Bucaramanga
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Teleparallel Relativity

and current challenges in cosmology

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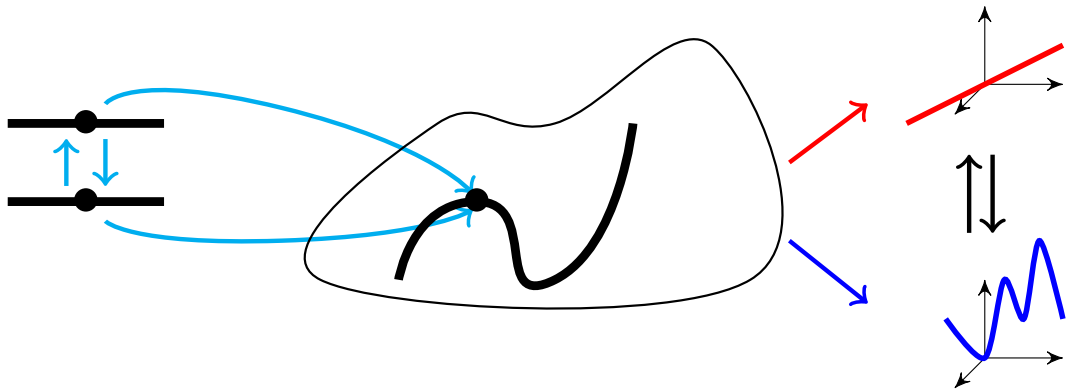
1. Spacetime geometry
   - Connection
   - Gravity
   - Teleparallelism
2. Energy
   - The field equation
   - Noether charges
   - Examples
3. $G_{||}R$ cosmology
   - The frame
   - Cosmological solution
   - Conclusion

Motivations for teleparallel gravity

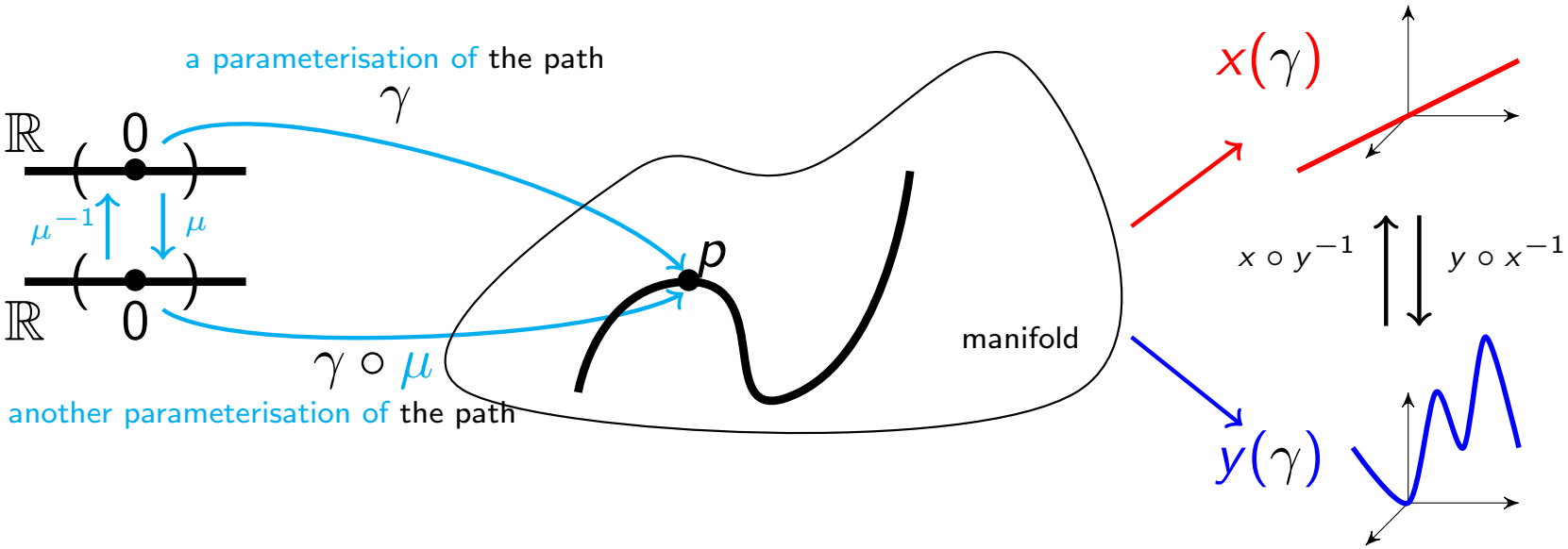
An alternative formulation of GR involving extra d.o.f's.

1. **Unification** of GR and electromagnetism
   - Einstein in the 1920's
   - Problem: the extra d.o.f's are pure gauge.
2. **Resolution** of foundational problems of GR
   - Møller in the 1960's, many authors since
   - The extra d.o.f.'s can be exploited to fix an inertial frame.
3. **Modifications** of GR
   - cf. Jose Beltrán's talk
   - Problem: the extra d.o.f's are ghost or strongly coupled.

