3rd Workshop on Current Challenges in Cosmology

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Bucaramanga, Colombia

Quadratic metric-affine gravity and stability of the vector sector

Alejandro Jiménez Cano
(Results from collaboration with **Fco. José Maldonado Torralba**)

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Structure of this presentation

1. The metric affine framework (yes... this thing again...)
2. Instabilities
3. The vector sector of quadratic MAG gravity
4. The case of quadratic Weyl-Cartan gravity
5. Final words

1. The metric affine framework (yes... this thing again...)

Geometric gravity (Einstein 1915) $\rightsquigarrow$ Spacetime is modelled as a *differentiable manifold* $\mathcal{M}$.

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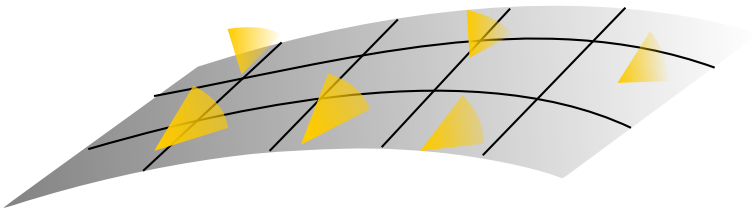
Geometric structures

- *Metric tensor:* $g_{\mu\nu}$
  - Measuring (length, volume...)

$$s[\gamma](\sigma) = \int_0^\sigma \sqrt{|g_{\mu\nu}(\sigma')\dot{x}^\mu(\sigma')\dot{x}^\nu(\sigma')|}\, \mathrm{d}\sigma' \,. \tag{1.1}$$

$$\mathrm{vol}(\mathcal{U}) = \int_{\mathcal{U}} \mathbf{vol}\,, \qquad \mathbf{vol} \coloneqq \sqrt{|g|}\, \mathrm{d}x^1 \wedge ... \wedge \mathrm{d}x^{\mathtt{D}} \qquad \mathtt{D} \coloneqq \dim(\mathcal{U}). \tag{1.2}$$

  - Module of a vector (not necessarily non-negative) $\Rightarrow$ light cones $\Rightarrow$ causality.



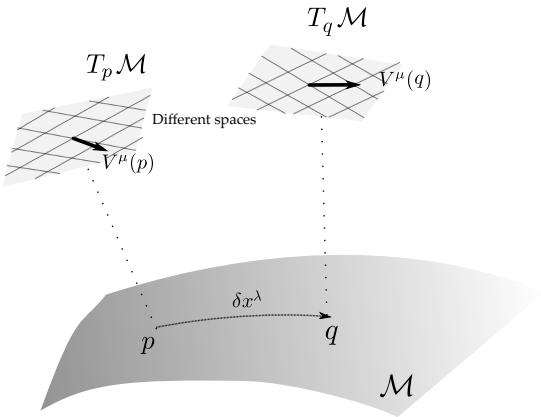
  - Notion of scale (conformal transformations...)

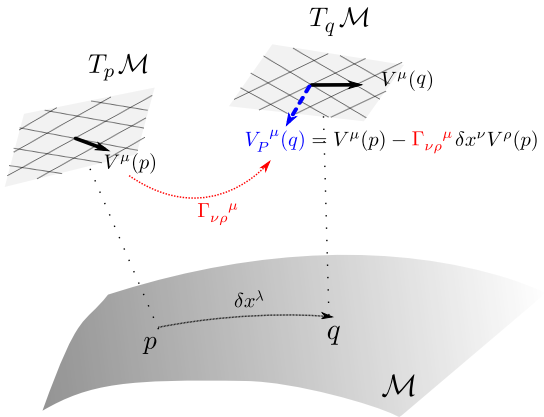
$$g_{\mu\nu} \to \mathrm{e}^{2\Omega} g_{\mu\nu} \,. \tag{1.3}$$

Geometric gravity (Einstein 1915) $\rightsquigarrow$ Spacetime is modelled as a *differentiable manifold* $\mathcal{M}$.

Geometric structures

- *Connection:* $\Gamma_{\mu\nu}{}^{\rho}$
  - Notion of parallel in $\mathcal{M}$ $\Rightarrow$ Covariant derivative ∇_μ





Geometric gravity (Einstein 1915) $\rightsquigarrow$ The spacetime is modelled as a *differentiable manifold* $\mathcal{M}$.

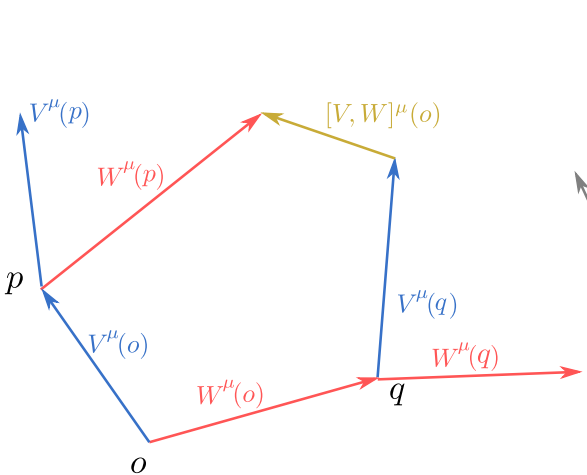
Geometric structures

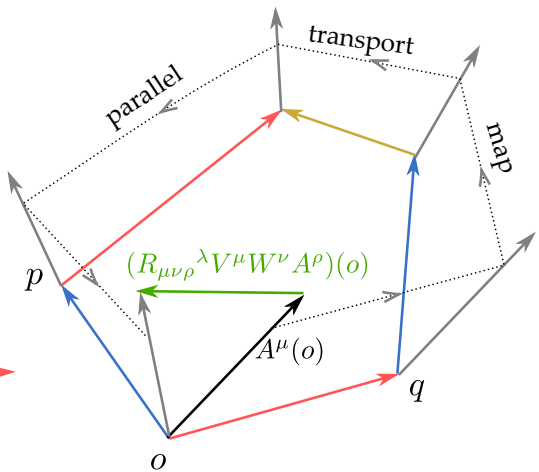
- *Connection:* $\Gamma_{\mu\nu}{}^{\rho}$
  - Geometrical objects:

$$\text{Curvature:} \qquad R_{\mu\nu\lambda}{}^{\rho} := \partial_{\mu}\Gamma_{\nu\lambda}{}^{\rho} - \partial_{\nu}\Gamma_{\mu\lambda}{}^{\rho} + \Gamma_{\mu\sigma}{}^{\rho}\Gamma_{\nu\lambda}{}^{\sigma} - \Gamma_{\nu\sigma}{}^{\rho}\Gamma_{\mu\lambda}{}^{\sigma}\,, \tag{1.4}$$

$$\text{Torsion:} \qquad T_{\mu\nu}{}^{\rho} := \Gamma_{\mu\nu}{}^{\rho} - \Gamma_{\nu\mu}{}^{\rho}\,. \tag{1.5}$$

[Talks by T. Koivisto & J. Beltrán]





