# Metric-Affine Gravity: from theory to applications in black holes and cosmology

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Third Workshop on Current Challenges in Cosmology, Bucaramanga, Colombia, 26/Oct/2023

JCAP **09** (2020), 057**+** JCAP **01** (2022) no.01, 011**+** JCAP **04** (2022) no.04, 011**+** JCAP **02**, 018 (2023)**+**Phys. Rev. D **108** (2023) no.4, 044037**+**arXiv:2310.16007

Jointly with **Jorge Gigante Valcarcel**, L. Järv, J. Chevrier, K.Aoki, M.A. Gorji.

KAVLI IPMU INSTITUTE FOR THE PHYSICS AND MATHEMATICS OF THE UNIVERSE

東京大学 THE UNIVERSITY OF TOKYO

# Overview of the Talk


1. Introduction to Metric-affine gravity
2. Metric-Affine gravity
   - Basic geometrical quantities
3. MAG models with dynamical torsion and nonmetricity (Weyl only)
   - Spherically symmetry in Weyl-Cartan geometry
   - Axial symmetry in Weyl-Cartan geometry
4. Algebraic classification in Weyl-Cartan
5. MAG models torsion and the complete nonmetricity (Traceless and Weyl)
6. Cosmological perturbations in MAG

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# How to modify it?

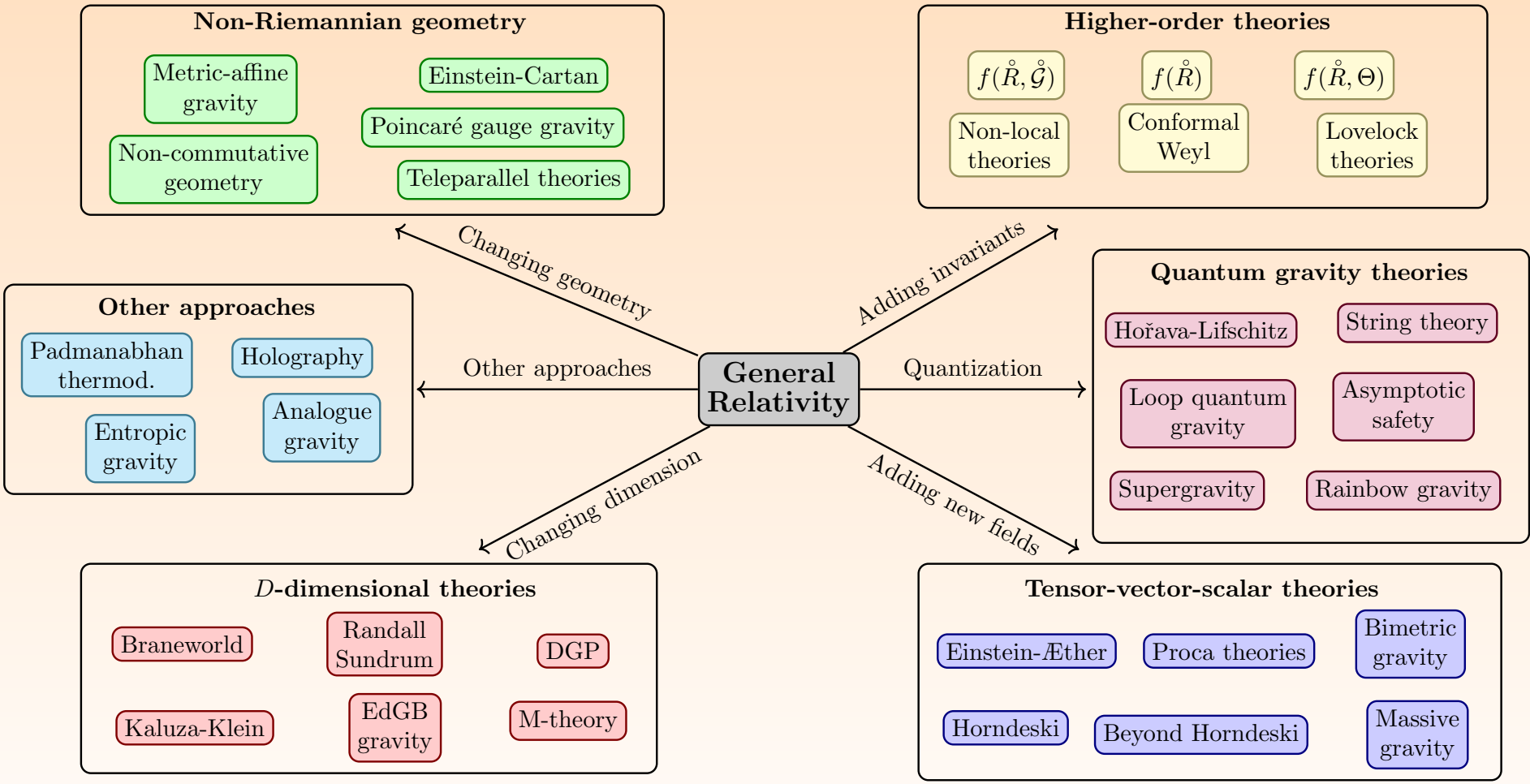


Figure: Classification of theories of gravity. (S. Bahamonde, K. F. Dialektopoulos, C. Escamilla-Rivera, G. Farrugia, V. Gakis, M. Hendry, M. Hohmann, J. Levi Said, J. Mifsud and E. Di Valentino, "Teleparallel gravity: from theory to cosmology," Rept. Prog. Phys. **86** (2023) no.2, 026901.)

# Post-Riemannian decomposition

- In the most general metric-affine setting, the fundamental variables are a **metric** $g_{\mu\nu}$ (10 comp.) as well as the coefficients $\tilde{\Gamma}^{\rho}{}_{\mu\nu}$ (64 comp.) of an **affine connection**.

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- Further, we can write

$$\tilde{\Gamma}^{\lambda}{}_{\mu\nu} = \Gamma^{\lambda}{}_{\mu\nu} + K^{\lambda}{}_{\mu\nu} + L^{\lambda}{}_{\mu\nu} = \Gamma^{\lambda}{}_{\mu\nu} + N^{\lambda}{}_{\mu\nu}$$
$$\tilde{R}^{\lambda}{}_{\rho\mu\nu} = R^{\lambda}{}_{\rho\mu\nu} + 2\nabla_{[\mu|}N^{\lambda}{}_{\rho|\nu]} + 2N^{\lambda}{}_{\sigma[\mu|}N^{\sigma}{}_{\rho|\nu]} \,.$$

# Decomposition into irreducible parts

- Irreducible decomposition of the torsion tensor:

$$T^{\lambda}{}_{\mu\nu} = \frac{1}{3}\left(\delta^{\lambda}{}_{\nu}T_{\mu} - \delta^{\lambda}{}_{\mu}T_{\nu}\right) + \frac{1}{6}\varepsilon^{\lambda}{}_{\rho\mu\nu}S^{\rho} + t^{\lambda}{}_{\mu\nu}\,.$$

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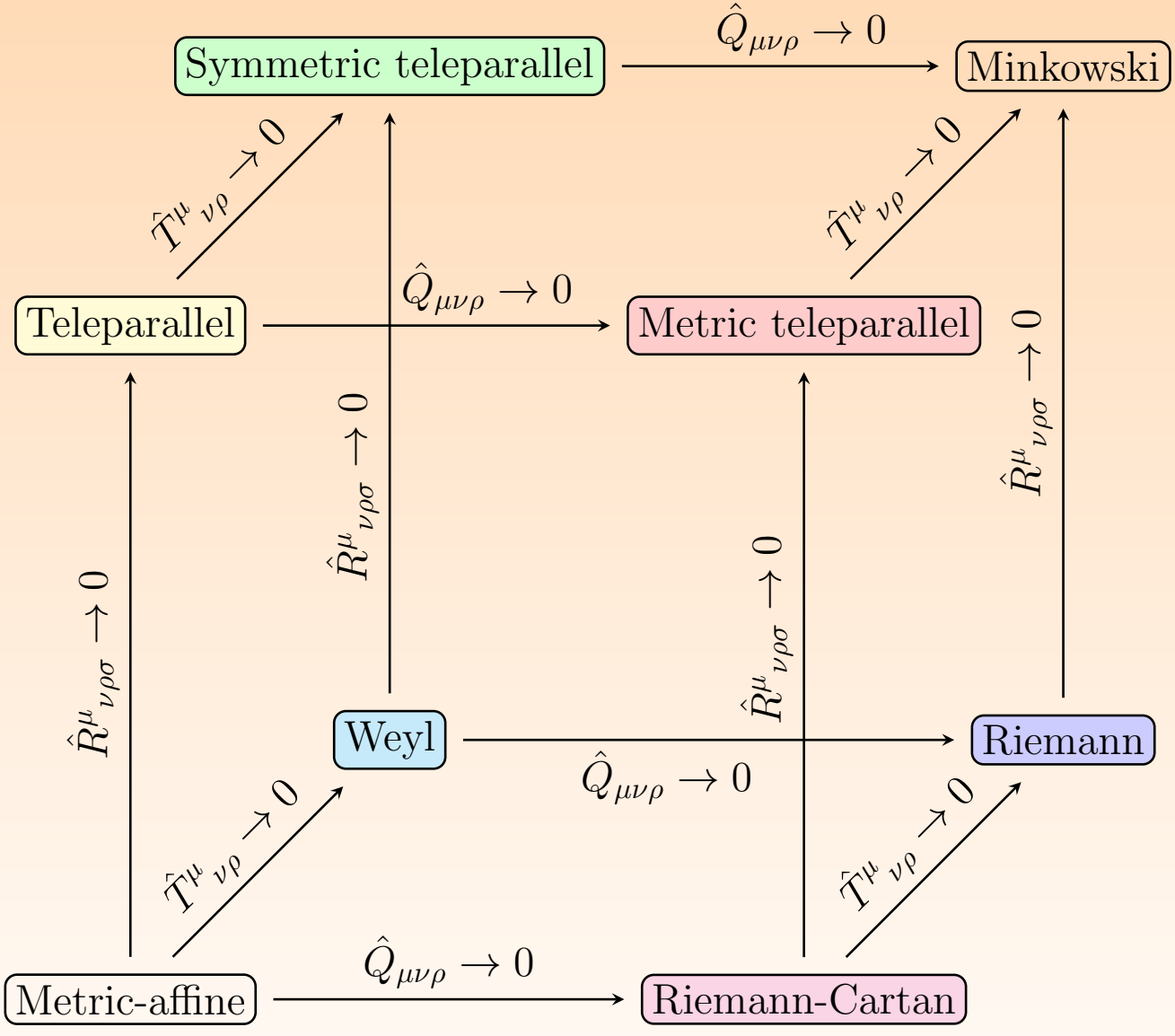


Figure: Classification of metric-affine geometries - Cube (S. Bahamonde, K. F. Dialektopoulos, C. Escamilla-Rivera, G. Farrugia, V. Gakis, M. Hendry, M. Hohmann, J. Levi Said, J. Mifsud and E. Di Valentino, "Teleparallel gravity: from theory to cosmology," Rept. Prog. Phys. **86** (2023) no.2, 026901.)