Manifold



Acceleration

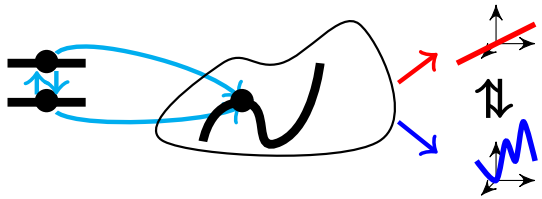


- A curve $\gamma(t) : \mathbb{R} \to M$

$$\left|\begin{array}{lcl} \gamma^\mu & = & x^\mu \circ \gamma \\ \dot{\gamma}^\mu & = & \frac{\mathrm{d}}{\mathrm{d}t}\gamma^\mu \\ \ddot{\gamma}^\mu & = & \frac{\mathrm{d}^2}{\mathrm{d}t^2}\gamma^\mu \end{array}\right| \quad \left|\begin{array}{lcl} \gamma^\mu & = & y^\mu \circ \gamma \\ \dot{\gamma}^\mu & = & \left(\frac{\partial y^\mu}{\partial x^\nu}\right)\dot{\gamma}^\nu \\ \ddot{\gamma}^\mu & = & \left(\frac{\partial y^\mu}{\partial x^\nu}\right)\ddot{\gamma}^\nu + \left(\frac{\partial^2 y^\mu}{\partial x^\alpha x^\beta}\right)\dot{\gamma}^\alpha\dot{\gamma}^\beta \end{array}\right.$$

- The connection: $\Gamma^\mu = \left(\frac{\partial y^\mu}{\partial x^\nu}\right)\Gamma^\nu + \left(\frac{\partial^2 y^\mu}{\partial x^\alpha x^\beta}\right)\dot{\gamma}^\alpha\dot{\gamma}^\beta = \Gamma^\mu{}_{\alpha\beta}\dot{\gamma}^\alpha\dot{\gamma}^\beta$

The force



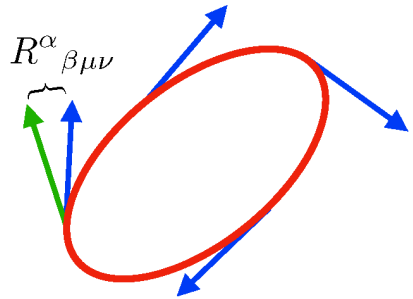
Field-body relations:

- **The law of inertia**, which introduces the Γ.
- **The definition of free motion**, upon the autoparallels of Γ.
- **The law of motion**, namely $m(\ddot{x}^i - \Gamma^i) = F^i$.

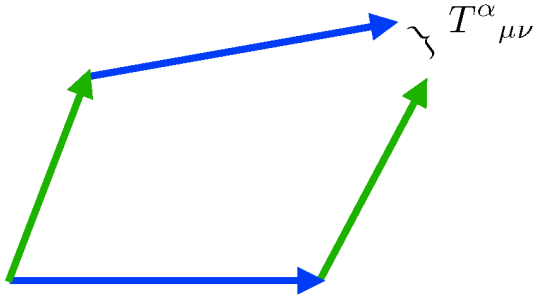
Gravity

- Newtonian: $\Gamma^i = 0$
- Einsteinian: $F^i = 0$

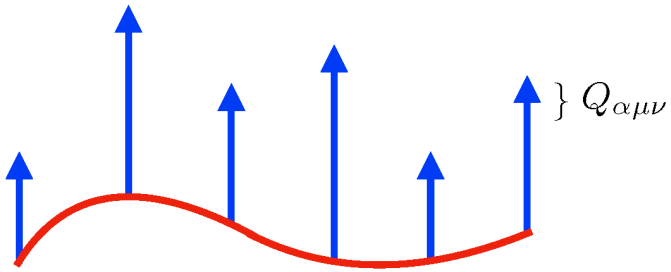
The affine connection $\Gamma^{\alpha}{}_{\mu\nu}$



The rotation of a vector transported along a closed curve is given by the curvature: General Relativity.



The non-closure of parallelograms formed when two vectors are transported along each other is given by the torsion: Teleparallel Equivalent of General Relativity.