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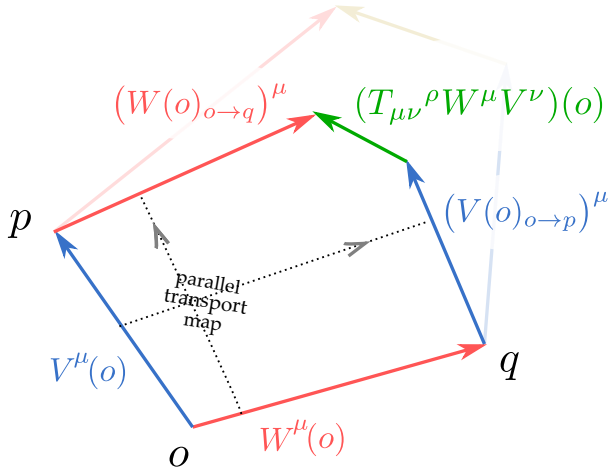
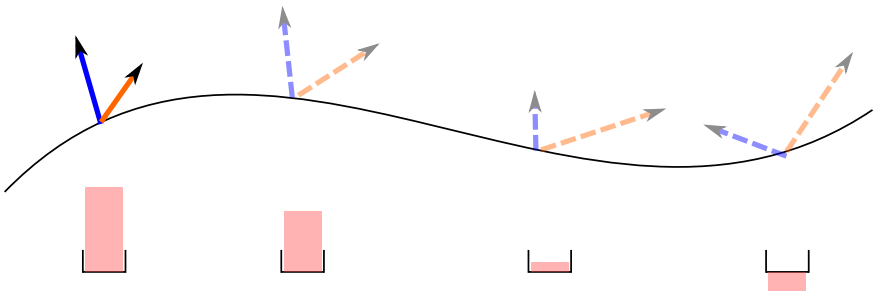


Def.: In the presence of metric and connection we define the *nonmetricity tensor*:

$$Q_{\mu\nu\rho} \coloneqq -\nabla_\mu g_{\nu\rho}\,. \tag{1.6}$$



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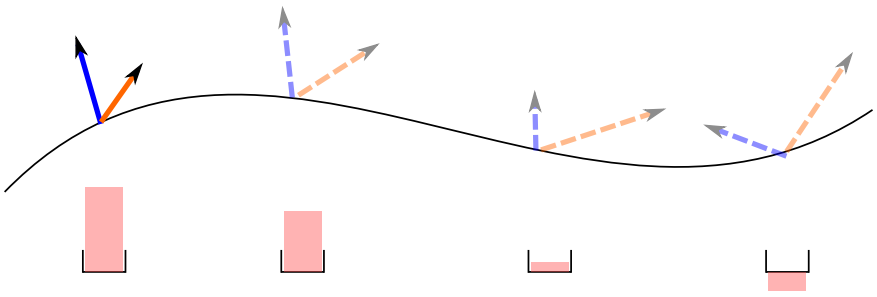


Theorem. Given $g_{\mu\nu}$, there is only one connection that satisfies

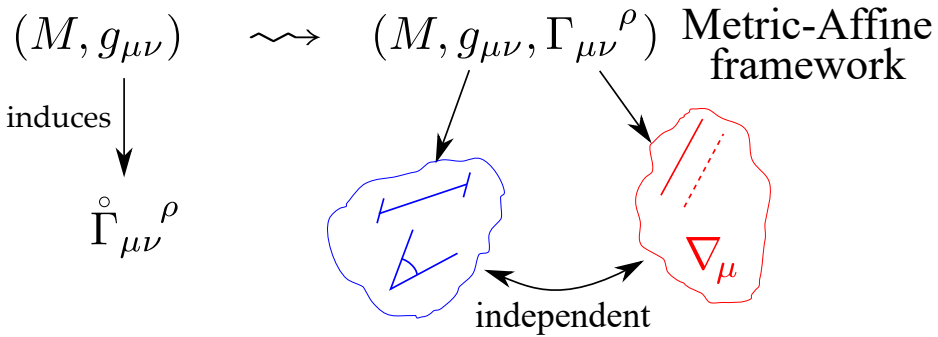
$$T_{\mu\nu}{}^{\rho} = 0 \quad \text{(torsionless condition)}, \tag{1.7}$$

$$Q_{\mu\nu\rho} = 0 \quad \text{(compatibility condition)}, \tag{1.8}$$

the *Levi-Civita connection:*

$$\mathring{\Gamma}_{\mu\nu}{}^{\rho} = \frac{1}{2} g^{\rho\sigma}\left(\partial_\mu g_{\sigma\nu} + \partial_\nu g_{\mu\sigma} - \partial_\sigma g_{\mu\nu}\right). \tag{1.9}$$

Metric-affine theories
Instead of choosing $\mathring{\Gamma}$, they consider the metric and the (general) connection as independent fields.



Γ, g

Curvature

$$R_{\mu\nu\rho}{}^{\lambda} := \partial_{\mu}\Gamma_{\nu\rho}{}^{\lambda} - \partial_{\nu}\Gamma_{\mu\rho}{}^{\lambda} + \Gamma_{\mu\sigma}{}^{\lambda}\Gamma_{\nu\rho}{}^{\sigma} - \Gamma_{\nu\sigma}{}^{\lambda}\Gamma_{\mu\rho}{}^{\sigma}$$

Torsion

$$T_{\mu\nu}{}^{\rho} := \Gamma_{\mu\nu}{}^{\rho} - \Gamma_{\nu\mu}{}^{\rho}$$

Nonmetricity

$$Q_{\mu\nu\rho} := -\nabla_{\mu} g_{\nu\rho}$$

For a given metric, a general connection can be split as follows:

$$\Gamma_{\mu\nu}{}^{\rho} = \mathring{\Gamma}_{\mu\nu}{}^{\rho} + \Xi_{\mu\nu}{}^{\rho}\,, \tag{1.10}$$

where the distorsion $\Xi_{\mu\nu}{}^{\rho}$ is given by

$$\Xi_{\mu\nu}{}^{\rho} := \frac{1}{2}g^{\rho\sigma}(T_{\mu\nu\sigma} + T_{\sigma\mu\nu} - T_{\nu\sigma\mu}) + \frac{1}{2}g^{\rho\sigma}(Q_{\mu\nu\sigma} + Q_{\nu\sigma\mu} - Q_{\sigma\mu\nu})\,. \tag{1.11}$$

New variables: $(g_{\mu\nu}, \Gamma_{\mu\nu}{}^{\rho}) \quad \rightarrow \quad (g_{\mu\nu}, T_{\mu\nu}{}^{\rho}, Q_{\mu\nu\rho})$

