parametrization for Bulk Viscosity Cosmology as Extension of the ΛCDM Model (Eur. Phys. J. Plus (2023) 101351)

$$\xi \equiv \xi_0 H^{1-2s} H_0^{2s} \left(\frac{\rho_m}{\rho_{m,0}}\right)^s = \hat{\xi}_0 H \Omega_m^s,$$

$$\hat{\xi}_0 \equiv \frac{\xi_0}{\Omega_{m,0}^s};\ \Omega_m \equiv \frac{\rho_m}{3H^2}.$$

Main Advantages of the parametrization

This parametrization has the main advantage to drive to an autonomous dynamical system of equations for any arbitrary.
Another remarkable property is that it consistently handle the viscosity effects turning them off when matter density vanishes.

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The model

Cosmological time scenarios

- We consider a universe composed by radiation ($\omega = 1/3$), (dark) matter ($\omega = 0$) and CC ($\omega = -1$).
- Dark Matter is described by a viscous fluid.

Note

Even when we are interested in recent and late times, the inclusion of radiation plays an important role as we will see later.

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Dynamical system analysis

Phase space variables

$$\Omega_r \equiv \frac{\rho_r}{3H^2}, \quad \Omega_m \equiv \frac{\rho_m}{3H^2}, \quad \text{y} \quad \Omega_\Lambda \equiv \frac{\Lambda}{3H^2}.$$

Dynamical system equations with viscosity

$$H' = \frac{H}{2}\left(-3 + 3\hat{\xi}_0\Omega_m^s - \Omega_r + 3\Omega_\Lambda\right),$$

$$\Omega_r' = \Omega_r\left(-1 - 3\hat{\xi}_0\Omega_m^s + \Omega_r - 3\Omega_\Lambda\right),$$

$$\Omega_m' = 3\hat{\xi}_0\Omega_m^s\left(1 - \Omega_m\right) + \Omega_m\left(\Omega_r - 3\Omega_\Lambda\right),$$

$$\Omega_\Lambda' = \Omega_\Lambda\left(-3\left(-1 + \Omega_\Lambda + \hat{\xi}_0\Omega_m^s\right) + \Omega_r\right).$$

Malkin's theorem of nonlinear stability

We have applied shown that the the stability properties of the fixed points hold beyond the usual linear Dynamical System analysis.

To be considered

- The dynamical equation for Ω_Λ is for completeness sake included because it it implicitly given from the first Friedmann's equation $\Omega_r + \Omega_m + \Omega_\Lambda = 1$.
- $'$ denotes a derivative with respect to $N = \ln(a)$.
- It turns out that the EoS becomes an effective barotropic coefficient $\omega_{\text{eff}} = -\frac{2H'}{3H} - 1$.
- Because of thermodynamical arguments $\hat{\xi}_0 > 0$, while from physical arguments it should hold that $\hat{\xi}_0 < 1$.
- We use some particular values for the exponent s of the bulk viscosity -2, $-3/2$, -1, $-1/2$, 0, $1/2$, 1, $3/2$ y 2.

An important result

For negative integer s-values and zero, i.e. -2, -1 y 0, all the critical points are characterized by $\Omega_r = 0$, and therefore they must be discarded as they don't describe a complete universe time evolution.

s	Ω_r	Ω_m	Ω_Λ	$\omega_{\rm eff}$	Stability
$-3/2$	0	1	0	$\hat{\xi}_0$	saddle*
	0	1	0	$-\hat{\xi}_0$	saddle
	0	$\hat{\xi}_0^{2/5}$	$1-\hat{\xi}_0^{2/5}$	-1	attractor
	$1-3^{2/5}\hat{\xi}_0^{2/5}$	$3^{2/5}\hat{\xi}_0^{2/5}$	0	$1/3$	repeller*
$-1/2$	0	1	0	$\hat{\xi}_0$	saddle*
	0	1	0	$-\hat{\xi}_0$	saddle
	0	$\hat{\xi}_0^{2/3}$	$1-\hat{\xi}_0^{2/3}$	-1	attractor
	$1-3^{2/3}\hat{\xi}_0^{2/3}$	$3^{2/3}\hat{\xi}_0^{2/3}$	0	$1/3$	repeller*
$1/2$	0	1	0	$\hat{\xi}_0$	saddle*
	0	1	0	$-\hat{\xi}_0$	saddle
	$1-(3\hat{\xi}_0)^2$	$(3\hat{\xi}_0)^2$	0	$1/3$	repeller*
	0	$\hat{\xi}_0^2$	$1-\hat{\xi}_0^2$	-1	attractor

TABLA: Critical points for the autonomous dynamical system and its stability properties as a function of the exponent parameter s of the bulk viscosity: $-3/2$, $-1/2$ y $1/2$. Only cases for which $0 < \hat{\xi}_0 < 1$ or $*0 < \hat{\xi}_0 < 1/3$ are shown.