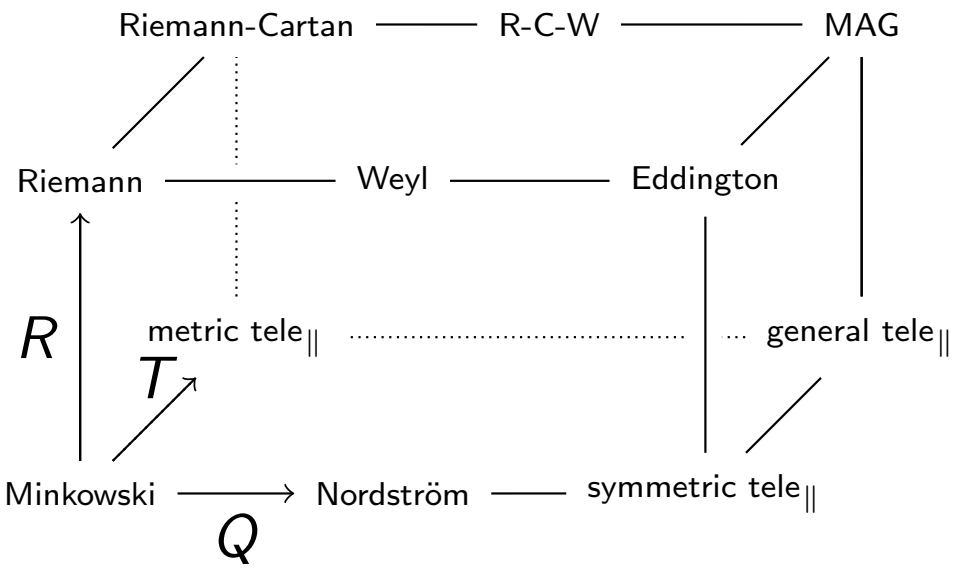


The variation of the length of a vector as it is transported is given by the non-metricity: Symmetric Teleparallel Equivalent of General Relativity.

Metric-affine gravity theories cf. Jiménez-Cano and Bahamonde on Friday

- **Torsion** $T^{\alpha}{}_{\mu\nu} = 2\Gamma^{\alpha}{}_{[\mu\nu]}$.
- **Curvature** $R^{\alpha}{}_{\beta\mu\nu} = 2\partial_{[\mu}\Gamma^{\alpha}{}_{\nu]\beta} + 2\Gamma^{\alpha}{}_{[\mu|\lambda|}\Gamma^{\lambda}{}_{\nu]\beta}$.
- **Non-metricity** $Q_{\alpha\mu\nu} = \nabla_{\alpha} g_{\mu\nu}$.

- $\boldsymbol{T}^a = \mathbf{D}\mathbf{e}^a$ of the **coframe** $\mathbf{e}^a$
- $\boldsymbol{R}_a{}^b = (\mathbf{D}\boldsymbol{\Gamma})_a{}^b$ of the the **connection** $\boldsymbol{\Gamma}_a{}^b$
- $\boldsymbol{Q}_{ab} = \mathbf{D}g_{ab}$ of the **metric** g_{ab}



Decomposition

$$\underbrace{\Gamma^{\alpha}{}_{\mu\nu}}_{\text{Affine connection}} = \underbrace{\left\{{\alpha \atop \mu\nu}\right\}}_{\text{Levi-Civita}} + \underbrace{\frac{1}{2}T^{\alpha}{}_{\mu\nu} - T_{(\mu\nu)}{}^{\alpha}}_{\text{Contortion}} + \underbrace{\frac{1}{2}Q^{\alpha}{}_{\mu\nu} - Q_{(\mu\nu)}{}^{\alpha}}_{\text{Disformation}}$$

$$\Rightarrow \quad R = \overset{\{\}}{R} - (T)^2 - (QT) - (Q)^2 - \overset{\{\}}{\nabla}_{\mu}\left(Q^{\mu} - \tilde{Q}^{\mu} + 2T^{\mu}\right)$$

Impose flatness: $R = 0$.

- Metric teleparallelism: $Q = 0 \;\Rightarrow\; (T)^2 = \overset{\{\}}{R}$ + boundary term
- Symmetric teleparallism: $T = 0 \;\Rightarrow\; (Q)^2 = \overset{\{\}}{R}$ + boundary t.