Post-riemannian expansion of the curvature

$$R_{\mu\nu\rho}{}^{\lambda} = \mathring{R}_{\mu\nu\rho}{}^{\lambda} + \mathring{\nabla}_{\mu}\Xi_{\nu\rho}{}^{\lambda} - \mathring{\nabla}_{\nu}\Xi_{\mu\rho}{}^{\lambda} + \Xi_{\mu\sigma}{}^{\lambda}\Xi_{\nu\rho}{}^{\sigma} - \Xi_{\nu\sigma}{}^{\lambda}\Xi_{\mu\rho}{}^{\sigma} \tag{1.12}$$

$$= \mathring{R}_{\mu\nu\rho}{}^{\lambda} + (\mathring{\nabla}T) + (\mathring{\nabla}Q) + (TT) + (TQ) + (QQ) \tag{1.13}$$

Teleparallel means $R_{\mu\nu\rho}{}^{\lambda} = 0$ (not necessarily $\mathring{R}_{\mu\nu\rho}{}^{\lambda} = 0$!)

$$T_{\mu\nu}{}^{\rho} := \Gamma_{\mu\nu}{}^{\rho} - \Gamma_{\nu\mu}{}^{\rho}\,, \qquad Q_{\rho\mu\nu} := -\nabla_{\rho} g_{\mu\nu}\,. \tag{1.14}$$

Irreducible parts of the torsion

$$T_{\mu\nu}{}^{\rho} = \underbrace{\frac{1}{3}(T_{\mu}\delta^{\rho}_{\nu} - T_{\nu}\delta^{\rho}_{\mu})}_{\text{trace}} \underbrace{-\frac{1}{3}\mathcal{E}_{\mu\nu}{}^{\rho\lambda}S_{\lambda}}_{\text{axial part}} + \underbrace{\mathsf{t}_{\mu\nu}{}^{\rho}}_{\text{rest}} \tag{1.15}$$

Irreducible parts of the nonmetricity

$$Q_{\mu\nu\rho} = \underbrace{\frac{1}{4}Q_{\mu}g_{\nu\rho}}_{\text{Weyl trace}} + \underbrace{\frac{1}{9}\left[2\Lambda_{\nu}g_{\rho\mu} + 2\Lambda_{\rho}g_{\nu\mu} - \Lambda_{\mu}g_{\nu\rho}\right]}_{\text{remaining trace}} + \underbrace{\underbrace{s_{\mu\nu\rho}}_{\text{tot. sym.}} + \underbrace{\Omega_{\mu\nu\rho}}_{\text{rest}}}_{\mathsf{q}_{\mu\nu\rho}} \tag{1.16}$$

Spin content

	Spin-0 modes	Spin-1 modes	Spin-2 modes	Spin-3 modes
$\mathsf{t}_{\mu\nu}{}^{\rho}$	0	2	2	0
T_μ	1	1	0	0
S_μ	1	1	0	0
$s_{\mu\nu\rho}$	1	1	1	1
$\Omega_{\mu\nu\rho}$	0	2	2	0
Q_μ	1	1	0	0
Λ_μ	1	1	0	0

In some gauge theories of gravity...

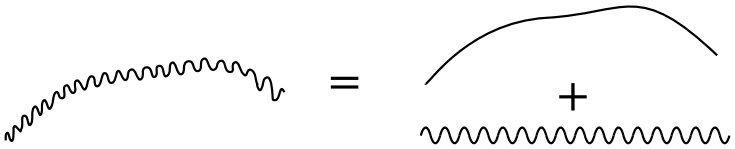
	T_μ	S_μ	$\mathsf{t}_{\mu\nu}{}^{\rho}$	Q_μ	Λ_μ	$\mathsf{q}_{\mu\nu\rho}$	
MAG	✓	✓	✓	✓	✓	✓	partially today
Weyl-Cartan	✓	✓	✓	✓	×	×	today
Poincaré	✓	✓	✓	×	×	×	already studied

2. Instabilities

Example: perturbation $\Phi = \Phi_0 + \phi$ around the background solution Φ_0,

$$\mathcal{L} = F_1(\Phi_0)\dot{\phi}^2 - F_2(\Phi_0)|\vec{\nabla}\phi|^2 - F_3(\Phi_0)\phi^2 \,.$$

Slow-varying background approx.



so

$$\mathcal{L} = \frac{1}{2a}\dot{\phi}^2 - \frac{1}{2}b|\vec{\nabla}\phi|^2 - \frac{1}{2}m^2\phi^2 \qquad \text{where} \quad b, m^2 \in \mathbb{R}, \quad a \in \mathbb{R} \setminus \{0\} \,.$$

EoM in Fourier space

$$\boxed{\omega^2 = a(b|\vec{k}|^2 + m^2)} \,.$$

ω imaginary $\rightsquigarrow$ $\mathrm{e}^{\pm|\omega|t}$ allowed

- **Tachyons** (wrong sign of the mass term $m^2 < 0$)
- **Gradient instabilities** (wrong sign of the gradient term $b < 0$)

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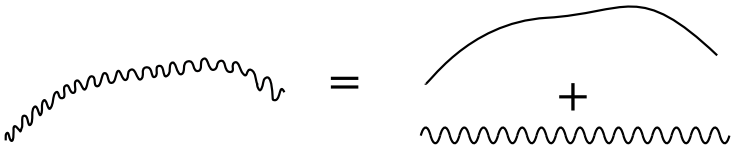

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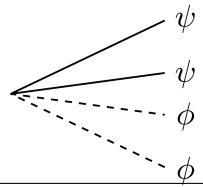
EoM in Fourier space

$$\boxed{\omega^2 = a(b|\vec{k}|^2 + m^2)}\,.$$

ω imaginary $\rightsquigarrow$ $\mathrm{e}^{\pm|\omega|t}$ allowed

- **Ghosts** (wrong sign of the kinetic term $a < 0$)
  - Unbounded H.
  - Certain sectors become highly excited (no viol. of E conservation).
  - Quantum-m.: couplings (healthy)+(ghost) make the vacuum unstable.

$$\mathcal{L} = \tfrac{1}{2}\partial_\mu\psi\partial^\mu\psi - \tfrac{1}{2}\partial_\mu\phi\partial^\mu\phi + \alpha\phi^2\psi^2 \qquad \rightsquigarrow$$



Ghosts are serious problems even at the classical level.

$$L = \frac{1}{2}\Big(\dot{x}(t)^2 - m^2 x(t)^2 \overset{!}{-} \dot{y}(t)^2 - n^2 y(t)^2 \dot{x}(t)^2 + a x(t)^2 y(t)^2\Big) \tag{2.1}$$