s	Ω_r	Ω_m	Ω_Λ	$\omega_{\rm eff}$	Stability
1	0	1	0	$-\hat{\xi}_0$	saddle
	1	0	0	$1/3$	repeller
	0	0	1	-1	attractor
3/2	0	1	0	$\hat{\xi}_0$	saddle*
	1	0	0	$1/3$	repeller
	0	0	1	-1	attractor
	0	1	0	$-\hat{\xi}_0$	saddle
	$1-(3\hat{\xi}_0)^{-2}$	$(3\hat{\xi}_0)^{-2}$	0	$1/3$	repeller*
	0	$\hat{\xi}_0^{-2}$	$1-\hat{\xi}_0^{-2}$	-1	attractor
2	0	1	0	$-\hat{\xi}_0$	saddle
	1	0	0	$1/3$	repeller
	0	0	1	-1	attractor
	0	$\hat{\xi}_0^{-1}$	$1-\hat{\xi}_0^{-1}$	-1	attractor

TABLA: Critical points and their stability properties for the dynamical system for integer and half-integer s-values: 1, 3/2 y 2. We use the physical intervals $0 < \hat{\xi}_0 < 1$ or $*0 < \hat{\xi}_0 < 1/3$ to illustrate.

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Cosmological constraints

Search or the best fit

- The best-value fit for the parameters of the viscous ΛCDM as well as their confidence intervals are found with the "affine-invariant Markov Chain Monte Carlo" (MCMC)[a], using the code *emcee*[b].
- Using as priors the output obtained by dynamical system analysis we restrict the late universe period where $\Omega_r = 0$.
- As catalog for SNe Ia we used Pantheon, with 1048 points in the range $0.01 \leq z \leq 2.3$[c].
- While for OHD we have used the Magaña et al. compilation, which has 51 points within the range $0.07 \leq z \leq 2.36$[d].

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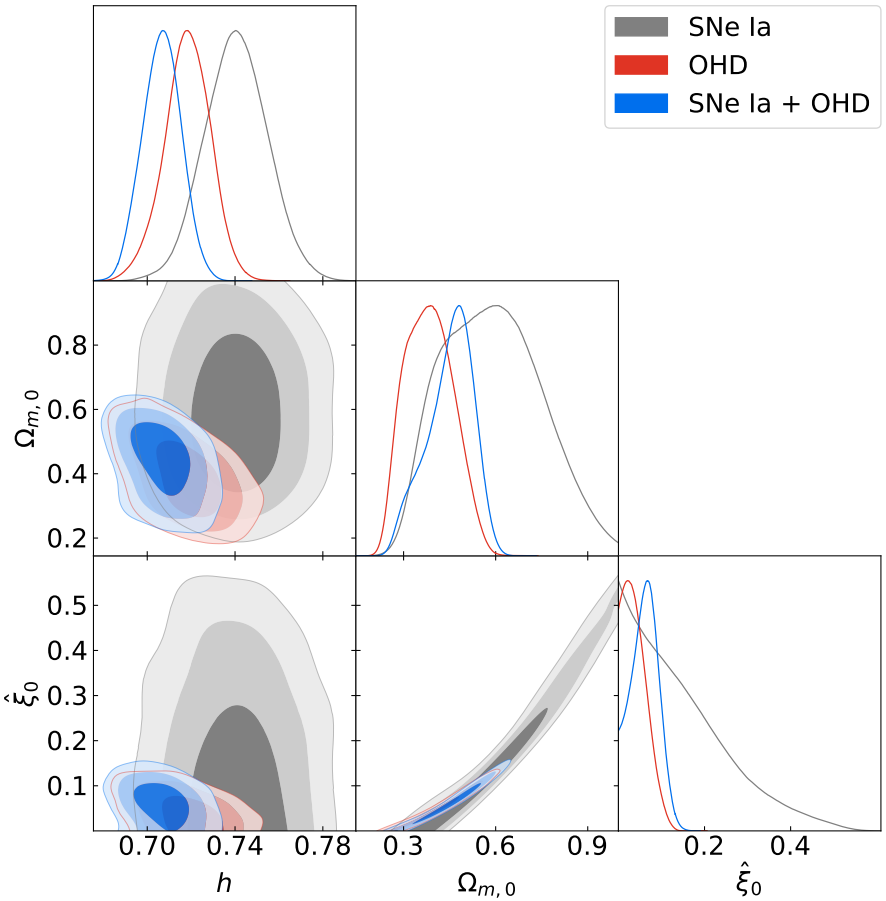
[a] J. Goodman et al., Comm. App. Math. Comp. Sci. **5**, 65 (2010).
[b] D. Foreman-Mackey et al., Publ. Astron. Soc. Pac. **125**, 306 (2013).
[c] M. Scolnic et al. (Pan-STARRS1), Astrophys. J. **859**, 101 (2018).
[d] J. Magaña et al., Mon. Not. Roy. Astron. Soc. **476**, 1036 (2018).

