Rethinking Stability

Verónica Errasti Díez vero.erdi@origins-cluster.de

Ludwig-Maximilians-Universität (LMU)
Excellence Cluster ORIGINS

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Outline

1. Motivation

2. Unstable constraints

3. Unstable equations of motion

4. Hippie twist

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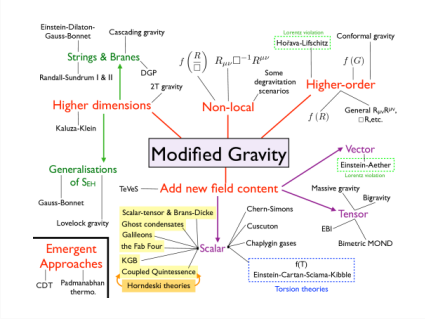
We need and construct new field theories

- General Relativity $\rightarrow$ **Modified Gravity**



[Bull et al. 1512.05356]

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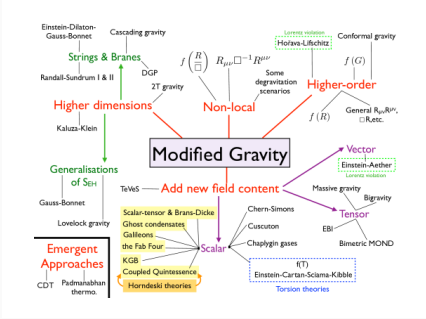

[Bull et al. 1512.05356]

- **Other examples:**
  - Standard Model → Beyond Standard Model [Langacker Ed.2, CRC Press (2017)]
  - Quantum Electrodyanamics → Non-linear electrodynamics

ModMax [Bandos et al. 2007.09092]

Related to LUXE [Abramowicz et al. 2102.02032]

The naive construction prescription is unreliable

- **Choose:** spacetime dimension, matter content and symmetries
- **Write:** most general Lagrangian upto a certain mass/length dimension

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- **Example** [Éboli & González-García 1604.03555]
  - $d = 4$, $(A_\mu, W_\mu^+, W_\mu^-, Z_\mu)$, $[M] = 6$
  - "we find 5 Lorentz structures without derivatives and 70 with two derivatives"
  - $\sim Z_\mu Z^\nu Y^{\rho\mu} Y_{\rho\nu}$, where $Y_{\mu\nu} = \partial_\mu Z_\nu + \partial_\nu Z_\mu$

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↳ Part of Generalized Proca [Tasinato 1402.6450] [Heisenberg 1402.7026] [Beltrán-Jiménez & Heisenberg 1610.08960]

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Non-linear interactions → Boulware-Deser Instabilities (1/2)

$$\underbrace{S = \int_{\mathcal{M}} d^d x \, \mathcal{L}}_{d \text{ finite, } \partial\mathcal{M}=\emptyset} \qquad \underbrace{\mathcal{L} = \mathcal{L}(Q^A, \partial_\mu Q^A)}_{\text{first-order, local}} \qquad \underbrace{A = 1, \ldots, N}_{N \text{ finite}}$$

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EoMs $\longrightarrow$ **Dynamical problem**

$$\left\{ \begin{array}{rcl} E_A := W_{AB}\ddot{Q}^B + U_A & \stackrel{!}{=} & 0 \\ E_I^{(2)} \equiv d_t\phi_I := \quad d_t(V_I^A U_A) & \stackrel{!}{=} & 0 \\ \vdots & & \\ E_R^{(n)} \equiv \quad d_t\phi_R^{(n-1)} & \stackrel{!}{=} & 0 \end{array} \right\} \longrightarrow \left\{ \begin{array}{c} \mathbb{E} := (\mathsf{V}_0^{(n_\mathrm{f})\downarrow})^\perp \cdot \mathsf{E}^{(n_\mathrm{f})\downarrow} = 0 \\ 2N_{\mathrm{DoF}} \quad \text{Initial Conditions} \end{array} \right\}$$

[VED et al. 2310.12218]

Non-linear interactions $\rightarrow$ Boulware-Deser Instabilities (2/2)

- $N_{\mathrm{Dof}} = N - \frac{1}{2}(L + g + e)$ [Díaz et. al. 1406.1156] [Díaz & Montesinos 1710.01371]
  - g, e related to gauge-like symmetries [Banerjee et al. 9906072 & 9907217]
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  - DBIs result in insufficient (functionally independent Lagrangian) constraints L
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- It can be shown that such extra DoF's are associated with $E_{\mathrm{kin}} < 0$
  - I will call negative kinetic energy modes **ghosts**

Example: Multi-field Generalized Proca

$$\mathcal{L} = -\frac{1}{4}F^{\alpha}_{\mu\nu}F^{\mu\nu}_{\alpha} + f(A^{\alpha}_{\mu}, F^{\alpha}_{\mu\nu}) + \sum_{n=1}^{4} \mathcal{T}^{\mu_1\ldots\mu_n\nu_1\ldots\nu_n}_{\alpha_1\ldots\alpha_n}(A^{\alpha}_{\mu})\partial_{\mu_1}A^{\alpha_1}_{\nu_1}\ldots\partial_{\mu_n}A^{\alpha_n}_{\nu_n} \qquad \alpha = 1, \ldots, P$$

[Allys et al. 1609.05870] [Beltrán-Jiménez & Heisenberg 1610.08960]