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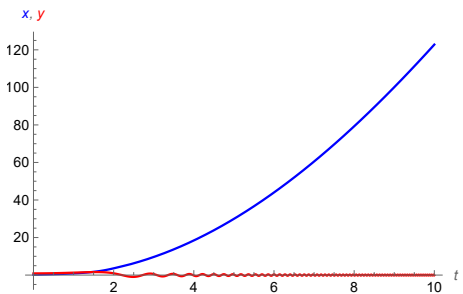
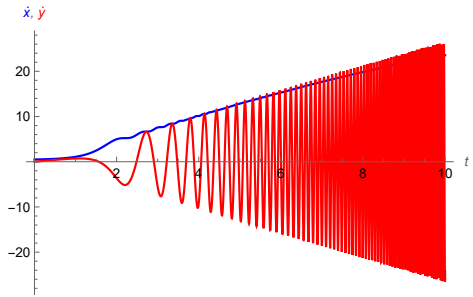
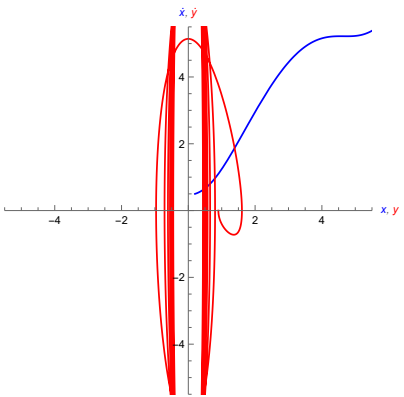
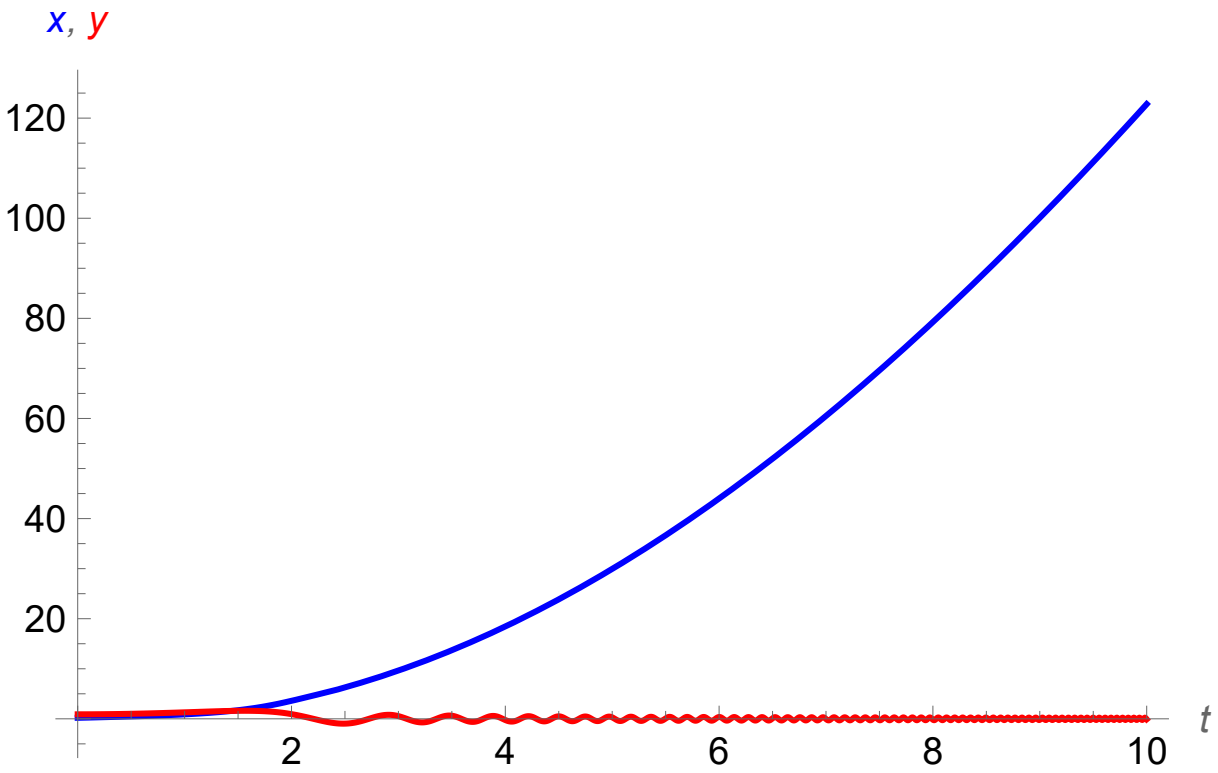
I took $(m, n, a) = (0.1, 1.0, 1.0)$ and initial conditions $(x_0, \dot{x}_0, y_0, \dot{y}_0) = (0.2, 0.5, 0.9, 0.0)$