The action

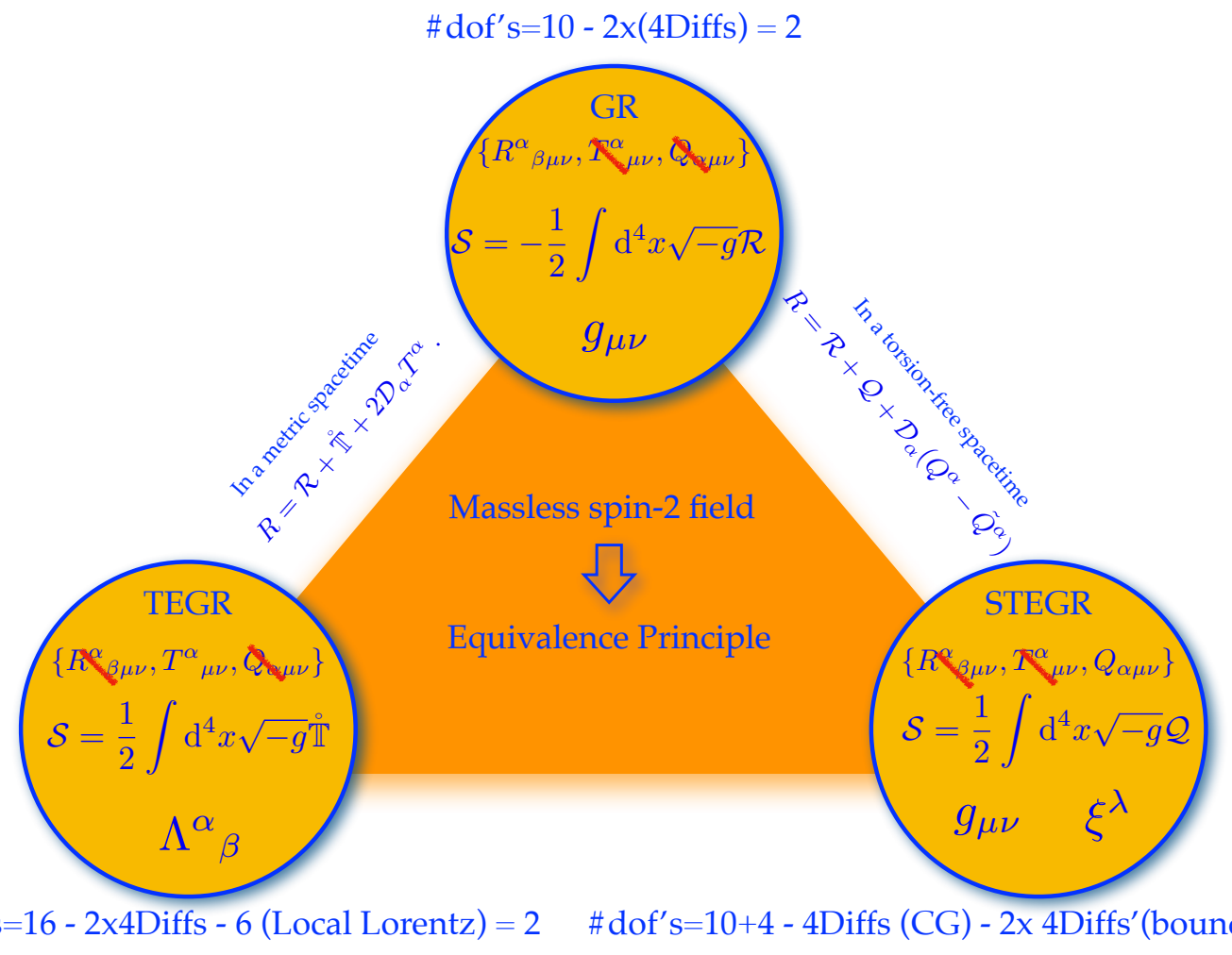
"Durch Ausführung der partiellen Integration" ...

[*Hamiltonsches Prinzip und allgemeine Relativitätstheorie* (1916)]

$$\overset{\{\}}{R} \overset{\text{part. int.}}{\rightarrow} L_E = g^{\mu\nu}\left(\left\{ {\alpha \atop \sigma\mu} \right\} \left\{ {\sigma \atop \nu\alpha} \right\} - \left\{ {\alpha \atop \sigma\alpha} \right\} \left\{ {\sigma \atop \mu\nu} \right\} \right) \overset{\text{gauge fix}}{\leftarrow} -L_{\|}$$

- The problem with L_E: it is not a scalar
- The problem with $\overset{\{\}}{R}$: it is higher derivative

The geometrical trinity [Jose Beltrán Jiménez, Lavinia Heisenberg, TSK: Universe 5 no.6, 153 (2019)]



(Re)interpretations

Einstein-Hilbert: gravity is geometry

$$\ddot{x}^\alpha + \overbrace{\left\{ {\alpha \atop \mu\nu} \right\}}^{\text{geometry}} \dot{x}^\mu \dot{x}^\nu = 0 \,.$$

Metric teleparallelism: gravity is force

$$\ddot{x}^\alpha + \overbrace{\Gamma^\alpha{}_{\mu\nu}}^{\text{Weitzenböck}} \dot{x}^\mu \dot{x}^\nu = \overbrace{-\frac{1}{2} T_{\mu\nu}{}^\alpha \dot{x}^\mu \dot{x}^\nu}^{\text{force } F^\alpha} \,.$$

Symmetric teleparallelism: gravity is inertia

$$\ddot{x}^\alpha + \overbrace{\Gamma^\alpha{}_{\mu\nu}}^{\text{pure gauge}} \dot{x}^\mu \dot{x}^\nu = \overbrace{\left(\frac{1}{2} Q^\alpha{}_{\mu\nu} - Q_{\mu\nu}{}^\alpha \right) \dot{x}^\mu \dot{x}^\nu}^{\text{inertia } I^\alpha} \,.$$