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- Quadratic gravitational action with dynamical torsion and nonmetricity in Weyl-Cartan geometry ($Q_{\lambda\mu\nu} = g_{\mu\nu}W_\lambda$ and $\mathring{Q}_{\lambda\mu\nu} = 0$)

$$
\begin{aligned}
S = \int d^4x\sqrt{-g}\Big\{\mathcal{L}_{\rm m} + \frac{1}{64\pi}\Big[ & -4R - 6d_1\tilde{R}_{\lambda[\rho\mu\nu]}\tilde{R}^{\lambda[\rho\mu\nu]} \\
& - 9d_1\tilde{R}_{\lambda[\rho\mu\nu]}\tilde{R}^{\mu[\lambda\nu\rho]} + 8\,d_1\tilde{R}_{[\mu\nu]}\tilde{R}^{[\mu\nu]} + \frac{1}{8}\left(32e_1 + 8e_2 + 17d_1\right)\tilde{R}^{\lambda}{}_{\lambda\mu\nu}\tilde{R}^{\rho}{}_{\rho}{}^{\mu\nu} \\
& - 7d_1\tilde{R}_{[\mu\nu]}\tilde{R}^{\lambda}{}_{\lambda}{}^{\mu\nu} + 3\left(1 - 2a_2\right)T_{[\lambda\mu\nu]}T^{[\lambda\mu\nu]}\Big]\Big\}\,.
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[1] S. Bahamonde and J. G. Valcarcel, JCAP **09**, 057 (2020).

[2] S. Bahamonde and J. G. Valcarcel, JCAP **01** (2022) no.01, 011.

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- Absence of a general Birkhoff's theorem in MAG: new spherically and axially symmetric vacuum solutions with independent dynamical torsion and nonmetricity fields[1,2]

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# Spherical symmetry

- Metric, torsion and nonmetricity in spherically symmetric space-times ($\#2 + \#8 + \#2 = \#12$):

$$\mathcal{L}_\xi g_{\mu\nu} = \mathcal{L}_\xi T^\lambda{}_{\mu\nu} = \mathcal{L}_\xi W_\mu = 0 \implies \mathcal{L}_\xi \tilde{R}_{\lambda\rho\mu\nu} = 0$$

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- By solving these equations we find that torsion and nonmetricity behave as

$$\begin{aligned}
T^t{}_{tr} &= a(r)\,, \quad T^r{}_{tr} = b(r)\,, \quad T^{\theta_k}{}_{t\theta_k} = f(r)\,, \quad T^{\theta_k}{}_{r\theta_k} = g(r) \\
T^{\theta_k}{}_{t\theta_l} &= e^{a\theta_k}\, e^b{}_{\theta_l}\, \epsilon_{ab}\, d(r)\,, \quad T^{\theta_k}{}_{r\theta_l} = e^{a\theta_k}\, e^b{}_{\theta_l}\, \epsilon_{ab}\, h(r)\,, \\
T^t{}_{\theta_k\theta_l} &= \epsilon_{kl}\, k(r)\, \sin\theta_1\,, \quad T^r{}_{\theta_k\theta_l} = \epsilon_{kl}\, l(r)\, \sin\theta_1\,, \\
W_\lambda &= (w_1(r), w_2(r), 0, 0)\,,
\end{aligned}$$

whereas the metric is in the standard spherically symmetric form:

$$ds^2 = \Psi_1(r)\, dt^2 - \frac{dr^2}{\Psi_2(r)} - r^2 \left(d\theta_1^2 + \sin^2\theta_1 d\theta_2^2\right)\,.$$

Here, $\epsilon_{kl}$ is the Levi-Civita symbol in two dimensions.

# Resolution of the field equations - solution with spin and dilation

- Reissner-Nordström solution with spin and dilation charges:

$$g_{tt} = -1/g_{rr} \equiv \Psi(r) = 1 - \frac{2m}{r} + \frac{d_1\kappa_{\rm s}^2 - 4e_1\kappa_{\rm d}^2}{r^2}\,.$$

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- Kinetic part of the Lagrangian:

$$16\pi\mathcal{L}_{\rm kin} = \frac{d_1}{2}\left[\frac{1}{9}F^{(\bar{S})}_{\mu\nu}F^{(\bar{S})\mu\nu} + \partial_\lambda \bar{t}^{\lambda}{}_{\mu\nu}\partial_\rho \bar{t}^{\rho\mu\nu} + 3\,\partial_\rho \bar{t}_{\lambda\mu\nu}\left(\partial^\rho \bar{t}^{\mu\nu\lambda} - \partial^\mu \bar{t}^{\lambda\nu\rho}\right)\right]$$
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- Ghost-free conditions for torsion and nonmetricity:

$$d_1 \leq 0\,, \quad e_1 \leq 0\,.$$

- Absence of the inner pathological Cauchy horizon in the Reissner-Nordström geometry if

$$d_1\kappa_{\rm s}^2 - 4e_1\kappa_{\rm d,e}^2 \leq 0\,, \quad \Psi(r) = 1 - \frac{2m}{r} - \frac{|d_1\kappa_{\rm s}^2 - 4e_1\kappa_{\rm d,e}^2|}{r^2}\,.$$

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  2. We know that the spin is a fundamental property of particles. Since their masses contribute to gravity, why their spin do not?

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1. Intrinsic spin generates gravitation. This effect does not exist in GR.
2. We know that the spin is a fundamental property of particles. Since their masses contribute to gravity, why their spin do not?
3. The solution is in vacuum and a charge $\kappa_s$ appears (spin charge). Analogue to the case of Schwarzschild where the mass $M$ appears.

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4. We expect that the spin charge might be important in certain astrophysical scenarios such as: highly mangnetized neutron stars; supermassive black holes with endowed spin.

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  2. dilation: deformation that involves only change of volume (in this case, intrinsic dilation!)
  3. Weyl part of nonmetricity is ”scale invariant”
  4. Do all particles in nature have different dilations? is this property important in particle physics?

# Extension to axisymmetric space-times

- Metric, torsion and nonmetricity tensors in symmetric space-times:

$$\mathcal{L}_\xi g_{\mu\nu} = \mathcal{L}_\xi T^\lambda{}_{\mu\nu} = \mathcal{L}_\xi Q^\lambda{}_{\mu\nu} = 0 \implies \mathcal{L}_\xi \tilde{R}^\lambda{}_{\rho\mu\nu} = 0\,.$$

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- Stationary and axisymmetric space-times:

$$\#10 \to \#4 \begin{cases} ds^2 = \Psi_1(r,\vartheta)\,dt^2 - \frac{dr^2}{\Psi_2(r,\vartheta)} \\ -\,r^2\Psi_3(r,\vartheta)\Big[d\vartheta^2 + \sin^2\vartheta(d\varphi - \Psi_4(r,\vartheta)dt)^2\Big] \end{cases};$$

$$\#24 \begin{cases} T^\lambda{}_{\mu\nu} = T^\lambda{}_{\mu\nu}(r,\vartheta) \end{cases}$$

$$\#4 \begin{cases} W_\mu = (W_t(r,\vartheta), W_r(r,\vartheta), W_\vartheta(r,\vartheta), W_\varphi(r,\vartheta))\,. \end{cases}$$

# Axisymmetric space-times - Kerr-Newmann de-Sitter

- Rotating Kerr-Newman metric structure[3]:

$$
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ds^2 = {} & \Psi(r,\vartheta)\, dt^2 - \frac{r^2 + a^2\cos^2\vartheta}{\left(r^2 + a^2\cos^2\vartheta\right)\Psi(r,\vartheta) + a^2\sin^2\vartheta}\, dr^2 \\
& - \left(r^2 + a^2\cos^2\vartheta\right) d\vartheta^2 + 2a\left(1 - \Psi(r,\vartheta)\right)\sin^2\vartheta\, dt d\varphi \\
& - \sin^2\vartheta\left[r^2 + a^2 + a^2\left(1 - \Psi(r,\vartheta)\right)\sin^2\vartheta\right] d\varphi^2 \,,
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$$

$$
\Psi(r,\vartheta) = 1 - \frac{\left[2mr + 4e_1(\kappa_{\mathrm{d,e}}^2 + \kappa_{\mathrm{d,m}}^2) - d_1\kappa_{\mathrm{s}}^2\right]}{r^2 + a^2\cos^2\vartheta} \,.
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[3] S. Bahamonde and J. G. Valcarcel, JCAP **01** (2022) no.01, 011.

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$$

- Field strength tensors:

$$
\begin{aligned}
\bar{R}_{[\mu\nu]} &= \frac{1}{12}\varepsilon^{\lambda}{}_{\sigma\mu\nu}\nabla_\lambda \bar{S}^\sigma + \frac{1}{2}\nabla_\lambda \bar{t}^{\lambda}{}_{\mu\nu} \,; \quad \tilde{R}^{\lambda}{}_{\lambda\mu\nu} = 4\nabla_{[\nu}W_{\mu]} \,; \\
\bar{R}^{\lambda}{}_{[\mu\nu\rho]} &= \frac{1}{6}\varepsilon^{\lambda}{}_{\sigma[\rho\nu}\nabla_{\mu]}\bar{S}^\sigma + \nabla_{[\mu}\bar{t}^{\lambda}{}_{\rho\nu]} + \frac{1}{4}\varepsilon^{\lambda}{}_{\omega\sigma[\rho}\mathring{t}_1^{\sigma}{}_{\mu\nu]}\bar{S}^\omega \\
&\quad - \frac{1}{18}\varepsilon_{\sigma\mu\nu\rho}\mathring{T}_1^{\lambda}\bar{S}^\sigma \,.
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- Later, in[4] we found that the Plebanski-Damianski solution (having mass,Taub-NUT charge, acceleration, angular momentum) can be also obtained in our model under a similar approximation.
- How can we go beyond the decoupling limit? why can this be important?

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# Gravitational spin-orbit interaction

- We found a solution in the decoupling limit $a\kappa_s \ll 1$, which ensures that the Maxwell equation and closure conditions are fulfilled by the field strength tensors of torsion

$$\nabla_\lambda \tilde{R}^\lambda{}_{[\rho\mu\nu]} = \nabla_\mu \tilde{R}^{[\mu\nu]} = 0\,, \quad \nabla_{[\sigma}\tilde{R}_{\lambda[\rho\mu\nu]]} = \nabla_{[\lambda}\tilde{R}_{[\mu\nu]]} = 0\,.$$

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- **Gravitational spin-orbit interaction:**

$$\mathcal{H}_{\rm LS} = \frac{1}{m_{\rm e}^2 r}\frac{\partial V}{\partial r}\mathbf{L}\cdot\mathbf{S} \approx \frac{d_1}{2r}\frac{\partial g_{tt}}{\partial r} a\kappa_{\rm s}\cos\vartheta$$

# Overview of the Talk


1. Introduction to Metric-affine gravity
2. Metric-Affine gravity
   - Basic geometrical quantities
3. MAG models with dynamical torsion and nonmetricity (Weyl only)
   - Spherically symmetry in Weyl-Cartan geometry
   - Axial symmetry in Weyl-Cartan geometry
4. Algebraic classification in Weyl-Cartan
5. MAG models torsion and the complete nonmetricity (Traceless and Weyl)
6. Cosmological perturbations in MAG

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- The most general Type D solution in Einstein-Maxwell is known as the Plebański-Demiański characterised by $\{M, a, \alpha, N\}$ (mass, angular momentum, acceleration and Nut charge) and the electromagnetic charges.
- *Goldberg-Sachs theorem:* A vacuum solution of the Einstein's field equations admits a shear-free null geodesic congruence if and only if the conformal part of the Riemann tensor is algebraically special.

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- **Result in Riemannian geometry:** Weyl has 6 types **(Petrov classification)**; Ricci traceless has 15 types**(Segre classification)**;
- **What happens in GR in vacuum?** $\not{R}_{\rho\nu} = R = 0$ and then the curvature is fully characterised by the Weyl tensor with their 6 types.

# Curvature decomposition in Metric-Affine geometry

- First, we need to find the building blocks of the general curvature tensor (recall that in Riemannan geometry we had 3: $W_{\alpha\beta\mu\nu}, \tilde{R}_{\mu\nu}, R$).