Priors

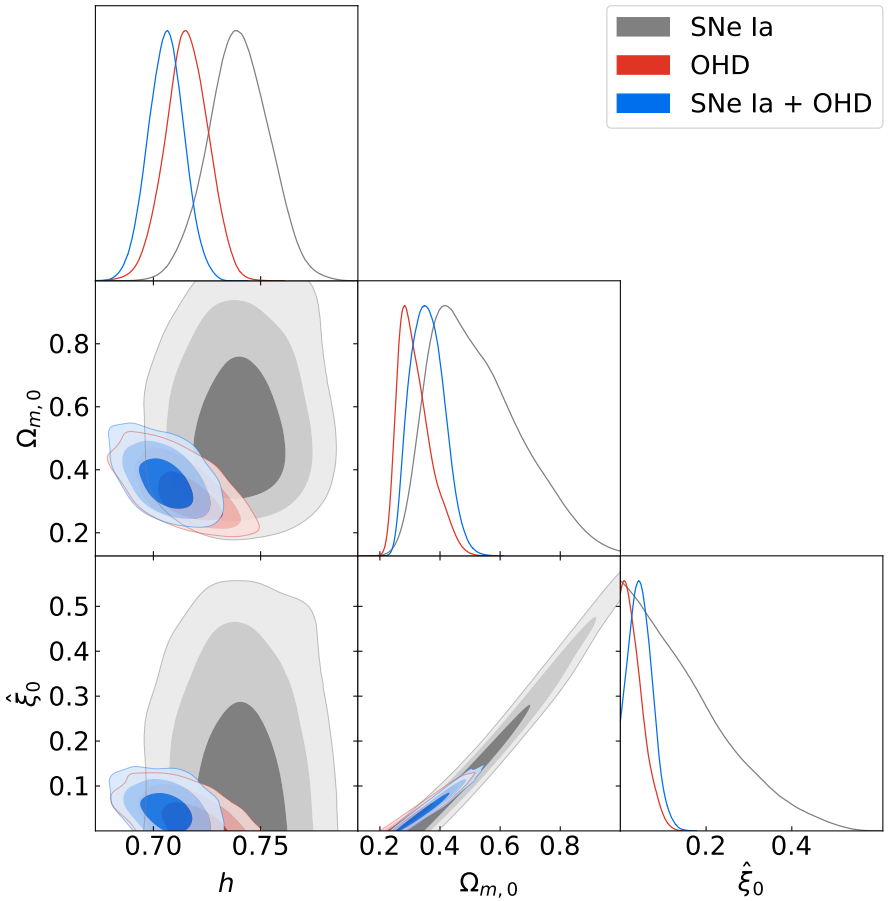
Independent of the model the following *priors* are used:
$h \in G(0.7403, 0.0142)$, $\Omega_{m,0} \in F(0,1)$ y $\hat{\xi}_0 \in F(0,1)$, con $H_0 = 100\frac{km/s}{Mpc}h$.

Model	Best values fit			$\chi^2_{\rm min}$
	h	$\Omega_{m,0}$	$\hat{\xi}_0$	
ΛCDM	$0.711^{+0.008}_{-0.008}$	$0.258^{+0.012}_{-0.011}$	$\cdots$	1058.4
$s=-3/2$	$0.707^{+0.009}_{-0.009}$	$0.460^{+0.064}_{-0.095}$	$0.062^{+0.031}_{-0.035}$	1055.0
$s=-1/2$	$0.706^{+0.008}_{-0.009}$	$0.355^{+0.058}_{-0.054}$	$0.046^{+0.029}_{-0.026}$	1055.2
$s=1/2$	$0.706^{+0.009}_{-0.008}$	$0.309^{+0.037}_{-0.032}$	$0.044^{+0.029}_{-0.026}$	1055.2
$s=1$	$0.706^{+0.008}_{-0.009}$	$0.298^{+0.031}_{-0.027}$	$0.045^{+0.030}_{-0.025}$	1055.2
$s=3/2$	$0.706^{+0.008}_{-0.009}$	$0.290^{+0.027}_{-0.022}$	$0.047^{+0.031}_{-0.027}$	1055.2
$s=2$	$0.706^{+0.009}_{-0.008}$	$0.285^{+0.022}_{-0.020}$	$0.051^{+0.033}_{-0.030}$	1055.1

Tabla: Best value fit for the viscous ΛCDM model and viscous exponent $s = -3/2, -1/2, 1/2, 1, 3/2$ y 2. The uncertainty corresponds to $1\sigma(68.3\,\%)$ of confidence level. We display data obtained from observational data from SNe Ia+OHD to illustrative purposes.