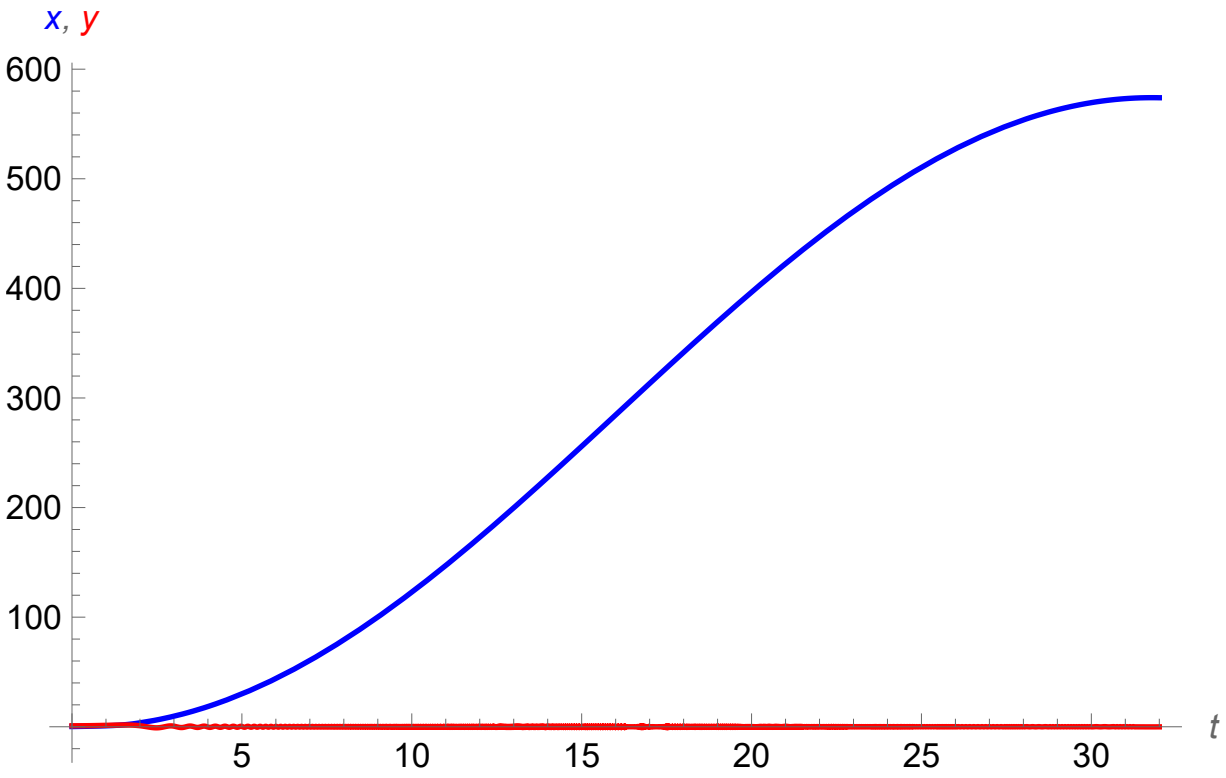


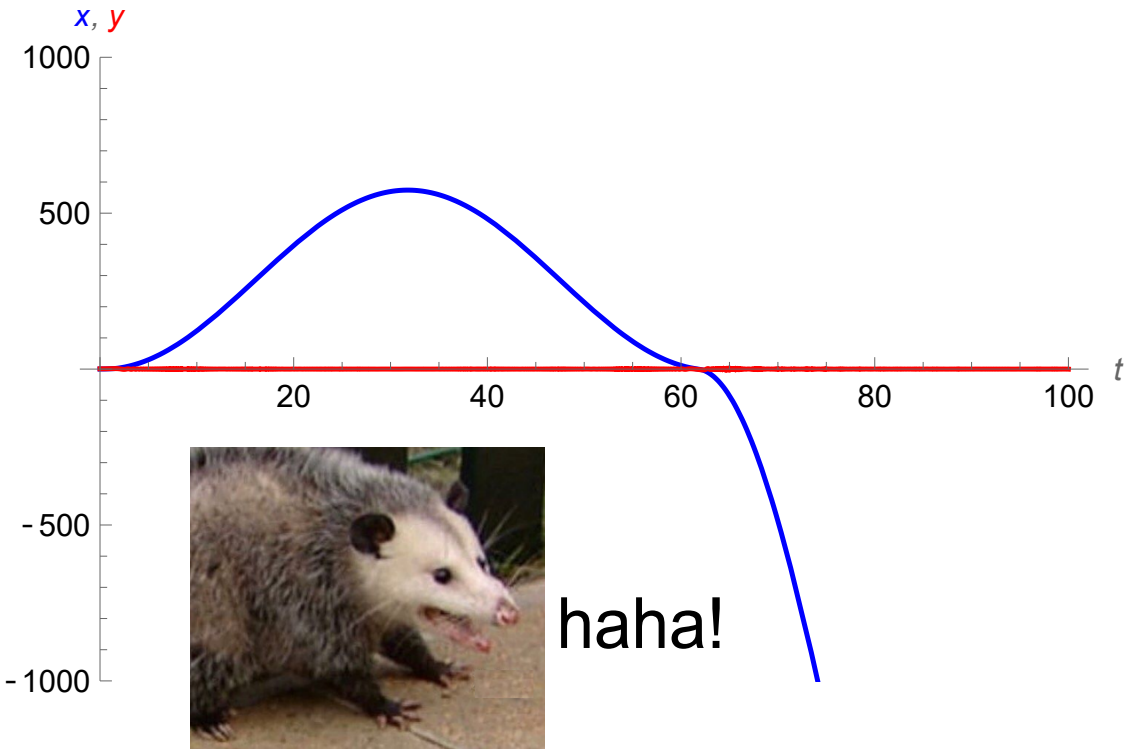


Wait...



Hey... not too bad




x, y
1000
500
-500
-1000
20
40
60
80
100
t
haha!

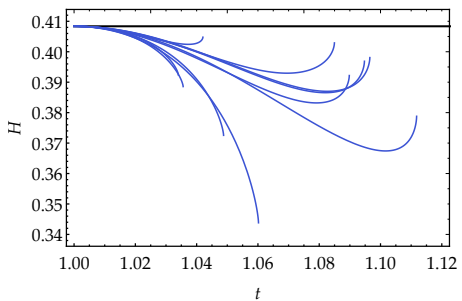
Strong coupling

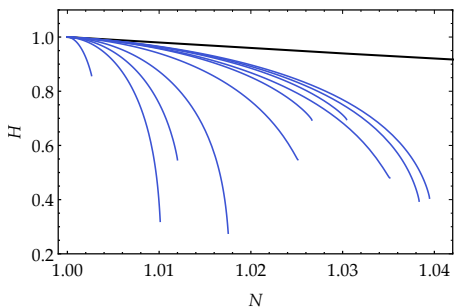
$$\mathcal{L} = \pm\frac{1}{2}F(\text{bg.})\dot{\phi}^2 + (\text{interactions}) \qquad \to 0 \times \dot{\phi}^2 + (\text{interactions})$$

$$\mathcal{L} = \pm\frac{1}{2}\dot{\phi}^2 + \frac{1}{F(\text{bg.})}(\text{interactions}) \qquad \to \pm\frac{1}{2}\dot{\phi}^2 + \infty \times (\text{interactions}) \tag{2.2}$$

Particular case of (cosmological) Einsteinian Cubic gravity From [Beltrán, AJC 2021]

Numerical result around the solution dominated by Λ and by radiation, respectively.





- Discontinuity in the number of dynamical degrees of freedom at different orders.
- Singular surface in phase space.
- Perturbation expansion does not capture the full solution.

Ostrogradski Theorem.
Let a Lagrangian involve n-th order finite time derivatives of variables. If $n \geq 2$ and the Lagrangian is non-degenerate with respect to the highest-order derivatives, the Hamiltonian of the system linearly depends on a canonical momentum.

$\rightsquigarrow$ (Generically) ghost instability [Vero's talk]

Theory Taken from [Joyce et al 2015]

$$\mathcal{L}_{\mathrm{A}} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi + \frac{\lambda}{2}(\Box\phi)^2, \qquad \Box \coloneqq \eta^{\mu\nu}\partial_\mu\partial_\nu. \tag{2.3}$$

$\lambda \neq 0 \quad \Rightarrow \quad$ non-degenerate.

Explicit ghost
Consider

$$\mathcal{L}_{\mathrm{B}} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi + \chi\Box\phi - \frac{1}{2\lambda}\chi^2\,. \tag{2.4}$$

On-shell $\left[\mathrm{EoM}(\chi) \,:\, \chi = \lambda\Box\phi\right]$ equivalent to $\mathcal{L}_{\mathrm{A}}$.

$$\mathcal{L}_{\mathrm{B}} \xrightarrow[\phi\to\chi-\alpha]{\text{field redef.}} \frac{1}{2}\partial_\mu\alpha\partial^\mu\alpha \overset{!!}{-} \frac{1}{2}\partial_\mu\chi\partial^\mu\chi - \frac{1}{2\lambda}\chi^2\,. \tag{2.5}$$

$\Rightarrow \quad \chi$ is the Ostrogradski ghost.

❒ Gauge symmetries in Yang-Mills $\leftrightarrow$ no longitudinal ghosts.