[5] S. Bahamonde and J. Gigante Valcarcel, "Algebraic classification of the gravitational field in Weyl-Cartan spacetimes," Phys. Rev. D **108** (2023) no.4, 044037.

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## General curvature

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- We find that there are 11 building blocks:[5]

| Building block | Number of independent components | Limit in Riemannian geometry |
|---|---|---|
| ${}^{(1)}\tilde{Z}_{\lambda\rho\mu\nu}$ | 30 | zero |
| ${}^{(1)}\tilde{W}_{\lambda\rho\mu\nu}$ | 10 | Weyl tensor $W_{\lambda\rho\mu\nu}$ |
| $\tilde{\mathring{R}}^{(T)}_{\lambda[\rho\mu\nu]}$ | 9 | zero |
| $\tilde{\mathring{R}}^{(Q)}_{\lambda[\rho\mu\nu]}$ | 9 | zero |
| $\tilde{\mathring{R}}_{(\mu\nu)}$ | 9 | Ricci traceless $\mathring{R}_{\mu\nu}$ |
| $\hat{\mathring{R}}^{(Q)}_{(\mu\nu)}$ | 9 | zero |
| $\tilde{R}^{(T)}_{[\mu\nu]}$ | 6 | zero |
| $\hat{R}^{(Q)}_{[\mu\nu]}$ | 6 | zero |
| $\tilde{R}^{\lambda}{}_{\lambda\mu\nu}$ | 6 | zero |
| $\tilde{R}$ | 1 | Ricci scalar $R$ |
| $*\tilde{R}$ | 1 | zero |

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- Using the 11 building blocks, one can express the curvature tensor into its irreducible decomposition.

# Curvature decomposition in Metric-Affine geometry

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- Antisymmetric components
$\tilde{R}_{[\lambda\rho]\mu\nu} = \tilde{W}_{\lambda\rho\mu\nu} = {}^{(1)}\tilde{W}_{\lambda\rho\mu\nu} + {}^{(2)}\tilde{W}_{\lambda\rho\mu\nu} + {}^{(3)}\tilde{W}_{\lambda\rho\mu\nu} + {}^{(4)}\tilde{W}_{\lambda\rho\mu\nu} + {}^{(5)}\tilde{W}_{\lambda\rho\mu\nu} + {}^{(6)}\tilde{W}_{\lambda\rho\mu\nu}$:

$$
\begin{aligned}
{}^{(1)}\tilde{W}_{\lambda\rho\mu\nu} &= \tilde{W}_{\lambda\rho\mu\nu} - \sum_{i=2}^{6} {}^{(i)}\tilde{W}_{\lambda\rho\mu\nu}\,, \\
{}^{(2)}\tilde{W}_{\lambda\rho\mu\nu} &= \frac{3}{4}\left(\cancel{\tilde{R}}^{(T)}_{\lambda[\rho\mu\nu]} + \cancel{\tilde{R}}^{(T)}_{\nu[\lambda\rho\mu]} - \cancel{\tilde{R}}^{(T)}_{\rho[\lambda\mu\nu]} - \cancel{\tilde{R}}^{(T)}_{\mu[\lambda\rho\nu]}\right) + \frac{1}{2}\left(\cancel{\tilde{R}}^{(Q)}_{\mu[\lambda\rho\nu]} - \cancel{\tilde{R}}^{(Q)}_{\nu[\lambda\rho\mu]}\right), \\
{}^{(3)}\tilde{W}_{\lambda\rho\mu\nu} &= -\frac{1}{24}\,{}^{*}\tilde{R}\,\varepsilon_{\lambda\rho\mu\nu}\,, \qquad {}^{(6)}\tilde{W}_{\lambda\rho\mu\nu} = \frac{1}{6}\,\tilde{R}\,g_{\lambda[\mu}g_{\nu]\rho}\,, \\
{}^{(4)}\tilde{W}_{\lambda\rho\mu\nu} &= \frac{1}{4}\Big[g_{\lambda\mu}\left(2\cancel{\tilde{R}}_{(\rho\nu)} + \cancel{\hat{R}}^{(Q)}_{(\rho\nu)}\right) + g_{\rho\nu}\left(2\cancel{\tilde{R}}_{(\lambda\mu)} + \cancel{\hat{R}}^{(Q)}_{(\lambda\mu)}\right) \\
&\quad - g_{\lambda\nu}\left(2\cancel{\tilde{R}}_{(\rho\mu)} + \cancel{\hat{R}}^{(Q)}_{(\rho\mu)}\right) - g_{\rho\mu}\left(2\cancel{\tilde{R}}_{(\lambda\nu)} + \cancel{\hat{R}}^{(Q)}_{(\lambda\nu)}\right)\Big], \\
{}^{(5)}\tilde{W}_{\lambda\rho\mu\nu} &= \frac{1}{4}\Big[g_{\lambda\mu}\left(2\tilde{R}^{(T)}_{[\rho\nu]} + \hat{R}^{(Q)}_{[\rho\nu]}\right) + g_{\rho\nu}\left(2\tilde{R}^{(T)}_{[\lambda\mu]} + \hat{R}^{(Q)}_{[\lambda\mu]}\right) + \tilde{R}^{\sigma}{}_{\sigma\lambda[\mu}g_{\nu]\rho} \\
&\quad - g_{\lambda\nu}\left(2\tilde{R}^{(T)}_{[\rho\mu]} + \hat{R}^{(Q)}_{[\rho\mu]}\right) - g_{\rho\mu}\left(2\tilde{R}^{(T)}_{[\lambda\nu]} + \hat{R}^{(Q)}_{[\lambda\nu]}\right) - \tilde{R}^{\sigma}{}_{\sigma\rho[\mu}g_{\nu]\lambda}\Big].
\end{aligned}
$$

# Curvature decomposition in Metric-Affine geometry

- Using the 11 building blocks, one can express the curvature tensor into its irreducible decomposition.
- Antisymmetric components
$\tilde{R}_{[\lambda\rho]\mu\nu} = \tilde{W}_{\lambda\rho\mu\nu} = {}^{(1)}\tilde{W}_{\lambda\rho\mu\nu} + {}^{(2)}\tilde{W}_{\lambda\rho\mu\nu} + {}^{(3)}\tilde{W}_{\lambda\rho\mu\nu} + {}^{(4)}\tilde{W}_{\lambda\rho\mu\nu} + {}^{(5)}\tilde{W}_{\lambda\rho\mu\nu} + {}^{(6)}\tilde{W}_{\lambda\rho\mu\nu}$:

$$
\begin{aligned}
{}^{(1)}\tilde{W}_{\lambda\rho\mu\nu} &= \tilde{W}_{\lambda\rho\mu\nu} - \sum_{i=2}^{6} {}^{(i)}\tilde{W}_{\lambda\rho\mu\nu}\,, \\
{}^{(2)}\tilde{W}_{\lambda\rho\mu\nu} &= \frac{3}{4}\left(\cancel{\tilde{R}}^{(T)}_{\lambda[\rho\mu\nu]} + \cancel{\tilde{R}}^{(T)}_{\nu[\lambda\rho\mu]} - \cancel{\tilde{R}}^{(T)}_{\rho[\lambda\mu\nu]} - \cancel{\tilde{R}}^{(T)}_{\mu[\lambda\rho\nu]}\right) + \frac{1}{2}\left(\cancel{\tilde{R}}^{(Q)}_{\mu[\lambda\rho\nu]} - \cancel{\tilde{R}}^{(Q)}_{\nu[\lambda\rho\mu]}\right), \\
{}^{(3)}\tilde{W}_{\lambda\rho\mu\nu} &= -\frac{1}{24}\,{}^{*}\tilde{R}\,\varepsilon_{\lambda\rho\mu\nu}\,, \quad {}^{(6)}\tilde{W}_{\lambda\rho\mu\nu} = \frac{1}{6}\,\tilde{R}\,g_{\lambda[\mu}g_{\nu]\rho}\,, \\
{}^{(4)}\tilde{W}_{\lambda\rho\mu\nu} &= \frac{1}{4}\Big[g_{\lambda\mu}\left(2\cancel{\tilde{R}}_{(\rho\nu)} + \cancel{\hat{R}}^{(Q)}_{(\rho\nu)}\right) + g_{\rho\nu}\left(2\cancel{\tilde{R}}_{(\lambda\mu)} + \cancel{\hat{R}}^{(Q)}_{(\lambda\mu)}\right) \\
&\quad - g_{\lambda\nu}\left(2\cancel{\tilde{R}}_{(\rho\mu)} + \cancel{\hat{R}}^{(Q)}_{(\rho\mu)}\right) - g_{\rho\mu}\left(2\cancel{\tilde{R}}_{(\lambda\nu)} + \cancel{\hat{R}}^{(Q)}_{(\lambda\nu)}\right)\Big], \\
{}^{(5)}\tilde{W}_{\lambda\rho\mu\nu} &= \frac{1}{4}\Big[g_{\lambda\mu}\left(2\tilde{R}^{(T)}_{[\rho\nu]} + \hat{R}^{(Q)}_{[\rho\nu]}\right) + g_{\rho\nu}\left(2\tilde{R}^{(T)}_{[\lambda\mu]} + \hat{R}^{(Q)}_{[\lambda\mu]}\right) + \tilde{R}^{\sigma}{}_{\sigma\lambda[\mu}g_{\nu]\rho} \\
&\quad - g_{\lambda\nu}\left(2\tilde{R}^{(T)}_{[\rho\mu]} + \hat{R}^{(Q)}_{[\rho\mu]}\right) - g_{\rho\mu}\left(2\tilde{R}^{(T)}_{[\lambda\nu]} + \hat{R}^{(Q)}_{[\lambda\nu]}\right) - \tilde{R}^{\sigma}{}_{\sigma\rho[\mu}g_{\nu]\lambda}\Big].
\end{aligned}
$$

- Note that the generalised Weyl tensor ${}^{(1)}\tilde{W}_{\lambda\rho\mu\nu}$ has the same symmetries as the Riemannian Weyl tensor ${}^{(1)}\tilde{W}_{\lambda\rho\mu\nu} = -\,{}^{(1)}\tilde{W}_{\rho\lambda\mu\nu} = -\,{}^{(1)}\tilde{W}_{\lambda\rho\nu\mu}\,, {}^{(1)}\tilde{W}_{\lambda[\rho\mu\nu]} = {}^{(1)}\tilde{W}^{\lambda}{}_{\mu\lambda\nu} = 0$.