Upshot of Part I

Gravity with flat connection

In general teleparallelism, there can be both nonmetricity and torsion. The dynamical equivalent to GR exists also in this set-up [Beltrán, Heisenberg, Iosifidis, Jiménez-Cano, TSK: PLB 2020]. Let's call it the $G_{||}R$.

$$L_{||} = \underbrace{g^{\mu\nu}\left(X^{\alpha}{}_{\sigma\mu}X^{\sigma}{}_{\nu\alpha} - X^{\alpha}{}_{\sigma\alpha}X^{\sigma}{}_{\mu\nu}\right)}_{\text{covariant version of Einstein's action}} + \underbrace{\lambda_{\alpha}{}^{\beta\mu\nu}R^{\alpha}{}_{\beta\mu\nu}}_{\text{flatness constraint}}$$

- $X^{\alpha}{}_{\mu\nu} = \Gamma^{\alpha}{}_{\mu\nu} - \left\{{\alpha \atop \mu\nu}\right\} = \frac{1}{2}T^{\alpha}{}_{\mu\nu} - T_{(\mu\nu)}{}^{\alpha} + \frac{1}{2}Q^{\alpha}{}_{\mu\nu} - Q_{(\mu\nu)}{}^{\alpha}$

News from ArXiv today

arXiv:2310.15595 [pdf, ps, other]

Motion of Test Particles in Spacetimes with Torsion and Nonmetricity

Danianos Iosifidis, Friedrich W. Hehl

Comments: 8 pages, no figures

Subjects: **General Relativity and Quantum Cosmology (gr-qc)**; High Energy Physics - Theory (hep-th)

We derive the equations of motion of a test particle with intrinsic hypermomentum in spacetimes with both torsion S and nonmetricity Q (along with curvature R). Accordingly, S and Q can be measured by tracing out the trajectory followed by a hypermomentum-charged test particle in such a non-Riemannian background. The test particle is approximated by means of a Dirac δ-function. Thus we find a tangible way to observe and measure the effects of torsion and nonmetricity. Our results are consistent with earlier ones derived by Obukhov and Puetzfeld (2014) by means of a different method. We apply our insight and evaluate how far-reaching the so-called `geometrical trinity of gravity' really is.

arXiv:2310.15507 [pdf, ps, other]

Non-propagating ghost in covariant $f(Q)$ gravity

Kun Hu, Makishi Yamakoshi, Taishi Katsuragawa, Shin'ichi Nojiri, Taotao Qiu

Comments: 23 pages, 0 figures

Subjects: **General Relativity and Quantum Cosmology (gr-qc)**; Cosmology and Nongalactic Astrophysics (astro-ph.CO); High Energy Physics - Theory (hep-th)

$f(Q)$ gravity is an extension of the symmetric teleparallel equivalent to general relativity (STEGR). This work shows that based on the scalar-nonmetricity formulation, a scalar mode in $f(Q)$ gravity has a negative kinetic energy. This conclusion holds regardless of the coincident gauge frequently used in STEGR and $f(Q)$ gravity. To study the scalar mode, we further consider the covariant $f(Q)$ gravity as a special class in Higher-Order Scalar Tensor (HOST) theory and rewrite the four scalar fields, which play a role of the St\"{u}eckelberg fields associated with the diffeomorphism, by vector fields. Applying the standard Arnowitt-Deser-Misner (ADM) formulation to the new formulation of the $f(Q)$ gravity, we demonstrate that the ghost scalar mode can be eliminated by the second-class constraints, thus ensuring that $f(Q)$ gravity is a healthy theory.

arXiv:2310.15322 [pdf, ps, other]

Conserved quantities for the plane waves in TEGR and STEGR

E. Emtsova, A. N. Petrov, A. V. Toporensky

Subjects: **General Relativity and Quantum Cosmology (gr-qc)**

We study the energy-momentum characteristics of the plane ``+"-polarised gravitational wave solution of general relativity in the Teleparallel Equivalent of General Relativity (TEGR) and the Symmetric Teleparallel Equivalent of General Relativity (STEGR) using the previously constructed Noether currents. These currents can describe locally measured by observer energy-momentum if the displacement vector ξ is equal to the observer's 4-velocity. To determine the non-dynamical connection in these theories we use the unified ``turning off" gravity principle. For a constructive analysis of the values of Noether currents and superpotentials in TEGR and STEGR, we use the concept of ``gauges". The gauge changing can affect the Noether current values. We study under what conditions the Noether current for the freely falling observer is zero because this can be interpreted as the equivalence principle. We highlight two important cases with positive and zero energy, which reproduce the results of previous works with a different approach to determine gravitational energy-momentum in TEGR, and give their interpretation.