3. The vector sector of quadratic MAG gravity

Requirements

1. only algebraic invariants in T, Q and R (+ cosmological constant)
2. at most quadratic in these quantities
3. no odd-parity invariants

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2. at most quadratic in these quantities
3. no odd-parity invariants

Result: quadratic MAG Lagrangian

$$
\begin{aligned}
2\kappa\mathcal{L}_{\text{MAG}} \coloneqq & \boxed{-2\kappa\Lambda + a_0 R} && \text{Einstein-Palatini}\\
& + \boxed{a_1 \mathsf{t}_{\mu\nu\rho}\mathsf{t}^{\mu\nu\rho} + a_2 T_\mu T^\mu + a_3 S_\mu S^\mu} && \text{TT terms}\\
& + \boxed{b_1 \mathsf{q}_{\mu\nu\rho}\mathsf{q}^{\mu\nu\rho} + b_2 \mathsf{q}_{\mu\nu\rho}\mathsf{q}^{\rho\mu\nu} + b_3 Q_\mu Q^\mu + b_4 \Lambda_\mu \Lambda^\mu + b_5 Q_\mu \Lambda^\mu} && \text{QQ terms}\\
& + \boxed{c_1 \mathsf{q}_{\mu\nu\rho}\mathsf{t}^{\mu\nu\rho} + c_2 Q_\mu T^\mu + c_3 \Lambda_\mu T^\mu} && \text{TQ terms}\\
& + \ell^2 \Big[R^{\mu\nu\rho\lambda}\big(d_1 R_{\mu\nu(\rho\lambda)} + d_2 R_{\mu\nu[\rho\lambda]} + d_3 R_{\mu(\rho\lambda)\nu} + d_4 R_{\mu[\rho\lambda]\nu} + d_5 R_{\rho\lambda\mu\nu}\big)\\
& \quad + R^{\mu\nu}(d_6 R_{(\mu\nu)} + d_7 R_{[\mu\nu]}) + \mathcal{C}^{\mu\nu}(d_8 R_{(\mu\nu)} + d_9 R_{[\mu\nu]})\\
& \quad + \mathcal{C}^{\mu\nu}(d_{10}\mathcal{C}_{(\mu\nu)} + d_{11}\mathcal{C}_{[\mu\nu]})\\
& \quad + \mathcal{P}^{\mu\nu}(d_{12} R_{\mu\nu} + d_{13}\mathcal{C}_{\mu\nu} + d_{14}\mathcal{P}_{\mu\nu}) + d_{15} R^2 + d_{16}\mathcal{H}^2 \Big] && \text{RR terms}
\end{aligned}
$$

(27 parameters beyond Einstein-Palatini!)

Notation:

$$
R_{\mu\nu} \coloneqq R_{\mu\lambda\nu}{}^{\lambda}\,, \qquad \mathcal{P}_{\mu\nu} \coloneqq R_{\mu\nu\lambda}{}^{\lambda}\,(= \tfrac{1}{2}(\partial_\mu Q_\nu - \partial_\nu Q_\mu))\,, \qquad \mathcal{C}_\mu{}^\nu \coloneqq g^{\rho\lambda} R_{\mu\rho\lambda}{}^{\nu}\,,
$$

$$
R \coloneqq g^{\mu\nu} R_{\mu\nu}\,, \qquad \mathcal{H} \coloneqq \mathcal{E}_{\mu\nu\rho\lambda} R^{\mu\nu\rho\lambda}\,.
$$

Quadratic MAG Lagrangian

$$\boxed{\mathcal{L}_{\text{MAG}} = -\Lambda + \frac{a_0}{2\kappa}R + (TT\,\text{terms}) + (QQ\,\text{terms}) + (TQ\,\text{terms}) + (RR\,\text{terms})} \tag{3.1}$$

The idea now is to take:

$$R_{\mu\nu\rho}{}^{\lambda} \to \mathring{R}_{\mu\nu\rho}{}^{\lambda} + (\mathring{\nabla}T) + (\mathring{\nabla}Q) + (TT) + (TT) + (QQ) \tag{3.2}$$

and then

$$T_{\mu\nu}{}^{\rho} \to T_{\mu}, S_{\mu}, \mathsf{t}_{\mu\nu}{}^{\rho}\,, \qquad \text{and} \qquad Q_{\mu\nu\rho} \to Q_{\mu}, \Lambda_{\mu}, \mathsf{q}_{\mu\nu\rho}. \tag{3.3}$$

Goal: Trying to get a stable theory containing at least one of the spin-1 in $\{T_\mu, S_\mu, Q_\mu, \Lambda_\mu\}$ and the graviton.

Method. Analysis based on:

- Previous results on stability of Proca fields in curved spacetime and multi-Proca. e.g. [Heisenberg 2017] [Beltran, Heisenberg 2016] [Beltran, Heisenberg 2017]

- A similar study performed in the Poincaré case: [Beltran, Maldonado 2020] (in agreement with Hamiltonian analysis: [Yo, Nester 1999] [Yo, Nester 2002])

Goal: Trying to get a stable theory containing at least one of the spin-1 in $\{T_\mu, S_\mu, Q_\mu, \Lambda_\mu\}$ and the graviton.

Structure of the Lagrangian

Everything in terms of $(\mathring{R}, T_\mu, S_\mu, \mathtt{t}_{\mu\nu}{}^{\rho}, Q_\mu, \Lambda_\mu, \mathtt{q}_{\mu\nu\rho})$

$$
\begin{aligned}
\mathcal{L}_{\text{MAG}} \quad \to \quad & \mathcal{L}_{\text{GR}} && = -\Lambda + \frac{a_0}{2\kappa}\mathring{R} \\
& + \mathcal{L}_{\mathring{R}\mathring{R}} && = \frac{\ell^2}{2\kappa}\left[\beta\mathring{R}^2 + \alpha\mathring{R}_{\mu\nu}\mathring{R}^{\mu\nu}\right] \; + \; (...)\mathring{\mathcal{L}}_{\text{GB}} \\
& + \mathcal{L}_{\text{v}} && = \mathcal{L}_T + \mathcal{L}_S + \mathcal{L}_Q + \mathcal{L}_\Lambda + \mathcal{L}_{\text{v, mix}} \\
& + \mathcal{L}_{*} && = \text{ten-(ten, vec, } g\text{) couplings}\,,
\end{aligned}
\tag{3.4}
$$

Steps to follow:

1. Stable graviton sector (Einstein-Hilbert + Gauss-Bonnet limit)
2. Stable isolated vector sectors (including non-min. couplings)
3. Stable interactions between vectors (including non-min. couplings)
4. Correct signs in the kinetic matrix to avoid ghosts

[J. Beltrán's talk]

→ To ensure that no ghosts come from the graviton sector, we study the Riemannian limit:

$$2\kappa\mathcal{L}_{\rm MAG} \quad \to \quad 2\kappa\big(\mathcal{L}_{\rm GR}+\mathcal{L}_{\mathring{R}\mathring{R}}\big) \tag{3.5}$$

$$= -2\kappa\Lambda + a_0\mathring{R} + \ell^2\left[\beta\mathring{R}^2 + \alpha\mathring{R}_{\mu\nu}\mathring{R}^{\mu\nu}\right] \; + \; (...)\mathring{\mathcal{L}}_{\rm GB}. \tag{3.6}$$

To recover Einstein-Gauss-Bonnet, one needs to impose:

$$\alpha = 0\,, \qquad \beta = 0\,. \tag{3.7}$$

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Equivalent to:

$$\text{(I)} := \begin{cases} d_6 \to -d_{10} - 4d_2 + 2d_4 - 4d_5 + d_8, \\ d_2 \to d_{15} - d_5 + \frac{d_4}{2}. \end{cases}$$

- We define:

$$ {}^{V}H_{\mu\nu} \coloneqq 2\mathring{\nabla}_{(\mu}V_{\nu)}, \qquad {}^{V}H \coloneqq 2\mathring{\nabla}_{\mu}V^{\mu}, \qquad {}^{V}F_{\mu\nu} \coloneqq 2\mathring{\nabla}_{[\mu}V_{\nu]} \quad (\equiv 2\partial_{[\mu}V_{\nu]}). \tag{3.8} $$

- Boundary terms to eliminate ${}^{V}H_{\mu\nu} \rightarrow$ Lagrangian depending on ${}^{V}H$ and ${}^{V}F_{\mu\nu}$.