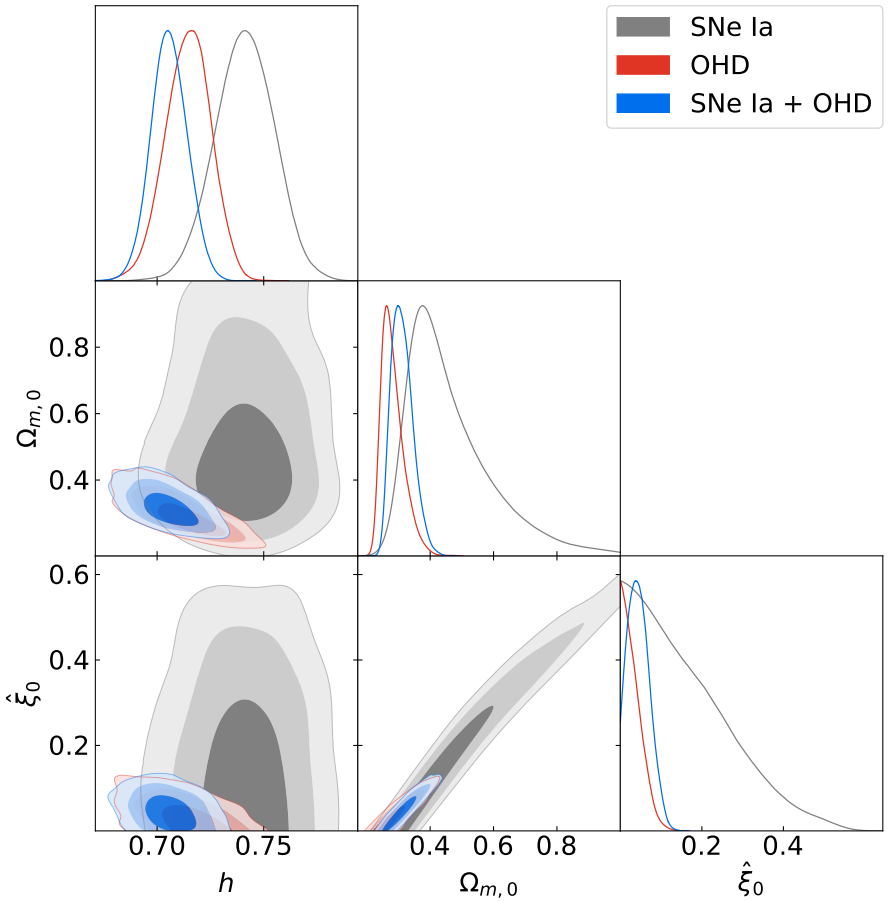


(a) Viscous model for $s = -3/2$.

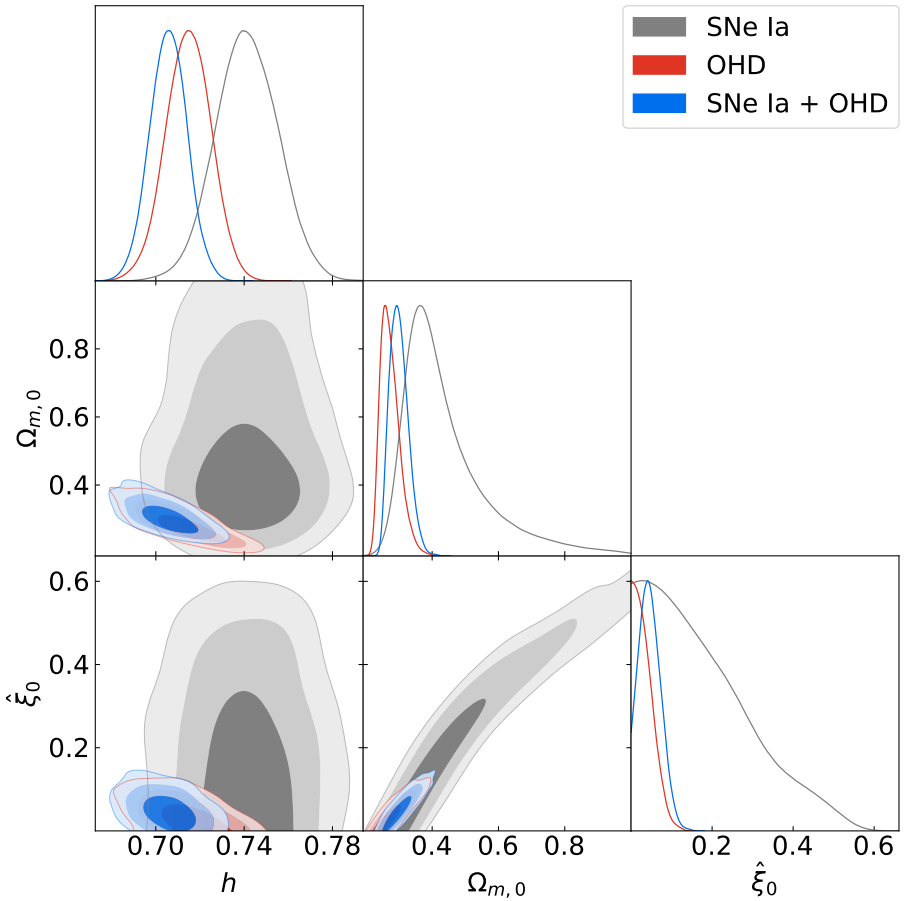


(b) Viscous model for $s = -1/2$.