A new view at the Einstein equation: wrt electromagnetism

Electromagnetism

$$\nabla_\alpha\left(\sqrt{-g}F^{\alpha\mu}\right)=\sqrt{-g}J^\mu \quad \Rightarrow \quad q=\oint\sqrt{-g}F^{\mu\nu}\mathrm{d}S_{\mu\nu}$$

- $J^\mu \to J^\mu{}_\nu$, because there are 4 translations
- $J^\mu{}_\nu = t^\mu{}_\nu + T^\mu{}_\nu$, because gravity self-couples

Gravity

$$\nabla_\alpha\left(\sqrt{-g}H^{\alpha\mu}{}_\nu\right)=\sqrt{-g}\left(T^\mu{}_\nu+t^\mu{}_\nu\right) \Rightarrow E_\alpha=\oint\sqrt{-g}H^{\mu\nu}{}_\alpha\mathrm{d}S_{\mu\nu}$$

A new view at the Einstein equation: wrt the old one

The divergence of the excitation tensor sourced by matter & inertia

$$\nabla_\alpha \left(\sqrt{-g} H^{\alpha\mu}{}_\nu\right) = \sqrt{-g}\left(T^\mu{}_\nu + t^\mu{}_\nu\right)$$

$$H^{\mu\nu}{}_\alpha = -m_P^2\left(Q^{[\mu\nu]}{}_\alpha - \delta^{[\mu}_\alpha V^{\nu]} + \tfrac{1}{2} T_\alpha{}^{\mu\nu} - T^{[\mu\nu]}{}_\alpha\right),$$

$$V^\mu = Q^\mu - \tilde{Q}^\mu + 2T^\mu\,, \quad Q_\mu = Q_\mu{}^\alpha{}_\alpha\,, \quad \tilde{Q}_\mu = Q^\alpha{}_{\mu\alpha}\,, \quad T_\mu = T^\alpha{}_{\mu\alpha}$$

Under frame transformation (=change of the connection)

,

- the excitation $H^{\mu\nu}{}_\alpha$ inhomogeneous.
- the inertial current $t^\mu{}_\nu$ inhomogeneous.
- $G^\mu{}_\nu = m_P^{-2}\left[\sqrt{-g}\nabla_\alpha\left(\sqrt{-g}H^{\alpha\mu}{}_\nu\right) - t^\mu{}_\nu\right]$ is invariant.
- the material current $T_{\mu\nu} = (2/\sqrt{-g})\delta(\sqrt{-g}L_M)/\delta g^{\mu\nu}$ is invariant.

An inertial frame [J. Beltrán Jiménez, L. Heisenberg, TSK: Int.J.Mod.Phys.D 28 (2019)]

The inertial frame is distinguished by $t^{\mu}{}_{\nu} = 0$.

The energy & momentum associated with the choice of reference (metric/tetrad) is inertial. $\forall g_{\mu\nu} : \exists\, \Gamma^{\alpha}{}_{\mu\nu}$ s.t. $t^{\mu}{}_{\nu} = 0$. It follows that:

$m_P^{-2} G^{\mu}{}_{\nu} = T^{\mu}{}_{\nu}$ describes the local energy-momentum density.

- So, there is energy density *only where there is matter.*
- H. A. Lorentz, T. Levi-Civita, F. Klein &tc&tc all argued this!
- F. Cooperstock even proposed: $t^{\mu}_{E\nu} = 0$.
- But R. Feynman presented the bead argument...
- so the story cannot end here. GW's? BH's?

Noether's 2nd theorem

Noether charges for gauge symmetries in general

- according to Noether "improper", the charge $\int \mathrm{d}x^3 J^\mu n_\mu \approx 0$
- This implies the Noether-Wald potential $j^{\mu\nu}$ s.t. $J^\mu \approx \partial_\nu j^{\mu\nu}$.
- Note: the quasilocal formula $\oint j^{\mu\nu}$ gives the physical charge
- ...and it can be nontrivial for gauge symmetries...

Noether charge for diffeomorphisms in particular

$$E_\alpha = \oint \mathrm{d}^2 S \sqrt{-g} H^{i0}{}_\alpha n_i$$

[J. Beltrán Jiménez, TSK: Phys.Rev.D 105 (2022) 2, L021502]

A new view at the Einstein equation: wrt Noether theorems

Proposition: an *inertial frame* is distinguished by $t^{\mu}{}_{\nu} = 0$.

The energy & momenta associated with the choice of reference (metric/tetrad) is inertial. $\forall g_{\mu\nu} : \exists\, \Gamma^{\alpha}{}_{\mu\nu}$ s.t. $t^{\mu}{}_{\nu} = 0$. This criterion fixes the translation gauge freedom.