# Curvature decomposition in Metric-Affine geometry

- Symmetric components
$\tilde{R}_{(\lambda\rho)\mu\nu} = \tilde{Z}_{\lambda\rho\mu\nu} = {}^{(1)}\tilde{Z}_{\lambda\rho\mu\nu} + {}^{(2)}\tilde{Z}_{\lambda\rho\mu\nu} + {}^{(3)}\tilde{Z}_{\lambda\rho\mu\nu} + {}^{(4)}\tilde{Z}_{\lambda\rho\mu\nu} + {}^{(5)}\tilde{Z}_{\lambda\rho\mu\nu}$:

$$
\begin{aligned}
{}^{(1)}\tilde{Z}_{\lambda\rho\mu\nu} &= \tilde{Z}_{\lambda\rho\mu\nu} - \sum_{i=2}^{5} {}^{(i)}\tilde{Z}_{\lambda\rho\mu\nu}\,, \\
{}^{(2)}\tilde{Z}_{\lambda\rho\mu\nu} &= \frac{1}{4}\left(\tilde{\not{R}}^{(Q)}_{\lambda[\rho\mu\nu]} + \tilde{\not{R}}^{(Q)}_{\rho[\lambda\mu\nu]}\right), \\
{}^{(3)}\tilde{Z}_{\lambda\rho\mu\nu} &= \frac{1}{6}\left(g_{\lambda\nu}\hat{R}^{(Q)}_{[\rho\mu]} + g_{\rho\nu}\hat{R}^{(Q)}_{[\lambda\mu]} - g_{\lambda\mu}\hat{R}^{(Q)}_{[\rho\nu]} - g_{\rho\mu}\hat{R}^{(Q)}_{[\lambda\nu]} + g_{\lambda\rho}\hat{R}^{(Q)}_{[\mu\nu]}\right), \\
{}^{(4)}\tilde{Z}_{\lambda\rho\mu\nu} &= \frac{1}{4}g_{\lambda\rho}\tilde{R}^{\sigma}{}_{\sigma\mu\nu}\,, \\
{}^{(5)}\tilde{Z}_{\lambda\rho\mu\nu} &= \frac{1}{8}\left(g_{\lambda\nu}\hat{\not{R}}^{(Q)}_{(\rho\mu)} + g_{\rho\nu}\hat{\not{R}}^{(Q)}_{(\lambda\mu)} - g_{\lambda\mu}\hat{\not{R}}^{(Q)}_{(\rho\nu)} - g_{\rho\mu}\hat{\not{R}}^{(Q)}_{(\lambda\nu)}\right).
\end{aligned}
$$

# Curvature decomposition in Metric-Affine geometry

- Symmetric components
$\tilde{R}_{(\lambda\rho)\mu\nu} = \tilde{Z}_{\lambda\rho\mu\nu} = {}^{(1)}\tilde{Z}_{\lambda\rho\mu\nu} + {}^{(2)}\tilde{Z}_{\lambda\rho\mu\nu} + {}^{(3)}\tilde{Z}_{\lambda\rho\mu\nu} + {}^{(4)}\tilde{Z}_{\lambda\rho\mu\nu} + {}^{(5)}\tilde{Z}_{\lambda\rho\mu\nu}$:

$$
\begin{aligned}
{}^{(1)}\tilde{Z}_{\lambda\rho\mu\nu} &= \tilde{Z}_{\lambda\rho\mu\nu} - \sum_{i=2}^{5} {}^{(i)}\tilde{Z}_{\lambda\rho\mu\nu}\,, \\
{}^{(2)}\tilde{Z}_{\lambda\rho\mu\nu} &= \frac{1}{4}\left(\tilde{\not{R}}^{(Q)}_{\lambda[\rho\mu\nu]} + \tilde{\not{R}}^{(Q)}_{\rho[\lambda\mu\nu]}\right), \\
{}^{(3)}\tilde{Z}_{\lambda\rho\mu\nu} &= \frac{1}{6}\left(g_{\lambda\nu}\hat{R}^{(Q)}_{[\rho\mu]} + g_{\rho\nu}\hat{R}^{(Q)}_{[\lambda\mu]} - g_{\lambda\mu}\hat{R}^{(Q)}_{[\rho\nu]} - g_{\rho\mu}\hat{R}^{(Q)}_{[\lambda\nu]} + g_{\lambda\rho}\hat{R}^{(Q)}_{[\mu\nu]}\right), \\
{}^{(4)}\tilde{Z}_{\lambda\rho\mu\nu} &= \frac{1}{4} g_{\lambda\rho}\tilde{R}^{\sigma}{}_{\sigma\mu\nu}\,, \\
{}^{(5)}\tilde{Z}_{\lambda\rho\mu\nu} &= \frac{1}{8}\left(g_{\lambda\nu}\hat{\not{R}}^{(Q)}_{(\rho\mu)} + g_{\rho\nu}\hat{\not{R}}^{(Q)}_{(\lambda\mu)} - g_{\lambda\mu}\hat{\not{R}}^{(Q)}_{(\rho\nu)} - g_{\rho\mu}\hat{\not{R}}^{(Q)}_{(\lambda\nu)}\right).
\end{aligned}
$$

- The tensor ${}^{(2)}\tilde{Z}_{\lambda\rho\mu\nu}$ satisfies ${}^{(2)}\tilde{Z}_{(\lambda\rho\mu)\nu} = 0\,, {}^{(2)}\tilde{Z}^{\lambda}{}_{\lambda\mu\nu} = 0\,, {}^{(2)}\tilde{Z}^{\lambda}{}_{\mu\lambda\nu} = 0.$

# Curvature decomposition in Metric-Affine geometry

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$\tilde{R}_{(\lambda\rho)\mu\nu} = \tilde{Z}_{\lambda\rho\mu\nu} = {}^{(1)}\tilde{Z}_{\lambda\rho\mu\nu} + {}^{(2)}\tilde{Z}_{\lambda\rho\mu\nu} + {}^{(3)}\tilde{Z}_{\lambda\rho\mu\nu} + {}^{(4)}\tilde{Z}_{\lambda\rho\mu\nu} + {}^{(5)}\tilde{Z}_{\lambda\rho\mu\nu}$:

$$
\begin{aligned}
{}^{(1)}\tilde{Z}_{\lambda\rho\mu\nu} &= \tilde{Z}_{\lambda\rho\mu\nu} - \sum_{i=2}^{5} {}^{(i)}\tilde{Z}_{\lambda\rho\mu\nu}\,, \\
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{}^{(3)}\tilde{Z}_{\lambda\rho\mu\nu} &= \frac{1}{6}\left(g_{\lambda\nu}\hat{R}^{(Q)}_{[\rho\mu]} + g_{\rho\nu}\hat{R}^{(Q)}_{[\lambda\mu]} - g_{\lambda\mu}\hat{R}^{(Q)}_{[\rho\nu]} - g_{\rho\mu}\hat{R}^{(Q)}_{[\lambda\nu]} + g_{\lambda\rho}\hat{R}^{(Q)}_{[\mu\nu]}\right), \\
{}^{(4)}\tilde{Z}_{\lambda\rho\mu\nu} &= \frac{1}{4}g_{\lambda\rho}\tilde{R}^{\sigma}{}_{\sigma\mu\nu}\,, \\
{}^{(5)}\tilde{Z}_{\lambda\rho\mu\nu} &= \frac{1}{8}\left(g_{\lambda\nu}\cancel{\hat{R}}^{(Q)}_{(\rho\mu)} + g_{\rho\nu}\cancel{\hat{R}}^{(Q)}_{(\lambda\mu)} - g_{\lambda\mu}\cancel{\hat{R}}^{(Q)}_{(\rho\nu)} - g_{\rho\mu}\cancel{\hat{R}}^{(Q)}_{(\lambda\nu)}\right).
\end{aligned}
$$

- The tensor ${}^{(2)}\tilde{Z}_{\lambda\rho\mu\nu}$ satisfies ${}^{(2)}\tilde{Z}_{(\lambda\rho\mu)\nu} = 0\,, {}^{(2)}\tilde{Z}^{\lambda}{}_{\lambda\mu\nu} = 0\,, {}^{(2)}\tilde{Z}^{\lambda}{}_{\mu\lambda\nu} = 0.$
- Recall that if $Q_{\alpha\beta\mu} = 0$, then $\tilde{R}_{(\lambda\rho)\mu\nu} = \tilde{Z}_{\lambda\rho\mu\nu} = 0.$

# Curvature decomposition in Metric-Affine geometry

- Symmetric components
$\tilde{R}_{(\lambda\rho)\mu\nu} = \tilde{Z}_{\lambda\rho\mu\nu} = {}^{(1)}\tilde{Z}_{\lambda\rho\mu\nu} + {}^{(2)}\tilde{Z}_{\lambda\rho\mu\nu} + {}^{(3)}\tilde{Z}_{\lambda\rho\mu\nu} + {}^{(4)}\tilde{Z}_{\lambda\rho\mu\nu} + {}^{(5)}\tilde{Z}_{\lambda\rho\mu\nu}$:

$$
\begin{aligned}
{}^{(1)}\tilde{Z}_{\lambda\rho\mu\nu} &= \tilde{Z}_{\lambda\rho\mu\nu} - \sum_{i=2}^{5} {}^{(i)}\tilde{Z}_{\lambda\rho\mu\nu}\,, \\
{}^{(2)}\tilde{Z}_{\lambda\rho\mu\nu} &= \frac{1}{4}\left(\cancel{\tilde{R}}^{(Q)}_{\lambda[\rho\mu\nu]} + \cancel{\tilde{R}}^{(Q)}_{\rho[\lambda\mu\nu]}\right), \\
{}^{(3)}\tilde{Z}_{\lambda\rho\mu\nu} &= \frac{1}{6}\left(g_{\lambda\nu}\hat{R}^{(Q)}_{[\rho\mu]} + g_{\rho\nu}\hat{R}^{(Q)}_{[\lambda\mu]} - g_{\lambda\mu}\hat{R}^{(Q)}_{[\rho\nu]} - g_{\rho\mu}\hat{R}^{(Q)}_{[\lambda\nu]} + g_{\lambda\rho}\hat{R}^{(Q)}_{[\mu\nu]}\right), \\
{}^{(4)}\tilde{Z}_{\lambda\rho\mu\nu} &= \frac{1}{4}g_{\lambda\rho}\tilde{R}^{\sigma}{}_{\sigma\mu\nu}\,, \\
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\end{aligned}
$$

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- Recall that if $Q_{\alpha\beta\mu} = 0$, then $\tilde{R}_{(\lambda\rho)\mu\nu} = \tilde{Z}_{\lambda\rho\mu\nu} = 0.$
- Special case that we will be interested: Weyl-Cartan geometry which is when $Q_{\alpha\beta\mu} = \frac{1}{4}g_{\mu\nu}Q_{\lambda\rho}{}^{\rho} + \cancel{Q}_{\lambda\mu\nu} = \frac{1}{4}g_{\mu\nu}Q_{\lambda\rho}{}^{\rho}$ (its traceless part is zero).

# Algebraic classification in Weyl-Cartan geometry

- As said before, Weyl-Cartan geometries are the ones where there is curvature, torsion and nonmetricity but $Q_{\alpha\beta\mu} = \frac{1}{4} g_{\mu\nu} Q_{\lambda\rho}{}^{\rho} + \mathring{Q}_{\lambda\mu\nu} = W_{\alpha} g_{\beta\mu}$, where $W_{\mu} = \frac{1}{4} Q_{\mu\nu}{}^{\nu}$ is called the Weyl vector.

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- In this particular geometry, only ${}^{(4)}\tilde{Z}_{\lambda\rho\mu\nu} = \frac{1}{4} g_{\lambda\rho} \tilde{R}^{\sigma}{}_{\sigma\mu\nu} = 4\nabla_{[\nu} W_{\mu]}$ is different from zero and all ${}^{(i)}\tilde{W}_{\lambda\rho\mu\nu}$ are also non-vanishing.