FIGURA: Common and marginalized regions for the parameters h, $\Omega_{m,0}$ and $\hat{\xi}_0$. The suited regions correspond to $1\sigma(68.3\,\%)$, $2\sigma(95.5\,\%)$ y $3\sigma(99.7\,\%)$ of the confidence level respectively.

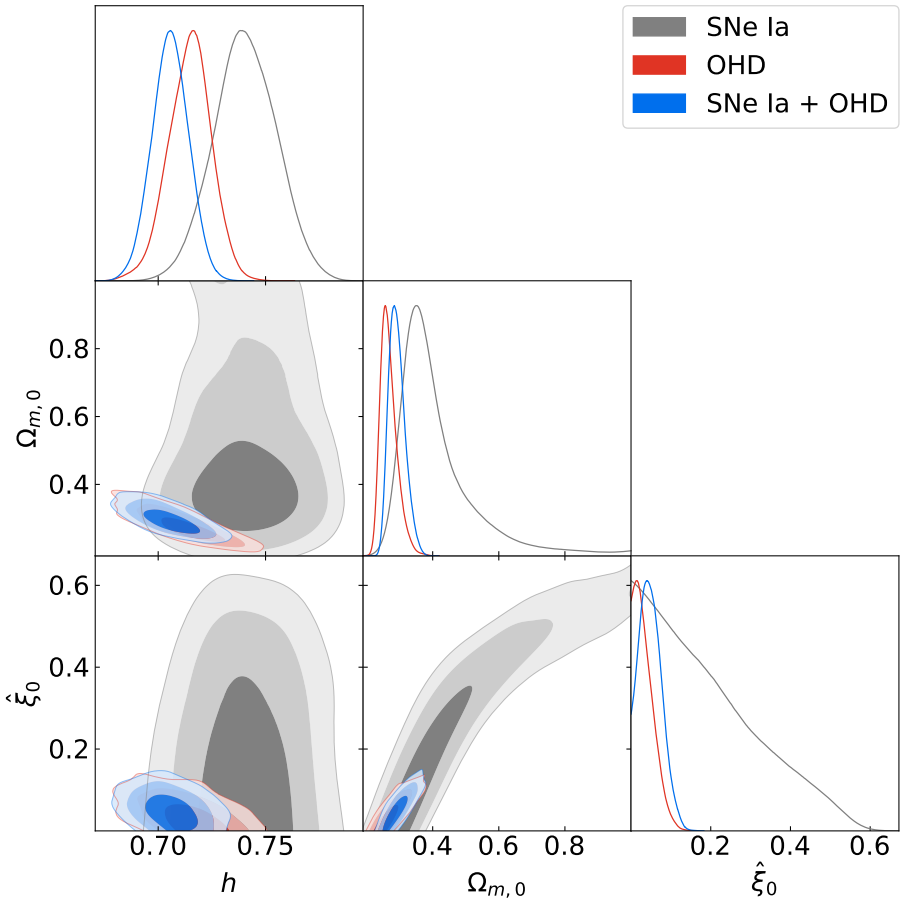


(a) Viscous model for $s = 1/2$.