> Example:
>
> $$ {}^{V}H_{\mu\nu}{}^{W}H^{\mu\nu} = 2\mathring{R}\,V_{\mu}W^{\mu} + {}^{V}H{}^{W}H + 4\mathring{G}_{\mu\nu}V^{\mu}W^{\nu} + {}^{V}F_{\mu\nu}{}^{W}F^{\mu\nu} + \text{b.t.} \tag{3.9} $$

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The self-interacting Lagrangian for each of the vectors are:

$$ \mathcal{L}_{T}|_{(\text{I})} = (\ldots){}^{T}F_{\mu\nu}{}^{T}F^{\mu\nu} + (\ldots)T_{\mu}T^{\mu} \tag{3.10} $$

$$ \mathcal{L}_{Q}|_{(\text{I})} = (\ldots){}^{Q}F_{\mu\nu}{}^{Q}F^{\mu\nu} + (\ldots)Q_{\mu}Q^{\mu} \tag{3.11} $$

$$ \mathcal{L}_{S}|_{(\text{I})} = (\ldots){}^{S}F_{\mu\nu}{}^{S}F^{\mu\nu} + (\ldots)S_{\mu}S^{\mu} + (\ldots)\mathring{G}_{\mu\nu}S^{\mu}S^{\nu} + \boxed{(\ldots){}^{S}H^{2} + (\ldots)\mathring{R}\,S_{\mu}S^{\mu}} \tag{3.12} $$

$$ \mathcal{L}_{\Lambda}|_{(\text{I})} = (\ldots){}^{\Lambda}F_{\mu\nu}{}^{\Lambda}F^{\mu\nu} + (\ldots)\Lambda_{\mu}\Lambda^{\mu} + (\ldots)(\Lambda_{\mu}\Lambda^{\mu})^{2} + (\ldots){}^{\Lambda}H\,\Lambda_{\mu}\Lambda^{\mu} + (\ldots)\mathring{G}_{\mu\nu}\Lambda^{\mu}\Lambda^{\nu} $$

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$$ \tag{3.14} $$

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$$ \mathcal{L}_{S}|_{(\text{I})} = (...)^{S}F_{\mu\nu}{}^{S}F^{\mu\nu} + (...)S_{\mu}S^{\mu} + (...)\mathring{G}_{\mu\nu}S^{\mu}S^{\nu} + \boxed{(...)^{S}H^{2} + (...)\mathring{R}\,S_{\mu}S^{\mu}} \tag{3.12} $$

$$ \mathcal{L}_{\Lambda}|_{(\text{I})} = (...)^{\Lambda}F_{\mu\nu}{}^{\Lambda}F^{\mu\nu} + (...)\Lambda_{\mu}\Lambda^{\mu} + (...)(\Lambda_{\mu}\Lambda^{\mu})^{2} + (...)^{\Lambda}H\,\Lambda_{\mu}\Lambda^{\mu} + (...)\mathring{G}_{\mu\nu}\Lambda^{\mu}\Lambda^{\nu} $$

$$ + \boxed{(...)^{\Lambda}H^{2} + (...)\mathring{R}\,\Lambda_{\mu}\Lambda^{\mu} + (...)\mathring{R}{}^{\Lambda}H} \tag{3.13} $$

$$ \tag{3.14} $$

$\Rightarrow$ 5 more conditions (7 in total):

$$ (\text{II}) \coloneqq \Big\{ d_2 \rightarrow d_5,\ d_3 \rightarrow -2(d_1 + 4d_{10}),\ d_4 \rightarrow 4d_5,\ d_6 \rightarrow d_{10}, $$

$$ d_8 \rightarrow 2d_{10},\ d_{15} \rightarrow 0,\ d_{16} \rightarrow \frac{1}{4}d_5 \Big\}. $$

Under these conditions the sector that mixes the vectors reads:

$$
\begin{aligned}
2\kappa\mathcal{L}_{\text{v,mix}}|_{\text{(II)}} = {} & (\ldots)^{\Lambda}F_{\mu\nu}{}^{Q}F^{\mu\nu} + (\ldots)^{\Lambda}F_{\mu\nu}{}^{T}F^{\mu\nu} + (\ldots)^{Q}F_{\mu\nu}{}^{T}F^{\mu\nu} + (\ldots)^{S}F_{\mu\nu}\Lambda^{\mu}S^{\nu} \\
& + (\ldots)(\star^{Q}F)_{\mu\nu}\Lambda^{\mu}S^{\nu} + (\ldots)(\star^{\Lambda}F)_{\mu\nu}\Lambda^{\mu}S^{\nu} + (\ldots)(\star^{T}F)_{\mu\nu}\Lambda^{\mu}S^{\nu} \\
& + (\ldots)Q_{\mu}\Lambda^{\mu} + (\ldots)Q_{\mu}T^{\mu} + (\ldots)\Lambda_{\mu}T^{\mu} \\
& + (\ldots)(\Lambda_{\mu}S^{\mu})^{2} + (\ldots)\Lambda_{\mu}\Lambda^{\mu}S_{\nu}S^{\nu} + (\ldots)\Lambda_{\mu}T^{\mu}S_{\nu}S^{\nu} \\
& + (\ldots)S_{\mu}T^{\mu}S_{\nu}\Lambda^{\nu} + (\ldots)S_{\mu}Q^{\mu}S_{\nu}\Lambda^{\nu} + (\ldots)\Lambda_{\mu}Q^{\mu}S_{\nu}S^{\nu} \\
& - \frac{1}{81}\ell^{2}(d_1+2d_5+4d_{10})({}^{S}H\Lambda_{\mu}S^{\mu} + 2^{\Lambda}HS_{\mu}S^{\mu})
\end{aligned}
\tag{3.15}
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Here $(\star^{V}F)^{\mu\nu} := \frac{1}{2}{}^{V}F_{\alpha\beta}\mathcal{E}^{\alpha\beta\mu\nu}$.

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& + (...)Q_{\mu}\Lambda^{\mu} + (...)Q_{\mu}T^{\mu} + (...)\Lambda_{\mu}T^{\mu} \\
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Here $(\star^{V}F)^{\mu\nu} := \frac{1}{2}{}^{V}F_{\alpha\beta}\mathcal{E}^{\alpha\beta\mu\nu}$.