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- Then, we only have 7 building blocks:

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- In this particular geometry, only ${}^{(4)}\tilde{Z}_{\lambda\rho\mu\nu} = \frac{1}{4} g_{\lambda\rho} \tilde{R}^{\sigma}{}_{\sigma\mu\nu} = 4\nabla_{[\nu} W_{\mu]}$ is different from zero and all ${}^{(i)}\tilde{W}_{\lambda\rho\mu\nu}$ are also non-vanishing.
- Then, we only have 7 building blocks:

## Building blocks for Weyl-Cartan geometry

$$\left\{ {}^{(1)}\tilde{W}_{\lambda\rho\mu\nu}, \not{\tilde{R}}^{(T)}_{\lambda[\rho\mu\nu]}, \not{\tilde{R}}_{(\mu\nu)}, \tilde{R}^{(T)}_{[\mu\nu]}, \tilde{R}^{\lambda}{}_{\lambda\mu\nu}, \tilde{R}, *\tilde{R} \right\} .$$

$$\#10 + \#9 + \#9 + \#6 + \#6 + \#1 + \#1 = \#42 \,\text{dof}$$

# Algebraic classification in Weyl-Cartan geometry

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## Building blocks for Weyl-Cartan geometry

$$\left\{ {}^{(1)}\tilde{W}_{\lambda\rho\mu\nu}, \tilde{\cancel{R}}^{(T)}_{\lambda[\rho\mu\nu]}, \tilde{\cancel{R}}_{(\mu\nu)}, \tilde{R}^{(T)}_{[\mu\nu]}, \tilde{R}^{\lambda}{}_{\lambda\mu\nu}, \tilde{R}, *\tilde{R} \right\}.$$

$$\#10 + \#9 + \#9 + \#6 + \#6 + \#1 + \#1 = \#42\,\text{dof}$$

- The classification of $\left\{ \tilde{\cancel{R}}^{(T)}_{\lambda[\rho\mu\nu]}, \tilde{\cancel{R}}_{(\mu\nu)} \right\}$ are equivalent since they carry the same number of dof $\#9$.

# Algebraic classification in Weyl-Cartan geometry

- As said before, Weyl-Cartan geometries are the ones where there is curvature, torsion and nonmetricity but $Q_{\alpha\beta\mu} = \frac{1}{4} g_{\mu\nu} Q_{\lambda\rho}{}^{\rho} + \cancel{Q}_{\lambda\mu\nu} = W_{\alpha} g_{\beta\mu}$, where $W_{\mu} = \frac{1}{4} Q_{\mu\nu}{}^{\nu}$ is called the Weyl vector.
- In this particular geometry, only ${}^{(4)}\tilde{Z}_{\lambda\rho\mu\nu} = \frac{1}{4} g_{\lambda\rho} \tilde{R}^{\sigma}{}_{\sigma\mu\nu} = 4\nabla_{[\nu} W_{\mu]}$ is different from zero and all ${}^{(i)}\tilde{W}_{\lambda\rho\mu\nu}$ are also non-vanishing.
- Then, we only have 7 building blocks:

## Building blocks for Weyl-Cartan geometry

$$\left\{ {}^{(1)}\tilde{W}_{\lambda\rho\mu\nu}, \tilde{\cancel{R}}^{(T)}_{\lambda[\rho\mu\nu]}, \tilde{\cancel{R}}_{(\mu\nu)}, \tilde{R}^{(T)}_{[\mu\nu]}, \tilde{R}^{\lambda}{}_{\lambda\mu\nu}, \tilde{R}, *\tilde{R} \right\}.$$

$$\#10 + \#9 + \#9 + \#6 + \#6 + \#1 + \#1 = \#42 \,\text{dof}$$

- The classification of $\left\{ \tilde{\cancel{R}}^{(T)}_{\lambda[\rho\mu\nu]}, \tilde{\cancel{R}}_{(\mu\nu)} \right\}$ are equivalent since they carry the same number of dof $\#9$.
- Further, $\left\{ \tilde{R}^{(T)}_{[\mu\nu]}, \tilde{R}^{\lambda}{}_{\lambda\mu\nu} \right\}$ are equivalent since they also carry the same number of dof $\#6$.

# Algebraic types of ${}^{(1)}\tilde{W}_{\lambda\rho\mu\nu}$ ($\#10$ dof)

| Algebraic type | Segre characteristic | Invariants | Constraints with the PNDs |
|---|---|---|---|
| $I$ | $[1\,1\,1]$ | $I^3 \neq 12J^2$ | No further constraints |
| $II$ | $[2\,1]$ | $I^3 = 12J^2$ | ${}^{(1)}\tilde{W}_{\lambda\rho\mu[\nu}l_{\omega]}l^\rho l^\mu = 0$ |
| $D$ | $[(1\,1)\,1]$ | $I^3 = 12J^2$ | ${}^{(1)}\tilde{W}_{\lambda\rho\mu[\nu}k_{\omega]}k^\rho k^\mu = {}^{(1)}\tilde{W}_{\lambda\rho\mu[\nu}l_{\omega]}l^\rho l^\mu = 0$ |
| $III$ | $[3]$ | $I = J = 0$ | ${}^{(1)}\tilde{W}_{\lambda\rho\mu[\nu}l_{\omega]}l^\mu = 0$ |
| $N$ | $[(2\,1)]$ | $I = J = 0$ | ${}^{(1)}\tilde{W}_{\lambda\rho\mu\nu}l^\mu = 0$ |
| $O$ | $[-]$ | $I = J = 0$ | ${}^{(1)}\tilde{W}_{\lambda\rho\mu\nu} = 0$ |

Table: Algebraic types for the tensor ${}^{(1)}\tilde{W}_{\lambda\rho\mu\nu}$. ( S. Bahamonde and J. Gigante Valcarcel, "Algebraic classification of the gravitational field in Weyl-Cartan spacetimes," Phys. Rev. D **108** (2023) no.4, 044037.)

# Algebraic types of $\tilde{R}^{(T)}_{[\mu\nu]}$ and $\tilde{R}^{\lambda}{}_{\lambda\mu\nu}$ ($\#9$ dof each)

| Algebraic type | Segre characteristic | Invariants | Constraints with the PNDs |
|---|---|---|---|
| $I$ | $[1\,1]$ | $X \neq 0$ | No further constraints |
| $N$ | $[2]$ | $X = 0$ | $\tilde{R}^{(T)}_{[\mu\nu]} l^{\mu} = 0$ |
| $O$ | $[-]$ | $X = 0$ | $\tilde{R}^{(T)}_{[\mu\nu]} = 0$ |

Table: Algebraic types for the tensor $\tilde{R}^{(T)}_{[\mu\nu]}$.

| Algebraic type | Segre characteristic | Invariants | Constraints with the PNDs |
|---|---|---|---|
| $I$ | $[1\,1]$ | $Y \neq 0$ | No further constraints |
| $N$ | $[2]$ | $Y = 0$ | $\tilde{R}^{\lambda}{}_{\lambda\mu\nu} l^{\mu} = 0$ |
| $O$ | $[-]$ | $Y = 0$ | $\tilde{R}^{\lambda}{}_{\lambda\mu\nu} = 0$ |

Table: Algebraic types for the tensor $\tilde{R}^{\lambda}{}_{\lambda\mu\nu}$.

# Application to stationary and axisymmetric space-times

- MAG model with dynamical torsion in $4$ dimensions[6]:

$$\mathcal{L} = \frac{1}{64\pi}\Big(-4R - 6d_1\tilde{R}_{\lambda[\rho\mu\nu]}\tilde{R}^{\lambda[\rho\mu\nu]} - 9d_1\tilde{R}_{\lambda[\rho\mu\nu]}\tilde{R}^{\mu[\lambda\nu\rho]} + 8\,d_1\tilde{R}_{[\mu\nu]}\tilde{R}^{[\mu\nu]}\Big)\,.$$

[6] S. Bahamonde, J. Chevrier and J. G. Valcarcel, JCAP **02**, 018 (2023).

[7] S. Bahamonde and J. G. Valcarcel,"Algebraic classification of the gravitational field in Weyl-Cartan spacetimes," Phys. Rev. D **108** (2023) no.4, 044037.

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- In terms of building blocks:

$$\mathcal{L} = \frac{1}{64\pi}\Big(-4R - 6d_1\tilde{\not{R}}^{(T)}_{\lambda[\rho\mu\nu]}\tilde{\not{R}}^{(T)\,\lambda[\rho\mu\nu]} - 9d_1\tilde{\not{R}}^{(T)}_{\lambda[\rho\mu\nu]}\tilde{\not{R}}^{(T)\,\mu[\lambda\nu\rho]} + 2d_1\tilde{R}^{(T)}_{[\mu\nu]}\tilde{R}^{(T)\,[\mu\nu]} - \frac{d_1}{8} * \tilde{R}^2\Big)\,.$$

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- In terms of building blocks:

$$\mathcal{L} = \frac{1}{64\pi}\Big(-4R - 6d_1\tilde{R}^{(T)}_{\lambda[\rho\mu\nu]}\tilde{R}^{(T)\,\lambda[\rho\mu\nu]} - 9d_1\tilde{R}^{(T)}_{\lambda[\rho\mu\nu]}\tilde{R}^{(T)\,\mu[\lambda\nu\rho]} + 2d_1\tilde{R}^{(T)}_{[\mu\nu]}\tilde{R}^{(T)\,[\mu\nu]} - \frac{d_1}{8} * \tilde{R}^2\Big)\,.$$

- We showed that the field strength tensors of torsion cannot be doubly aligned with the principal null directions of the Riemannian Weyl tensor in generic stationary and axisymmetric space-times[7]:

$$W_{\lambda\rho\mu[\nu}k_{\omega]}k^\rho k^\mu = W_{\lambda\rho\mu[\nu}l_{\omega]}l^\rho l^\mu = 0\,,$$
$$\big(\tilde{R}^{(T)}_{[\mu\nu]}k_\lambda - \tilde{R}^{(T)}_{[\mu\lambda]}k_\nu\big)k^\mu = \big(\tilde{R}^{(T)}_{[\mu\nu]}l_\lambda - \tilde{R}^{(T)}_{[\mu\lambda]}l_\nu\big)l^\mu = 0\,.$$

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[7] S. Bahamonde and J. G. Valcarcel,"Algebraic classification of the gravitational field in Weyl-Cartan spacetimes," Phys. Rev. D **108** (2023) no.4, 044037.

# Overview of the Talk


1. Introduction to Metric-affine gravity
2. Metric-Affine gravity
   - Basic geometrical quantities
3. MAG models with dynamical torsion and nonmetricity (Weyl only)
   - Spherically symmetry in Weyl-Cartan geometry
   - Axial symmetry in Weyl-Cartan geometry
4. Algebraic classification in Weyl-Cartan
5. MAG models torsion and the complete nonmetricity (Traceless and Weyl)
6. Cosmological perturbations in MAG

# The traceless part of nonmetricity and shears

- Recall that the complete nonmetricity is $Q_{\lambda\mu\nu} = g_{\mu\nu}W_\lambda + \mathring{Q}_{\lambda\mu\nu}$, where $\mathring{Q}_{\lambda\mu\nu}$ being its traceless part.

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- If this quantity is different to zero, when we parallel transport a vector, not only its norm changes but also its angle.
- It is invariant under shears transformations.
- Shears: Deformations without changing the volume.
- Up to now, there were not exact solutions with shears in MAG.

# MAG theory with shears

- Let us first consider a simple model where torsion is not propagating and the traceless part of nonmetricity is dynamical:

$$S = \frac{1}{16\pi}\int d^4x\sqrt{-g}\Big[-R + 2f_1\tilde{R}_{(\lambda\rho)\mu\nu}\tilde{R}^{(\lambda\rho)\mu\nu} + 2f_2\left(\tilde{R}_{(\mu\nu)} - \hat{R}_{(\mu\nu)}\right)\left(\tilde{R}^{(\mu\nu)} - \hat{R}^{(\mu\nu)}\right)\Big],$$

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- As can be seen, the propagation of the nonmetricity field described in the action is carried out by the symmetric part of the curvature tensor and its symmetric contraction:

$$\tilde{R}^{(\lambda\rho)}{}_{\mu\nu} = \tilde{\nabla}_{[\nu}Q_{\mu]}{}^{\lambda\rho} + \frac{1}{2}T^{\sigma}{}_{\mu\nu}Q_{\sigma}{}^{\lambda\rho},$$

$$\tilde{R}_{(\mu\nu)} - \hat{R}_{(\mu\nu)} = \tilde{\nabla}_{(\mu}Q^{\lambda}{}_{\nu)\lambda} - \tilde{\nabla}_{\lambda}Q_{(\mu\nu)}{}^{\lambda} - Q^{\lambda\rho}{}_{\lambda}Q_{(\mu\nu)\rho} + Q_{\lambda\rho(\mu}Q_{\nu)}{}^{\lambda\rho} + T_{\lambda\rho(\mu}Q^{\lambda\rho}{}_{\nu)},$$

which in turn constitute deviations from the third Bianchi of GR.