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For $N_{\rm dof} = N - \frac{L}{2} = 3P$ **it is necessary that** (perhaps not necessary **and** sufficient)

$$\begin{cases} \dfrac{\partial^2\mathcal{L}}{\partial\dot{A}^{\alpha}_0\partial\dot{A}^{\beta}_{\mu}} = 0 \quad \text{and} \quad \det\left(\dfrac{\partial^2\mathcal{L}}{\partial\dot{A}^{\alpha}_i\partial\dot{A}^{\beta}_j}\right) \neq 0 \\ 2\dfrac{\partial^2\mathcal{L}}{\partial\dot{A}^{[\alpha}_0\partial\dot{A}^{\beta]}_0} + \partial_i\left(\dfrac{\partial^2\mathcal{L}}{\partial\partial_i A^{\alpha}_0\partial\dot{A}^{\beta}_0}\right) = 0 \end{cases}$$

[VED et al. 1905.06967 & 1905.06968] [Janaun & Vanichchapongjaroen 2303.15261]

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Higher-order terms $\rightarrow$ Ostrogradsky instabilities

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A. Lower the order via Lagrange multipliers [Crisostomi et al. 1703.01623]

B. Ostrogradsky prescription [Delhom et al. 2207.13431]

C. Work directly with M-order [Skinner & Rusk 1983] [Batlle et al. 1988] [Gràcia et al. 1991]

- Most non-trivial **ghost** killing [Langlois & Noui 1510.06930] [Achour et al. 1608.08135]

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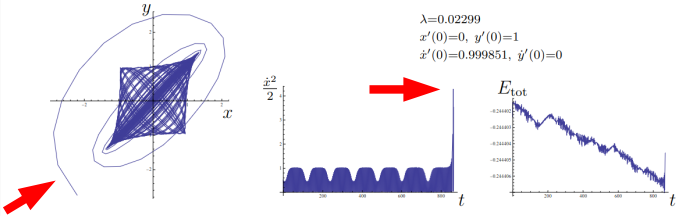
What is the problem with ghosts $\equiv E_{\mathbf{kin}} < 0$ modes?

- **Classically**, their trajectories tend to diverge



[Pavšič 1302.5257]

- **In the quantum regime**, unitarity tends to break down

Quantum detour: the second-order harmonic oscillator case

- Breaking of unitarity [Stelle 1977]

$$\alpha_+|\psi_0\rangle = 0 = \alpha_-^\dagger|\psi_0\rangle \quad \to \psi_0 \text{ NOT normalizable}$$

$$|N_+, N_-\rangle \propto (\alpha_+^\dagger)^{N_+}(\alpha_-)^{N_-}|\psi_0\rangle \quad \text{span a "formal" space of states}$$

$$H|N_+, N_-\rangle \propto (N_+ k_+ + N_- k_-)\,|N_+, N_-\rangle \quad \text{but}$$

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- Embrace negative energies [Woodard 1506.02210]

$$\alpha_+|\psi\rangle = 0 = \alpha_-|\psi\rangle \quad \to \psi \text{ normalizable}$$

$$|N_+, N_-\rangle \propto (\alpha_+^\dagger)^{N_+}(\alpha_-^\dagger)^{N_-}|\psi\rangle \quad \text{complete set of normalized states}$$

$$H|N_+, N_-\rangle \propto (N_+ k_+ - N_- k_-)\,|N_+, N_-\rangle$$

Trickle of counterexamples

- **Pais-Uhlenbeck oscillator** [Pais & Uhlenbeck 1950] [Pavšič 1607.06589]

  Second-order harmonic oscillator – containts an OI

  2 harmonic oscillators, one with the wrong sign in its kinetic term – manifest BDI

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- **Interacting extensions and thus inspired beyond scenarios**

  Weak stability [Smilga (et al.) 0407231, 0808.0139, 1306.6066, 2110.11175]

  Lagrange stability [Kaparulin et. al. 1407.8481], [Deffayet et.al. 2305.09631]

  A bit of everything [Fring & Turner 2301.11317]

Lagrange stability: why does energy usually work?

- Let $f : \mathbb{R}^n \to \mathbb{R}$ be $\mathcal{C}^2$ with $\nabla f(x^*) = 0$ for some $x^* \in \mathbb{R}^n$
- Let $\nabla^2 f(x)$ be positive definte $\forall x \in \mathbb{R}^n$

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- Let $x, y \in \mathbb{R}^n$ and $t \in [0, 1]$
- Taylor: $f(y) = f(x) + \nabla f(x)^T (y - x) + \frac{1}{2}(y - x)^T \nabla^2 f(x + t(y - x))(y - x)$

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- f conserved quantity $\equiv$ its curves are inside level sets
- f has bounded curves **QED**

Lagrange stability: when do ghosts work?

- Quick answer: **Integrable systems**
- In this case, f before is not the energy
  f is some linear combination of the energy and additional conserved quantities
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- Beyond this framework
  - Weak stability The Smilga school
  - Asymptotically "sufficiently close" to working ghosts examples
    [Deffayet et.al. 2305.09631]

¡Muchas gracias por vuestra atención!

Take homes

- We remain in pursuit of physically viable new theories (MG, BSM...)
- Main challenge: evasion of instabilities (Boulware-Deser, Ostrogradsky)

Or is it?

- Recent examples strongly suggest instability avoidance is too strong a criterion
  Negative energy modes, but:
  - Bounded or only at infinity diverging solutions
  - No break down of unitarity at the quantum level
- Can we generalize those examples to more realistic scenarios?
  - Challenge accepted! [Work in progress with J. Gaset Rifà]