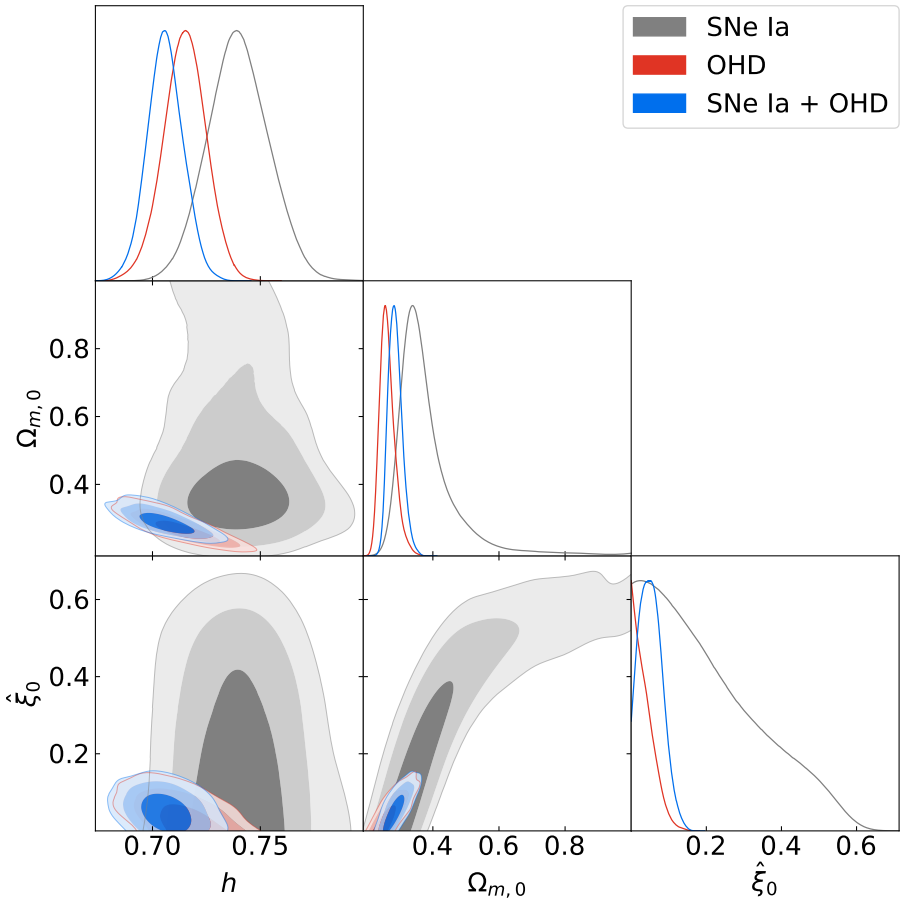


(b) Viscous model for $s = 1$.

FIGURA: Common and marginalized regions for the parameters h, $\Omega_{m,0}$ and $\hat{\xi}_0$. The suited regions correspond to $1\sigma(68.3\,\%)$, $2\sigma(95.5\,\%)$ y $3\sigma(99.7\,\%)$ of the confidence level respectively.



(a) Viscous model for $s = 3/2$.



(b) Viscous model for $s = 2$.

FIGURA: Common and marginalized regions for the parameters h, $\Omega_{m,0}$ and $\hat{\xi}_0$. The suited regions correspond to $1\sigma(68.3\,\%)$, $2\sigma(95.5\,\%)$ y $3\sigma(99.7\,\%)$ of the confidence level respectively.

Running vacuum energy density

An expanding universe is not expected to have a static vacuum energy density. We will use a two-parameter model inspired by a phenomenological application of Renormalization Group analysis, whose cosmological consequences has been studied[a]. The vacuum energy density can be written as

$$\rho_{\rm vac}(H) = \frac{3}{8\pi G_N}\left(c_0 + \nu H^2 + \tilde{\nu}\dot{H}\right) + \mathcal{O}(H^4), \qquad (1)$$

[a] Solà, J. and Gómez-Valent, A., Int. J. Mod. Phys D**24** 1541003(2015), 151.03832

From the different possibilities for the coefficients appearing in $\rho_{\rm vac}(H, \dot{H})$, we classified them into 4 models.

Label	Class of model	Bulk viscosity exponent s
Model 1	First class $\tilde{\nu} = 0$	$1/2$
Model 2	Second class $\tilde{\nu} = \nu/2$	1
Model 3	Second class $\tilde{\nu} = \nu/2$	$1/2$
Model 4	Second class $\tilde{\nu} = \nu/2$	0

Dynamical system equations for Model 1

We will concentrate for the sake of simplicity on model 1. In terms of the phase space variables,

$$\Omega_r \equiv \frac{\rho_r}{3H^2}, \quad \Omega_m \equiv \frac{\rho_m}{3H^2}, \quad \text{y} \quad \Omega_\Lambda \equiv \frac{\rho_{vac}(H)}{3H^2}.$$

The dynamical system equations are:

$$\begin{aligned}
\Omega_r' &= \Omega_r(-1+4\nu-3\hat{\xi}_0\Omega_m^s+\Omega_r-3\Omega_{\rm vac}),\\
\Omega_m' &= -3(-1+\nu)\hat{\xi}_0\Omega_m^s-3\hat{\xi}_0\Omega_m^{1+s}+\Omega_m(3\nu+\Omega_r-3\Omega_{\rm vac}),\\
\Omega_{\rm vac}' &= -3\nu\Omega_m+3\hat{\xi}_0\Omega_m^s(\nu-\Omega_{\rm vac})-3(-1+\Omega_{\rm vac})\Omega_{\rm vac}+\Omega_r(-4\nu+\Omega_{\rm vac})
\end{aligned} \tag{2}$$

The structure of the fixed points and their stability properties is illustrated in the next page.