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# Spherical symmetry with nonmetricity and torsion

- Metric, torsion and nonmetricity in spherically symmetric space-times ($\#2 + \#8 + \#12 = \#22$):

$$\mathcal{L}_\xi g_{\mu\nu} = \mathcal{L}_\xi T^\lambda{}_{\mu\nu} = \mathcal{L}_\xi Q_{\alpha\mu\nu} = 0 \implies \mathcal{L}_\xi \tilde{R}_{\lambda\rho\mu\nu} = 0$$

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- Nonmetricity now contains all the 12 dof:

$$\begin{aligned}
&Q_{ttt} = q_1(r)\,, \quad Q_{trr} = q_2(r)\,, \quad Q_{ttr} = q_3(r)\,,\\
&Q_{t\vartheta\vartheta} = Q_{t\varphi\varphi}\csc^2\vartheta = q_4(r)\,, \quad Q_{rtt} = q_5(r)\,, \quad Q_{rrr} = q_6(r)\,,\\
&Q_{rtr} = q_7(r)\,, \quad Q_{r\vartheta\vartheta} = Q_{r\varphi\varphi}\csc^2\vartheta = q_8(r)\,,\\
&Q_{\vartheta t\vartheta} = Q_{\varphi t\varphi}\csc^2\vartheta = q_9(r)\,, \quad Q_{\vartheta r\vartheta} = Q_{\varphi r\varphi}\csc^2\vartheta = q_{10}(r)\,,\\
&Q_{\vartheta t\varphi} = -\,Q_{\varphi t\vartheta} = q_{11}(r)\sin\vartheta\,, \quad Q_{\vartheta r\varphi} = -\,Q_{\varphi r\vartheta} = q_{12}(r)\sin\vartheta\,,
\end{aligned}$$

whereas the metric and torsion are the same as before.

# How to find a solution with all of these dof?

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q_1(r) &= \frac{\Psi_1(r)}{r^2}\left(r^2 q_2(r)\Psi_2(r) + 2q_4(r)\right) , \\
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  2. We imposed $N_{[\lambda\rho]\mu} = 0$ which is equivalent to $T_{\lambda\mu\nu} = Q_{[\mu\nu]\lambda}$:
     $\rightarrow$ Shear transformations in the general linear group involves the part of the anholonomic connection describing a shear current or charge to take values in the **symmetric traceless part** of the Lie algebra.

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  3. We demand the corresponding torsion and nonmetricity scalars of the solution to be regular.
- After following these three steps we end up with 2 dof (metric)+ 5 dof (torsion/nonmetricity) which is only 7 dof.

# New solution only with shears

- By plugging these conditions in the field equations, there are several branches but only one has solutions with dynamical shears. This gives

$$g_{tt} = -1/g_{rr} \equiv \Psi(r) = 1 - \frac{2m}{r} - \frac{2f_1\kappa_{\rm sh}^2}{r^2}\,, \text{ with } f_2 = -f_1/4.$$

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  where $\kappa_{\rm sh}$ is interpreted as a new charge, "shear charge".
- Kinetic part of the Lagrangian indicates that $f_1 \leq 0$

## Reissner-Nordström-like solutions with spin, dilation and shear charges

- After finding the shear part alone, we found a theory having the first solution (with spin+dilation) plus the second (with only shears).

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- The action of the full model is

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S = \frac{1}{64\pi}\int\Big[ & -4R - 6d_1\tilde{R}_{\lambda[\rho\mu\nu]}\tilde{R}^{\lambda[\rho\mu\nu]} - 9d_1\tilde{R}_{\lambda[\rho\mu\nu]}\tilde{R}^{\mu[\lambda\nu\rho]} \\
& + 2d_1\left(\tilde{R}_{[\mu\nu]} + \hat{R}_{[\mu\nu]}\right)\left(\tilde{R}^{[\mu\nu]} + \hat{R}^{[\mu\nu]}\right) + 18d_1\tilde{R}_{\lambda[\rho\mu\nu]}\tilde{R}^{(\lambda\rho)\mu\nu} \\
& - 3d_1\tilde{R}_{(\lambda\rho)\mu\nu}\tilde{R}^{(\lambda\rho)\mu\nu} + 6d_1\tilde{R}_{(\lambda\rho)\mu\nu}\tilde{R}^{(\lambda\mu)\rho\nu} + 2\left(2e_1 - f_1\right)\tilde{R}^{\lambda}{}_{\lambda\mu\nu}\tilde{R}^{\rho}{}_{\rho}{}^{\mu\nu} \\
& + 8f_1\tilde{R}_{(\lambda\rho)\mu\nu}\tilde{R}^{(\lambda\rho)\mu\nu} - 2f_1\left(\tilde{R}_{(\mu\nu)} - \hat{R}_{(\mu\nu)}\right)\left(\tilde{R}^{(\mu\nu)} - \hat{R}^{(\mu\nu)}\right) \\
& + 3\left(1 - 2a_2\right)T_{[\lambda\mu\nu]}T^{[\lambda\mu\nu]}\Big]d^4x\sqrt{-g}\,.
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- When traceless part of nonmetricity is zero, the above action coincides with the first one.

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- In this case, all nonmetricity components are fully set by the field equations (remember that in the shear case, one component was free)
- The solution gives us the following metric

$$ds^2 = \Psi_1(r)\, dt^2 - \frac{dr^2}{\Psi_2(r)} - r^2 \left(d\theta_1^2 + \sin^2\theta_1 d\theta_2^2\right) .$$

with

$$\Psi_1(r) = \Psi_2(r) = 1 - \frac{2m}{r} + \frac{d_1\kappa_{\rm s}^2 - 4e_1\kappa_{\rm d}^2 - 2f_1\kappa_{\rm sh}^2}{r^2} ,$$

having the three possible charges: spin, dilation and shear.

- On the other hand, the solution can also be trivially generalised to include the cosmological constant and Coulomb electromagnetic fields with electric and magnetic charges $q_e$ and $q_m$, which are decoupled from torsion under the assumption of the minimal coupling principle. This natural extension is then described by a Reissner-Nordström-de Sitter-like geometry

$$\Psi(r) = 1 - \frac{2m}{r} + \frac{d_1\kappa_{\rm s}^2 - 4e_1\kappa_{\rm d}^2 - 2f_1\kappa_{\rm sh}^2 + q_e^2 + q_m^2}{r^2} + \frac{\Lambda}{3}r^2 \,,$$

which turns out to represent **the broadest family of static and spherically symmetric black hole solutions obtained in MAG** so far.

# Overview of the Talk


1. Introduction to Metric-affine gravity
2. Metric-Affine gravity
   - Basic geometrical quantities
3. MAG models with dynamical torsion and nonmetricity (Weyl only)
   - Spherically symmetry in Weyl-Cartan geometry
   - Axial symmetry in Weyl-Cartan geometry
4. Algebraic classification in Weyl-Cartan
5. MAG models torsion and the complete nonmetricity (Traceless and Weyl)
6. Cosmological perturbations in MAG

# $3+1$ decomposition - settling notation

- Let us first introduce a projector tensor

$$P_{\mu\nu} \equiv g_{\mu\nu} + n_\mu n_\nu \, ,$$

which is orthogonal ($P_{\mu\nu} n^\mu = 0$) to a timelike vector $n_\mu$ satisfying $n_\mu n^\mu = -1$.

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- Let us introduce arrows on top of quantities to denote traceless-spatial and symmetric quantities, i.e.:

$$\vec{X}_{\mu_1 \cdots \mu_n} \equiv P_{\mu_1}{}^{\alpha_1} \cdots P_{\mu_n}{}^{\alpha_n} X_{\alpha_1 \cdots \alpha_n} \,,$$
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- One can then straightforwardly decompose scalar and vectors:

$$\phi = \phi \,, \quad X_\mu = \vec{X}_\mu - n_\mu X_* \,.$$

# Cosmology in MAG

- Let us assume that the metric, torsion and nonmetricity have the same cosmological symmetries (isotropy and homogeneity)

$$\mathcal{L}_\xi \bar{g}_{\mu\nu} = \mathcal{L}_\xi \bar{T}^\lambda{}_{\mu\nu} = \mathcal{L}_\xi \bar{Q}^\lambda{}_{\mu\nu} = 0\,.$$

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- By solving those equations one get the FLRW metric and torsion+nonmetricity satisfying homogeneity and isotropy:

$$\begin{aligned}
\bar{g} &= -\bar{n}_\mu \bar{n}_\nu \mathrm{d}x^\mu \otimes \mathrm{d}x^\nu + \bar{P}_{\mu\nu} \mathrm{d}x^\mu \otimes \mathrm{d}x^\nu = -N^2 \mathrm{d}t \otimes \mathrm{d}t + a^2 \gamma_{ij} \mathrm{d}x^i \otimes \mathrm{d}x^j\,, \\
\bar{T}^\lambda{}_{\mu\nu} &= 2T_1(t)\, \bar{n}_{[\mu} \bar{P}_{\nu]}{}^\lambda + 2T_2(t)\, \bar{\varepsilon}^\lambda{}_{\mu\nu\rho} \bar{n}^\rho\,, \\
\bar{Q}_{\lambda\mu\nu} &= 2Q_1(t) \bar{n}_\lambda \bar{n}_\mu \bar{n}_\nu + 2Q_2(t) \bar{n}_\lambda \bar{P}_{\mu\nu} + 2Q_3(t) \bar{P}_{\lambda(\mu} \bar{n}_{\nu)}\,,
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where $\gamma_{ij}\mathrm{d}x^i \otimes \mathrm{d}x^j = \frac{\mathrm{d}r^2}{1-Kr^2} + r^2 \mathrm{d}\Omega^2$. Note that there are 5 independent functions coming from Post-Riemannian.

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- Our aim is to perform a $3+1$+SVT-decomposition for all the tensors in MAG up to linear perturbations to then write:

$$g_{\mu\nu} = \bar{g}_{\mu\nu} + \delta g_{\mu\nu}\,, \quad \bar{T}^\lambda{}_{\mu\nu} = \bar{T}^\lambda{}_{\mu\nu} + \delta T^\lambda{}_{\mu\nu}\,, \quad \bar{Q}^\lambda{}_{\mu\nu} = \bar{Q}^\lambda{}_{\mu\nu} + \delta Q^\lambda{}_{\mu\nu}\,.$$

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- The majority of these studies predominantly focus on torsion.
- Usually, it is possible to avoid cosmological singularities and replace them a cosmic bounce.
- Dark energy can be explained by the scalar modes of torsion.
- It is possible to find inflationary models such as Einstein-Cartan couple to Higgs or other more complicated ones which are compatible with observations.