$\Rightarrow$ 1 more condition (8 in total):

$$
\begin{aligned}
\text{(III)} := \Big\{ & d_2 \to d_5,\ d_3 \to 4d_5,\ d_4 \to 4d_5,\ d_6 \to -\frac{1}{4}(d_1+2d_5), \\
& d_8 \to -\frac{1}{2}(d_1+2d_5),\ d_{10} \to -\frac{1}{4}(d_1+2d_5),\ d_{15} \to 0,\ d_{16} \to \frac{1}{4}d_5 \Big\}.
\end{aligned}
$$

$$\mathcal{L}_{\rm v}|_{\rm (III)} \supset ({}^{S}F_{\mu\nu}, {}^{T}F_{\mu\nu}, {}^{Q}F_{\mu\nu}, {}^{\Lambda}F_{\mu\nu}) \begin{pmatrix} \zeta & 0 & 0 & 0 \\ 0 & -4\zeta & K_1 & K_2 \\ 0 & K_1 & K_3 & K_4 \\ 0 & K_2 & K_4 & K_5 \end{pmatrix} \begin{pmatrix} {}^{S}F^{\mu\nu} \\ {}^{T}F^{\mu\nu} \\ {}^{Q}F^{\mu\nu} \\ {}^{\Lambda}F_{\mu\nu} \end{pmatrix} \tag{3.16}$$

$\Rightarrow$ No ghost-free theory with 4 vectors $\Rightarrow \zeta = 0$ (one more condition!)

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Under $\zeta = 0$

$$\mathcal{L}_{\rm v}|_{\rm (III),\zeta=0} \supset ({}^{S}F_{\mu\nu}, {}^{T}F_{\mu\nu}, {}^{Q}F_{\mu\nu}, {}^{\Lambda}F_{\mu\nu}) \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & K_1 & K_2 \\ 0 & K_1 & K_3 & K_4 \\ 0 & K_2 & K_4 & K_5 \end{pmatrix} \begin{pmatrix} {}^{S}F^{\mu\nu} \\ {}^{T}F^{\mu\nu} \\ {}^{Q}F^{\mu\nu} \\ {}^{\Lambda}F_{\mu\nu} \end{pmatrix} \tag{3.17}$$

$\Rightarrow$ Ghostly unless $K_1 = K_2 = 0$ (2 more conditions!)

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To sum up:
$\Rightarrow$ 1+2 more conditions (11 in total):

$$\text{(IV)} = \Big\{ d_2 \to d_5,\ d_3 \to 4d_5,\ d_4 \to 4d_5,\ d_6 \to -\frac{1}{4}(d_1 + 2d_5),\ d_8 \to -\frac{1}{2}(d_1 + 2d_5),$$
$$d_9 \to 2d_7,\ d_{10} \to -\frac{1}{4}(d_1 + 2d_5),\ d_{11} \to d_7,\ d_{13} \to d_{12},\ d_{15} \to 0,\ d_{16} \to \frac{1}{4}d_5 \Big\}.$$

$$\mathcal{L}_{\text{v}}|_{(\text{IV})} = -\frac{1}{4}({}^{Q}F_{\mu\nu}, {}^{\Lambda}F_{\mu\nu}) \begin{pmatrix} (\ldots) & (\ldots) \\ (\ldots) & (\ldots) \end{pmatrix} \begin{pmatrix} {}^{Q}F^{\mu\nu} \\ {}^{\Lambda}F_{\mu\nu} \end{pmatrix}$$
$$+ \text{masses}$$
$$+ \text{quartic terms}$$
$$+ \text{coupling vector-}(\star F)$$
$$+ \text{terms with } T_\mu, S_\mu \text{ (as auxiliary fields/Lagrange multipliers)} \qquad (3.18)$$

Important remarks:

- The conditions (IV) are necessary in the full theory.
- 16 parameters in the RR sector $\rightarrow$ 5 surviving ones.
- This is a partial analysis: we still need to analyze the tensor sector!

4. The case of quadratic Weyl-Cartan gravity

The starting point (now $\Lambda_\mu = \mathtt{q}_{\mu\nu\rho} = 0$) is a Lagrangian with just 7 parameters in the RR sector.

Only: $(\mathring{R}_{\mu\nu\rho}{}^{\lambda}, Q_\mu, T_\mu, S_\mu, \mathtt{t}_{\mu\nu}{}^{\rho})$

We again follow:

1. Stable graviton sector (Einstein-Hilbert + Gauss-Bonnet limit)
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Result:

- The RR sector is reduced to just 1 parameter:

$$\mathcal{P}_{\mu\nu}\mathcal{P}^{\mu\nu} \quad \sim \quad {}^{Q}F_{\mu\nu}{}^{Q}F^{\mu\nu} \,. \tag{4.1}$$

- S_μ and $\mathfrak{t}_{\mu\nu}{}^{\rho}$ only enter quadratically $\Rightarrow$ non-dynamical / vanishing.
- T_μ can be eliminated $\quad \Rightarrow \quad$ GR + Proca(Q_μ) (!)

(Non-trivial possibilities! This does not happen in Poincaré gravity)

5. Final words

Conclusion:

- The stability analysis of $\{T_\mu, S_\mu, Q_\mu, \Lambda_\mu\}$ is enough to reduce the par. of the RR sector...

  - ⇨ from 16 to 5 in quadratic MAG $\rightarrow$ (IV) necessary for stable spin-1 (incomplete!)

  - ⇨ from 7 to 1 in quadratic Weyl-Cartan $\rightarrow$ GR+Proca (complete!)

Conclusion:

- The stability analysis of $\{T_\mu, S_\mu, Q_\mu, \Lambda_\mu\}$ is enough to reduce the par. of the RR sector...

  ⇨ from 16 to 5 in quadratic MAG → (IV) necessary for stable spin-1 (incomplete!)

  ⇨ from 7 to 1 in quadratic Weyl-Cartan → GR+Proca (complete!)

Limitations of this work

- We follow the standard 'gauge' prescription for the Lagrangian → we are ignoring other dimension-4 invariants
- The method is oblivious to stable theories without vectors: Ricci based, purely scalar theories ($f(R)$, Holst...).
- The analysis of MAG is incomplete ⇝ tensor sector! ⇒ further restrictions
  + cross-check: [Percacci, Sezgin 2020] [Marzo 2021]
- Odd parity sector? ← we do not expect it to solve the problems [Beltran, Maldonado 2020]

1. Interested in building modifications? $\rightarrow$ stability analysis.
   Some useful tips:
   - ⇨ Choose a better set of variables.
   - ⇨ Check single field sectors, decoupling limits, etc.
   - ⇨ Check non-minimal couplings.
   - ⇨ Check kinetic matrix: determinant, diagonal elements, etc.
   - ⇨ Do it around specific backgrounds of interest.

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   $\Rightarrow$ Field theory-like modifications are more reasonable a priori (personal opinion!)

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Thanks for your attention!
¡Muchas gracias!