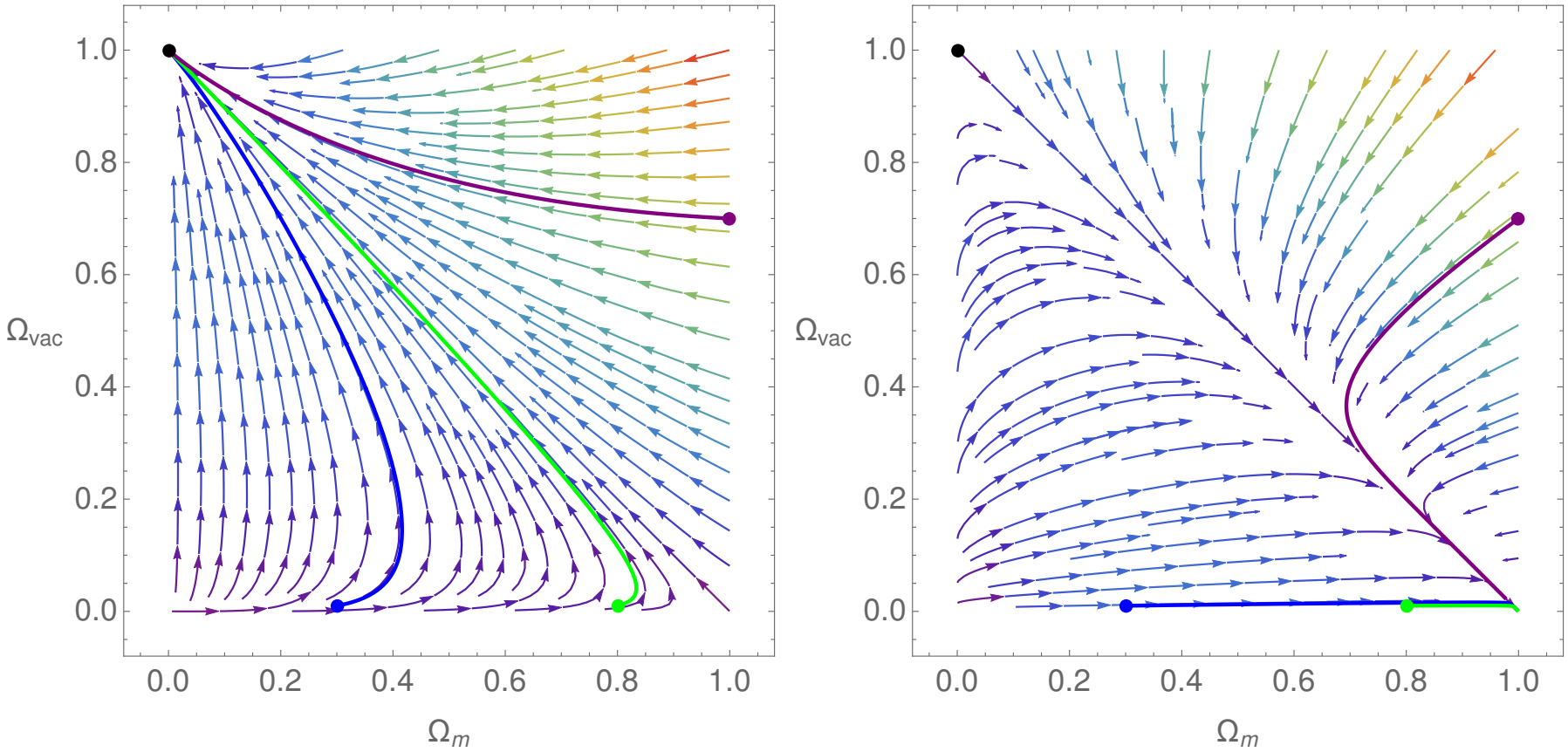
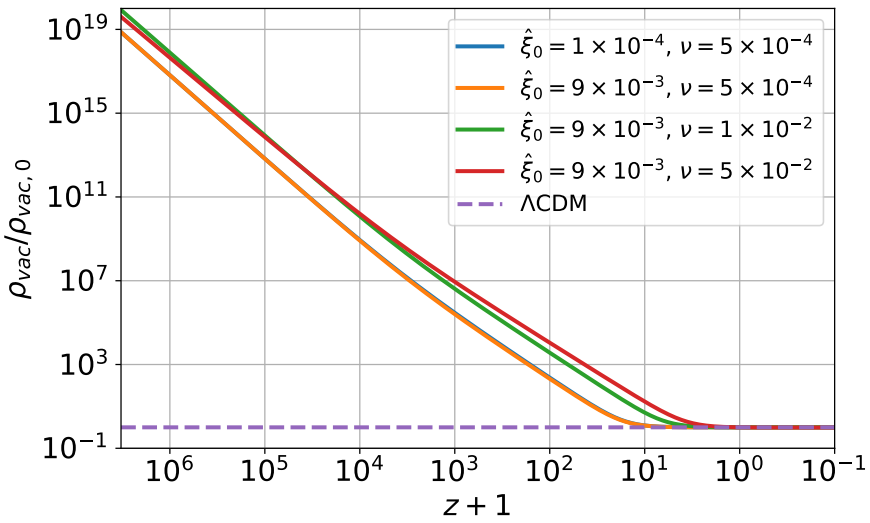
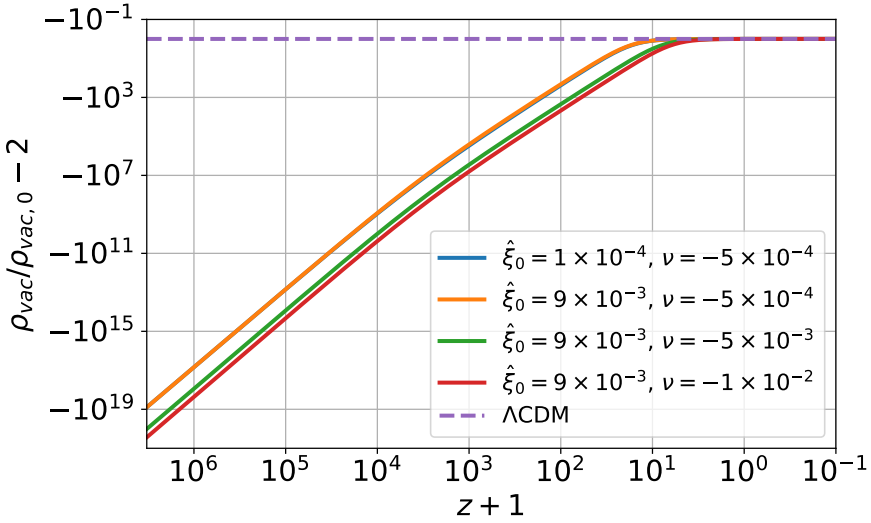