# $3+1$ decomposition for FLRW - metric sector

- A symmetric rank-2 tensor (as the metric) can be easily decomposed as 1 tensor, 1 vector and 2 scalars:

$$X_{(\mu\nu)} = n_\mu n_\nu X_{**} + \frac{1}{3} P_{\mu\nu} X_{***} + \vec{X}_{\mu\nu} - 2 n_{(\mu} \vec{X}_{*\nu)} \, .$$

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- Then, the metric perturbation can be decomposed as

$$\delta g_{\mu\nu} = - \bar{n}_\mu \bar{n}_\nu \alpha + \bar{P}_{\mu\nu} \zeta + \vec{h}_{\mu\nu} - 2 \bar{n}_{(\mu} \vec{\beta}_{\nu)} \, ,$$

in such a way that the full metric tensor is then given by

$$\bar{g}_{\mu\nu} + \delta g_{\mu\nu} = g_{\mu\nu} = \begin{pmatrix} -N^2 \left(1+\alpha\right) & a N \vec{\beta}_i \\ a N \vec{\beta}_j & a^2 \left[ \left(1+\zeta\right) \gamma_{ij} + \vec{h}_{ij} \right] \end{pmatrix} \, .$$

# $3+1$ decomposition for FLRW - Torsion sector

- The irreducible modes of antisymmetric rank-3 tensor (as torsion) can be decomposed as

$$T_\mu = \vec{T}_\mu - n_\mu \phi \,,$$
$$S_\mu = \vec{\mathcal{S}}_\mu - n_\mu \varrho \,,$$
$$t_{\mu\nu\rho} = 2n_{[\nu}\vec{A}_{\rho]\mu} + 2\Big(n_\mu n_{[\nu} + \frac{1}{2}P_{\mu[\nu}\Big)\vec{B}_{\rho]} + \frac{1}{2}\varepsilon_{\nu\rho}{}^{\alpha\beta}\Big[-n_\alpha \vec{\mathcal{A}}_{\beta\mu} + \Big(n_\beta n_\mu + \frac{1}{2}P_{\beta\mu}\Big)\vec{\mathcal{B}}_\alpha\Big] \,.$$

# $3+1$ decomposition for FLRW - Torsion sector

- The irreducible modes of antisymmetric rank-3 tensor (as torsion) can be decomposed as

$$T_\mu = \vec{T}_\mu - n_\mu \phi \,,$$
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- Then, torsion is decomposed as

$$T^\lambda{}_{\mu\nu} = \bar{T}^\lambda{}_{\mu\nu} + \delta T^\lambda{}_{\mu\nu} = \begin{cases} T^0{}_{0i} = -\dfrac{a}{3}\left(\vec{T}_i + 3\vec{B}_i\right) \\ T^0{}_{ij} = -\dfrac{a^2}{6N}\varepsilon_{ijk}\left(\vec{\mathcal{S}}^k + 3\vec{\mathcal{B}}^k\right) \\ T^i{}_{0j} = -N\Big[\vec{A}^i{}_j - \dfrac{1}{3}\delta^i{}_j\left(\bar{\phi}+\phi\right) + \dfrac{1}{6}\varepsilon^i{}_{jk}\Big(\vec{\mathcal{S}}^k - \dfrac{3}{2}\vec{\mathcal{B}}^k\Big)\Big] \\ T^i{}_{jk} = -\dfrac{a}{6}\Big\{4\delta^i{}_{[j}\vec{T}_{k]} - 6\delta^i{}_{[j}\vec{B}_{k]} + \Big[3\varepsilon_{jkl}\vec{\mathcal{A}}^{il} - \varepsilon^i{}_{jk}\left(\bar{\varrho}+\varrho\right)\Big]\Big\} \end{cases} ,$$

where we have introduced the background $T_i$ in the scalars $\phi, \varrho$.

# $3+1$ decomposition for FLRW - Nonmetricity sector

- The irreducible modes of symmetric rank-3 tensor (as nonmetricity) can be decomposed as

$$W_\mu = \vec{W}_\mu - n_\mu \theta \,,$$

$$\Lambda_\mu = \vec{\Lambda}_\mu - n_\mu \psi \,,$$

$$\Omega_{\mu\nu\rho} = 2n_{[\nu}\vec{\mathcal{Q}}_{\rho]\mu} + 2\Big(n_\mu n_{[\nu} + \frac{1}{2}P_{\mu[\nu}\Big)\vec{\mathcal{Y}}_{\rho]} + \frac{1}{2}\varepsilon_{\nu\rho}{}^{\alpha\beta}\Big[-n_\alpha \vec{Q}_{\beta\mu} + \Big(n_\beta n_\mu + \frac{1}{2}P_{\beta\mu}\Big)\vec{Y}_\alpha\Big] \,,$$

$$q_{\mu\nu\rho} = \vec{C}_{\mu\nu\rho} - 3n_{(\mu}\vec{\kappa}_{\nu\rho)} + \frac{3}{5}\big(5n_{(\mu}n_\nu + P_{(\mu\nu}\big)\vec{Z}_{\rho)} - \big(n_{(\mu}n_\nu + P_{(\mu\nu}\big)\, n_{\rho)}\xi \,.$$

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- Then, nonmetricity is decomposed as

$$Q^\lambda{}_{\mu\nu} = \begin{cases} Q_{000} = -N^3\Big[\bar{\theta} + \theta + \frac{3}{4}\,(\bar{\psi} + \psi) - (\bar{\xi} + \xi)\Big] \\ Q_{00i} = -\frac{N^2 a}{2}\Big(\vec{\Lambda}_i - \frac{1}{3}\vec{Y}_i - 2\vec{Z}_i\Big) \\ Q_{0ij} = -\frac{Na^2}{3}\Big\{\vec{Q}_{ij} - 3\vec{\kappa}_{ij} - 3\Big[\bar{\theta} + \theta - \frac{1}{4}\,(\bar{\psi} + \psi) + \frac{1}{3}\,(\bar{\xi} + \xi)\Big]\gamma_{ij}\Big\} \\ Q_{i00} = -N^2 a\Big(\vec{W}_i - \frac{1}{4}\vec{\Lambda}_i + \frac{1}{3}\vec{Y}_i - \vec{Z}_i\Big) \\ Q_{i0j} = \frac{Na^2}{6}\Big\{\big[\vec{Q}_{ij} + 6\vec{\kappa}_{ij} + (3\,(\bar{\psi} + \psi) + 2\,(\bar{\xi} + \xi))\gamma_{ij}\big] - 3\varepsilon_{ijk}\vec{\mathcal{Y}}^k\Big\} \\ Q_{ijk} = a^3\Big\{\vec{C}_{ijk} + \frac{1}{4}(4\vec{W}_i - \vec{\Lambda}_i)\gamma_{jk} + \gamma_{i(j}\vec{\Lambda}_{k)} - \frac{1}{6}(\gamma_{jk}\vec{Y}_i - \gamma_{i(j}\vec{Y}_{k)}) + \frac{3}{5}\gamma_{(ij}\vec{Z}_{k)} + \frac{2}{3}\varepsilon_{li(j}\vec{\mathcal{Q}}^l{}_{k)} \end{cases}$$

where we have introduced the background $Q_i$ in the scalars $\theta, \psi, \xi$.

# Summary of modes

- We can group the modes as:

$$\delta\vec{\mathbf{X}} = \{\alpha, \zeta, \phi, \varrho, \theta, \psi, \xi\}\,,$$
$$\delta\vec{\mathbf{X}}_i = \{\vec{\beta}_i, \vec{T}_i, \vec{\mathcal{S}}_i, \vec{B}_i, \vec{\mathcal{B}}_i, \vec{W}_i, \vec{\Lambda}_i, \vec{Y}_i, \vec{\mathcal{Y}}_i, \vec{Z}_i\}\,,$$
$$\delta\vec{\mathbf{X}}_{ij} = \{\vec{h}_{ij}, \vec{A}_{ij}, \vec{\mathcal{A}}_{ij}, \vec{Q}_{ij}, \vec{\mathcal{Q}}_{ij}, \vec{\kappa}_{ij}\}\,,$$
$$\delta\vec{\mathbf{X}}_{ijk} = \{\vec{C}_{ijk}\}\,.$$

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| **Spin and Parity** | $\mathbf{0}^+$ | $\mathbf{0}^-$ | $\mathbf{1}^+$ | $\mathbf{1}^-$ | $\mathbf{2}^+$ | $\mathbf{2}^-$ | $\mathbf{3}^-$ |
|---|---|---|---|---|---|---|---|
| **Metric sector** $g_{\mu\nu}$ | $\alpha, \zeta$ | - | - | $\vec{\beta}_i$ | $\vec{h}_{ij}$ | - | - |
| **Torsion sector** $T^{\lambda}{}_{\mu\nu}$ | $\phi$ | $\varrho$ | $\vec{\mathcal{S}}_i, \vec{\mathcal{B}}_i$ | $\vec{T}_i, \vec{B}_i$ | $\vec{A}_{ij}$ | $\vec{\mathcal{A}}_{ij}$ | - |
| **Nonmetricity sector** $Q_{\lambda\mu\nu}$ | $\theta, \psi, \xi$ | - | $\vec{\mathcal{Y}}_i$ | $\vec{W}_i, \vec{\Lambda}_i, \vec{Y}_i, \vec{Z}_i$ | $\vec{Q}_{ij}, \vec{\kappa}_{ij}$ | $\vec{\mathcal{Q}}_{ij}$ | $\vec{C}_{ijk}$ |

Table: Species in MAG. K. Aoki, S. Bahamonde, J. Gigante Valcarcel and M. A. Gorji, "Cosmological Perturbation Theory in Metric-Affine Gravity," [arXiv:2310.16007 [gr-qc]].

# Helicity (SVT) decomposition

- After the $3+1$ decomposition, we only need to deal with the spatial scalar and tensors with Latin indices belonging to the maximally symmetric space $\gamma_{ij}$ (and with the "external" parameter $t$).

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- After the $3+1$ decomposition, we only need to deal with the spatial scalar and tensors with Latin indices belonging to the maximally symmetric space $\gamma_{ij}$ (and with the "external" parameter $t$).
- Let us decompose them into different helicity sectors:

$$\delta\vec{\mathbf{X}}_i = \mathrm{D}_i\mathbf{S} + \mathbf{V}_i^{(\mathrm{T})}\,,$$

$$\delta\vec{\mathbf{X}}_{ij} = \left(\mathrm{D}_{(i}\mathrm{D}_{j)} - \frac{1}{3}\gamma_{ij}\mathrm{D}^2\right)\mathbf{S} + \mathrm{D}_{(i}\mathbf{V}_{j)}^{(\mathrm{T})} + \mathbf{T}_{ij}^{(\mathrm{TT})}\,,$$

$$\delta\vec{\mathbf{X}}_{ijk} = \left[\mathrm{D}_{(i}\mathrm{D}_j\mathrm{D}_{k)} - \frac{1}{5}\gamma_{(ij}\mathrm{D}_{k)}\left(3\mathrm{D}^2+4K\right)\right]\mathbf{S} + \left[\mathrm{D}_{(i}\mathrm{D}_j - \frac{1}{5}\gamma_{(ij}\left(\mathrm{D}^2+2K\right)\right]\mathbf{V}_{k)}^{(\mathrm{T})} + \mathrm{D}_{(i}\mathbf{T}_{jk)}^{(\mathrm{TT}}$$

Here, the superscript "$(\mathrm{T})$" refers that the quantity is transverse, while the tensors denoted by the superscript "$(\mathrm{TT})$" are transverse-traceless and symmetric:

$$\mathrm{D}^i\mathbf{V}_i^{(\mathrm{T})} = 0\,,$$

$$\mathrm{D}^i\mathbf{T}_{ij}^{(\mathrm{TT})} = \mathbf{T}_{ij}^{(\mathrm{TT})}\gamma^{ij} = 0\,, \quad \mathbf{T}_{ij}^{(\mathrm{TT})} = \mathbf{T}_{(ij)}^{(\mathrm{TT})}\,,$$

$$\mathrm{D}^i\mathbf{TT}_{ijk}^{(\mathrm{TT})} = \mathbf{TT}_{ijk}^{(\mathrm{TT})}\gamma^{ij} = 0\,, \quad \mathbf{TT}_{ijk}^{(\mathrm{TT})} = \mathbf{TT}_{(ijk)}^{(\mathrm{TT})}\,.$$

# SVT decomposition around FRW

- Thereby, the 10 dof described by the metric perturbations are split in terms of four scalars $\{\alpha, \beta, \zeta, h\}$ (1 dof each), two transverse vectors $\{\beta_i^{(\mathrm{T})}, h_i^{(\mathrm{T})}\}$ (2 dof each), and one symmetric and transverse-traceless tensor $h_{ij}^{(\mathrm{TT})}$ (2 dof).