The excitation is sourced by material ~~and inertial~~ currents

$$\underbrace{\nabla_{\alpha}\left(\sqrt{-g}H^{\alpha\mu}{}_{\nu}\right)}_{\text{Noether's 2nd theorem}} = \underbrace{\sqrt{-g}\left(T^{\mu}{}_{\nu} + \cancel{t^{\mu}{}_{\nu}}\right)}_{\text{Noether's 1st theorem}} \underbrace{= \sqrt{-g}\,T^{\mu}{}_{\nu}}_{\text{in an inertial frame}} .$$

- $H^{\alpha\mu}{}_{\nu}$ is the canonical superpotential in $\mathrm{GR}_{||}$
- $T^{\mu}{}_{\nu} + t^{\mu}{}_{\nu}$ is the (should-be-)canonical energy-momentum current

[Bessel-Hagen (1921)] ... [D. Aguiar Gomes, J. Beltrán, TK (2022)]

The Universal Energy Formula

$$E_\alpha = \oint \sqrt{-g} H^{\mu\nu}{}_\alpha \mathrm{d}S_{\mu\nu}$$

What's the big deal? -Let's recall some conventional approaches:

- Noether charges in different frameworks
- Superpotential approaches
- Boundary constructions

Superpotential approaches: Weinberg's example

- Many alternative superpotentials have been proposed, e.g.:

Gravitation and Cosmology: Principles and Applications of the General Theory of Relativity (Weinberg 1972)

$$\underline{\mathfrak{h}}_W^{\mu\nu\alpha} = \delta^{\mu\nu}_{\gamma\lambda}\delta^{\alpha\rho}_{\beta\sigma}\underline{g}^{\lambda\beta}\sqrt{-\underline{g}}\left(\tfrac{1}{4}\underline{g}^{\gamma\sigma}\underline{g}^{\tau\delta} - \tfrac{1}{2}\underline{g}^{\gamma\tau}\underline{g}^{\sigma\delta}\right)\overset{\{\}}{\underline{\nabla}}_\rho g_{\tau\sigma}$$

- $\underline{\mathfrak{h}}_W^{\mu\nu\alpha}$ is a pseudotensor defined wrt $\underline{g}_{\mu\nu} = \eta_{\mu\nu}$
- "Covariantised" but reference-dependent if $\underline{g}_{\mu\nu}$ arbitrary
- The symmetry of a solution can provide the natural reference

$\Rightarrow$ energy is unique only in special cases (ADM & its generalisations)

Pesudotensors $\rightarrow$ tensors

Proposal: minimal coupling $\underline{g}_{\mu\nu} \rightarrow g_{\mu\nu}$ and $\underline{\nabla}_\mu \rightarrow \nabla_\mu$ s.t. $\underline{\mathfrak{h}}^{\mu\nu\alpha}_W \rightarrow$

$$\mathfrak{h}^{\mu\nu\alpha}_W = \delta^{\mu\nu}_{\gamma\lambda}\delta^{\alpha\rho}_{\beta\sigma}g^{\lambda\beta}\sqrt{-g}\left(\frac{1}{4}g^{\gamma\sigma}g^{\tau\delta} - \frac{1}{2}g^{\gamma\tau}g^{\sigma\delta}\right)Q_{\rho\tau\sigma} = \sqrt{-g}H^{\mu\nu\alpha}$$

- The same occurs with several different superpotentials:
  - von Freud: $\underline{\mathfrak{h}}^{\mu\nu}_{vF\ \alpha} = -\frac{1}{2}\sqrt{-g}\delta^{\mu\nu\gamma}_{\lambda\sigma\alpha}g^{\beta\lambda}\left(\left\{{\lambda \atop \beta\gamma}\right\} - \underline{\left\{{\lambda \atop \beta\gamma}\right\}}\right) \rightarrow \sqrt{-g}H^{\mu\nu}{}_\alpha$
  - Landau and Lifshitz: $\underline{\mathfrak{h}}^{\mu\nu\alpha}_{LL} = \frac{1}{2}(-\underline{g})^{-\frac{1}{2}}\delta^{\mu\nu}_{\gamma\rho}\underline{\overset{\{\}}{\nabla}}_\beta\left(-gg^{\rho\alpha}g^{\gamma\beta}\right) \rightarrow \sqrt{-g}H^{\mu\nu\alpha}$
  - Bergmann and Thomson: $\underline{\mathfrak{h}}^{\mu\nu\alpha}_{BT} = \sqrt{\underline{g}/g}\ \underline{\mathfrak{h}}^{\mu\nu\alpha}_{LL} \rightarrow \sqrt{-g}H^{\mu\nu\alpha}$
  - Papapetrou: $\underline{\mathfrak{h}}^{\mu\nu\alpha}_{P} = \frac{1}{2}\delta^{\mu\nu}_{\gamma\lambda}\delta^{\alpha\rho}_{\beta\sigma}g^{\gamma\beta}\underline{\overset{\{\}}{\nabla}}_\rho\left(\sqrt{-g}g^{\gamma\sigma}\right) \rightarrow \sqrt{-g}H^{\mu\nu\alpha}$
- So $\exists_1$ background-independent unification of these approaches!