FIGURA: For the extended model including viscosity and running, the phase space of the system along with some trajectories for different initial conditions: $(\Omega_m^{(i)} = 0.3,\ \Omega_{\rm vac}^{(i)} = 0.01)$; $(\Omega_m^{(i)} = 0.8,\ \Omega_{\rm vac}^{(i)} = 0.01)$; $(\Omega_m^{(i)} = 1,\ \Omega_{\rm vac}^{(i)} = 0.7)$, for blue, green and purple curves, respectively. Left panel evidences the attractor character of the system after evolving with $\nu = \hat{\xi}_0 = 10^{-4}$ while right panel shows the saddle-like behavior for $\nu = 10^{-4}$ and $\hat{\xi}_0 = 1.05$.



(a)



(b)

FIGURA: Plots of vacuum energy density $\rho_{\rm vac}$ for **Model 1**, normalized with respect to their current value $\rho_{\rm vac,0}$, as a function of redshift z. Positive and negative ν-values are respectively considered in **(a)** and **(b)**, for different $\hat{\xi}_0$-values. Notice that we have plotted $\rho_{\rm vac}/\rho_{\rm vac,0} - 2$ in order to obtain a better representation in the symmetrical logarithm scale. As a reference, we have used $\Lambda = 4.24 \pm 0.11 \times 10^{-66}$ eV^2 to compute the present vacuum energy density $\rho_{\rm vac,0}$.

From this figures we can see how the bulk viscosity and the running vacuum affect the evolution of the vacuum energy density. It follows that an increment in $\hat{\xi}_0$ does not appreciably affect the evolution of $\rho_{\rm vac}$, contrary to what happens when we increment the values of $|\nu|$. This behaviour is due to the fact that the bulk viscosity affects the evolution of $\rho_{\rm vac}$ indirectly through the Hubble parameter and, therefore, it is necessary a remarkably difference in the evolution of H that does not appear due to the small values of $\hat{\xi}_0$. On the other hand, depending on the sign of ν, it is possible to obtain an always positive vacuum energy density when $\nu > 0$ or a vacuum energy density that experiences a transition between positive to negative values when $\nu < 0$. this transition occurs when

$$H = H_0 \sqrt{\frac{\Omega_{\rm vac,0} + |\nu|}{|\nu|}}, \tag{3}$$

and therefore, considering that $H = H(z)$, the redshift at which this change of sign occur depends strongly on the values of ν.

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Dynamical system analysis and cosmological constraints: a combined strategy

- The viscous model presented can fit quite well the observational data from SNe Ia and OHD. Nevertheless, it is only due to the dynamical system analysis that allows to guaranty that a particular model can indeed describe the whole cosmological evolution of the universe.
- Moreover, the dynamical system analysis shows the existence of two family of one-parameter solutions. One corresponds to the ΛCDM model, and the other one has the structure associated to the radiation dominated era of the form $\Omega_r = 1 - (3\hat{\xi}_0)^{\frac{1}{(1-s)}}$ y $\Omega_m = (3\hat{\xi}_0)^{\frac{1}{(1-s)}}$; while the shape of the CC dominated era behaves as $\Omega_m = \hat{\xi}_0^{\frac{1}{(1-s)}}$ and $\Omega_\Lambda = 1 - \hat{\xi}_0^{\frac{1}{(1-s)}}$.
- We note that the viscous model is very similar to the ΛCDM for large s values, which can be seen by considering for instance the particular case $s = 2$.

- The inclusion of both, a running vacuum energy density and a dissipative DM component enrich the whole physical scenario by adding non-trivial critical points absent in their non-running and non-viscous counterparts.
- For example, the inclusion of bulk viscosity can drive the current acceleration expansion of the Universe and provide a new phantom-like behaviour, depending on the values of $\hat{\xi}_0$ and ν.
- As an overall and encouraging conclusion from the dynamical system analysis is that we have obtained new critical points of cosmological relevance that arise from the combined effects of a running vacuum energy density and dissipative DM component. This is appealing to the light of current tensions permeated in the ΛCDM model, because of the potentiality to successfully describe the current observations due to an enlarged parameter phase space characterizing these extended models.
- Further constraints on the model parameters can be obtained by means of high-quality observational constraints by using the derived parameter regions from the dynamical system analysis as priors for the statistical treatment. This task will be addressed in future works.