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| SVT | Quantities | dof | Total dof |
|---|---|---|---|
| 5 scalars | $\{T, B, \phi, A, \varrho\}$ | 1 dof each | 5 |
| 3 pseudoscalars | $\{\mathcal{S}, \mathcal{B}, \mathcal{A}\}$ | 1 dof each | 3 |
| 3 vectors | $\{T_i^{(\mathrm{T})}, B_i^{(\mathrm{T})}, A_i^{(\mathrm{T})}\}$ | 2 dof each | 6 |
| 3 pseudovectors | $\{\mathcal{S}_i^{(\mathrm{T})}, \mathcal{B}_i^{(\mathrm{T})}, \mathcal{A}_i^{(\mathrm{T})}\}$ | 2 dof each | 6 |
| 1 rank-2 tensor | $\{A_{ij}^{(\mathrm{TT})}\}$ | 2 dof each | 2 |
| 1 rank-2 pseudotensor | $\{\mathcal{A}_{ij}^{(\mathrm{TT})}\}$ | 2 dof each | 2 |

Table: Perturbation spectrum for the torsion tensor.K. Aoki, S. Bahamonde, J. Gigante Valcarcel and M. A. Gorji, "Cosmological Perturbation Theory in Metric-Affine Gravity," [arXiv:2310.16007 [gr-qc]].

# SVT decomposition around FRW

| SVT | Quantities | dof | Total dof |
|---|---|---|---|
| 10 scalars | $\{\theta, \psi, \xi, \Lambda, Y, Z, \kappa, Q, W, C\}$ | 1 dof each | 10 |
| 2 pseudoscalars | $\{\mathcal{Y}, \mathcal{Q}\}$ | 1 dof each | 2 |
| 7 vectors | $\{\Lambda^{(\mathrm{T})}{}_i, Y^{(\mathrm{T})}{}_i, Z^{(\mathrm{T})}{}_i, \kappa^{(\mathrm{T})}{}_i, Q^{(\mathrm{T})}{}_i, W^{(\mathrm{T})}{}_i, C^{(\mathrm{T})}{}_i\}$ | 2 dof each | 14 |
| 2 pseudovectors | $\{\mathcal{Y}^{(\mathrm{T})}{}_i, \mathcal{Q}^{(\mathrm{T})}{}_i\}$ | 2 dof each | 4 |
| 3 rank-2 tensor | $\{\kappa^{(\mathrm{TT})}{}_{ij}, Q^{(\mathrm{TT})}{}_{ij}, C^{(\mathrm{TT})}{}_{ij}\}$ | 2 dof each | 6 |
| 1 rank-2 pseudotensor | $\{\mathcal{Q}^{(\mathrm{TT})}{}_{ij}\}$ | 2 dof each | 2 |
| 1 rank-3 tensor | $\{C^{(\mathrm{TT})}{}_{ijk}\}$ | 2 dof each | 2 |

Table: Perturbation spectrum for the nonmetricity tensor.

# SVT decomposition around FRW

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Table: Perturbation spectrum for the nonmetricity tensor.

- Note that the spin-3 carries a helicity-3 part and sice the metric does not have such terms, it totally decouples from the other modes in FLRW.

# Dynamics of metric-affine geometry

- General quadratic parity preserving action with dynamical torsion and nonmetricity:

$$
\begin{aligned}
S = \int d^4x\sqrt{-g}\Bigg\{\mathcal{L}_m + \frac{1}{16\pi}\Big[&-\tilde{R} + a_1\tilde{R}^2 + a_2\tilde{R}_{\lambda\rho\mu\nu}\tilde{R}^{\lambda\rho\mu\nu} + a_3\tilde{R}_{\lambda\rho\mu\nu}\tilde{R}^{\rho\lambda\mu\nu} \\
&+ a_4\tilde{R}_{\lambda\rho\mu\nu}\tilde{R}^{\mu\nu\lambda\rho} + a_5\tilde{R}_{\lambda\rho\mu\nu}\tilde{R}^{\lambda\mu\rho\nu} + a_6\tilde{R}_{\lambda\rho\mu\nu}\tilde{R}^{\mu\lambda\rho\nu} + a_7\tilde{R}_{\rho\lambda\mu\nu}\tilde{R}^{\mu\lambda\rho\nu} \\
&+ a_8\tilde{R}_{\mu\nu}\tilde{R}^{\mu\nu} + a_9\tilde{R}_{\mu\nu}\tilde{R}^{\nu\mu} + a_{10}\hat{R}_{\mu\nu}\hat{R}^{\mu\nu} + a_{11}\hat{R}_{\mu\nu}\hat{R}^{\nu\mu} + a_{12}\tilde{R}_{\mu\nu}\hat{R}^{\mu\nu} \\
&+ a_{13}\tilde{R}_{\mu\nu}\hat{R}^{\nu\mu} + a_{14}\tilde{R}^{\lambda}{}_{\lambda\mu\nu}\tilde{R}^{\rho}{}_{\rho}{}^{\mu\nu} + a_{15}\tilde{R}^{\lambda}{}_{\lambda\mu\nu}\tilde{R}^{\mu\nu} + a_{16}\tilde{R}^{\lambda}{}_{\lambda\mu\nu}\hat{R}^{\mu\nu} \\
&+ b_1 T_{\lambda\mu\nu}T^{\lambda\mu\nu} + b_2 T_{\lambda\mu\nu}T^{\mu\lambda\nu} + b_3 T^{\lambda}{}_{\lambda\nu}T^{\mu}{}_{\mu}{}^{\nu} + c_1 T_{\lambda\mu\nu}Q^{\mu\lambda\nu} \\
&+ c_2 T^{\lambda}{}_{\lambda\nu}Q^{\nu\mu}{}_{\mu} + c_3 T^{\lambda}{}_{\lambda\nu}Q^{\mu\nu}{}_{\mu} + d_1 Q_{\lambda\mu\nu}Q^{\lambda\mu\nu} + d_2 Q_{\lambda\mu\nu}Q^{\mu\lambda\nu} \\
&+ d_3 Q^{\lambda}{}_{\lambda\nu}Q^{\mu}{}_{\mu}{}^{\nu} + d_4 Q_{\nu}{}^{\lambda}{}_{\lambda}Q^{\nu\mu}{}_{\mu} + d_5 Q^{\lambda}{}_{\lambda\nu}Q^{\nu\mu}{}_{\mu}\Big]\Bigg\} .
\end{aligned}
$$

- Contribution of the spin-3 field (helicity-3 part) around flat FRW up to second order for the most general parity-preserving quadratic MAG action:

$$
\begin{aligned}
S_{\rm C}^{(2)} &= \frac{1}{16\pi}\int \mathrm{d}t\,\mathrm{d}^3x\,a^3\mathcal{G}_T^{(C)}\left[\left(\dot{C}_{ijk}^{(\mathrm{TT})}\right)^2-\frac{1}{a^2}\left(\mathrm{D}_l C_{ijk}^{(\mathrm{TT})}\right)^2+\frac{4}{a}T_2\varepsilon_{ijk}C^{(\mathrm{TT})\,i}{}_{lm}\mathrm{D}^j C^{(\mathrm{TT})\,klm}\right.\\
&\quad \left.-m_{C,\mathrm{eff}}^2\left(C_{ijk}^{(\mathrm{TT})}\right)^2\right]\\
&= \frac{1}{16\pi}\sum_{A=L,R}\int \mathrm{d}t\,\frac{\mathrm{d}^3k}{(2\pi)^3}\,a^3\mathcal{G}_T^{(C)}\left[[\dot{C}^{(A)}]^2-[\omega_{C,k}^{(A)}]^2[C^{(A)}]^2\right],
\end{aligned}
$$

where $[\omega_{C,k}^{(A)}]^2 \equiv \frac{k^2}{a^2}-4\lambda_A T_2\frac{k}{a}+m_{C,\mathrm{eff}}^2$ and

$$
\mathcal{G}_T^{(C)} = -\frac{1}{4}(2a_2+2a_3+a_5+a_6+a_7)\,.
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where $[\omega_{C,k}^{(A)}]^2\equiv\frac{k^2}{a^2}-4\lambda_A T_2\frac{k}{a}+m_{C,{\rm eff}}^2$ and

$$\mathcal{G}_T^{(C)}=-\frac{1}{4}(2a_2+2a_3+a_5+a_6+a_7)\,.$$

- Field equation of the spin-3 field around flat FRW:

$$\ddot{C}^{(A)}+3H\dot{C}^{(A)}+[\omega_{C,k}^{(A)}]^2C^{(A)}=J^{(A)}\,.$$

# Cosmological perturbations of the spin-3 field (helicity-3 part)

- it turns out that the helicity-3 modes do not present ghost nor gradient instabilities if

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- By choosing the parameters such that we recover the previously MAG model containing the black holes solutions, we find that $f_1 \leq 0$ to avoid instabilities (compatible with the vector sector).

# Cosmological perturbations of the spin-3 field (helicity-3 part)

- The quadratic action contains the parity-violating term

$$T_2 \varepsilon_{ijk} C^{(\mathrm{TT})i}{}_{lm} \mathrm{D}^j C^{(\mathrm{TT})klm} \to \lambda_A T_2 k [C^{(A)}]^2 \quad \text{(momentum space)}\,,$$

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- Recall that $\lambda_L = -1$ and $\lambda_R = +1$ so this term yields different contributions to the L and R modes.

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- Recall that $\lambda_L = -1$ and $\lambda_R = +1$ so this term yields different contributions to the L and R modes.
- While the original gravitational action is parity-preserving, the background evolution of the universe breaks the time reflection symmetry and then $S^0 \propto T_2 \neq 0$ spontaneously breaks the parity invariance of the perturbations, i.e. the left-handed and right-handed modes of $C^{(\mathrm{TT})}_{ijk}$ obey different equations of motion.

- The quadratic action contains the parity-violating term

$$T_2 \varepsilon_{ijk} C^{(\mathrm{TT})i}{}_{lm} \mathrm{D}^j C^{(\mathrm{TT})klm} \to \lambda_A T_2 k [C^{(A)}]^2 \quad \text{(momentum space)}\,,$$

  which originates from the covariant interaction $\varepsilon_{\alpha\nu\lambda\mu} q^{\rho\tau\nu} S^\alpha \nabla^\mu q_{\rho\tau}{}^\lambda$.
- Recall that $\lambda_L = -1$ and $\lambda_R = +1$ so this term yields different contributions to the L and R modes.
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- Note that if the background function $T_2 \neq 0$, we will always have such terms.

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- Another direction is to study cosmology: there are many new fields in MAG and the cosmological perturbation theory was remained unexplored in the literature.
- We have derived all the possible cosmological perturbations arising from MAG, by performing 3+1 and SVT decompositions over the metric, torsion and nonmetricity tensors.

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- What is the role of dilations/shears in particle physics?
- Some possible new quantum gravity theories with torsion and nonmetricity?