Noether currents in the Geometrical Trinity

formulation	constraints	superpotential	canonical frame
symm. tele$_{||}$	$R^{\alpha}{}_{\beta\mu\nu} = T^{\alpha}{}_{\mu\nu} = 0$	$m_P^{-2} h^{\mu\nu}{}_{\alpha} = \delta^{[\mu}_{\alpha} \tilde{Q}^{\nu]} - \delta^{[\mu}_{\alpha} Q^{\nu]} - Q^{[\mu\nu]}{}_{\alpha}$	$t^{\mu}{}_{\nu} = q^{\mu}{}_{\alpha\beta} Q_{\nu}{}^{\alpha\beta} - \frac{1}{2}\delta^{\mu}_{\nu} q^{\alpha}{}_{\beta\gamma} Q_{\alpha}{}^{\beta\gamma} = 0$
metric tele$_{||}$	$R^{\alpha}{}_{\beta\mu\nu} = Q_{\alpha}{}^{\mu\nu} = 0$	$m_P^{-2} t_{\alpha}{}^{\mu\nu} = \frac{1}{2} T^{\mu}{}_{\alpha}{}^{\nu} + T^{[\mu\nu]}{}_{\alpha} + 2\delta^{[\mu}_{\alpha} T^{\nu]}$	$t^{\mu}{}_{\nu} = 2 t_{\alpha}{}^{\beta\mu} T^{\alpha}{}_{\nu\beta} - \frac{1}{2}\delta^{\mu}_{\nu} t_{\alpha}{}^{\beta\gamma} T^{\alpha}{}_{\beta\gamma} = 0$
Palatini	–	$m_P^{-2} h_K^{\mu\nu}{}_{\alpha} = \mathring{\nabla}^{[\mu} \delta^{\nu]}_{\hat{\alpha}}$	$t^{\mu}_{K\nu} \overset{?}{=} \frac{m_P^2}{2} \left[\left(\mathring{R} - \mathring{\Box} \right) \delta^{\mu}_{\nu} + \left(2\mathring{\nabla}^{\mu}\mathring{\nabla}_{\alpha} - \mathring{\nabla}_{\alpha}\mathring{\nabla}^{\mu} \right) \delta^{\alpha}_{\nu} \right]$

The Komar charge: $E_{\xi}^{K} = \oint \sqrt{-g} \overset{\{\}}{\nabla}{}^{[\mu} \xi^{\nu]} \mathrm{d}S_{\mu\nu}$ Einstein-Hilbert/Einstein-Palatini

- Frame-invariant (unlike the tele$_{||}$ charges)
- Non-canonical form (linear in the derivative)
- Yields unphysical results in most cases ($E_0^K \neq$ ADM energy)

The conventional approach in GR

Hamiltonian vs Noether charge

The Noether charge of the **Einstein-Hilbert** action is given by the **Komar** expression. Adding the **Gibbons-Hawking-York** boundary term renders the action well-defined for Killing vectors tangential to the boundary. The resulting Hamiltonian charge is given by the **Brown-York** energy-momentum tensor.

It's used often even though...

it seems rather arbitrary construction and the results seem arbitrary as well!

Black hole

The energy is determined uniquely in an inertial frame

$$E = \oint \mathrm{d}^2 S \sqrt{-g} H^{i0}{}_0 n_i \,.$$

Black hole

- The metric $\mathrm{d}s^2 = -f(r)\mathrm{d}t^2 + f^{-1}(r)\mathrm{d}r^2 + r^2\mathrm{d}\Omega^2$
- The connection $\Gamma^{\alpha}{}_{\mu\nu} \neq 0$ in an inertial frame
- The excitation $H^{or}{}_0 = m_P^2 \left[1 - f(r)\right]/r$

The Schwarzschild case: $f(r) = 1 - \frac{m_S}{4\pi m_P^2 r}$

- $\Rightarrow$ the Newtonian form $H^{or}{}_0 = m_S/4\pi r^2$ note: $T^{\mu}{}_{\nu} = 0$!
- $\Rightarrow$ $E(r) = m_S$ note: it is (quasi)localised

Charged black hole in dS space

The energy is determined uniquely in an inertial frame

$$E = \oint \mathrm{d}^2 S \sqrt{-g} H^{i0}{}_0 n_i \,.$$

The Schwarzschild-Reissner-Nordström-de Sitter black hole

- $f(r) = 1 - \frac{m_S}{4\pi m_P^2 r} - \frac{q^2}{8\pi m_P^2 r^2} + \frac{\Lambda}{3} r^2$
- Requires the electromagnetic field: $q = \oint \mathrm{d}^2 S \sqrt{-g} F^{0i} n_i$
- Requires the cosmological constant Λ

The result $E = m_S - \frac{q^2}{2r} + \frac{4}{3}\pi r^3 \rho_\Lambda$ consists of

- Mass energy
- Electrostatic energy of a shell of radius r and total charge q
- Vacuum energy with density $\rho_\Lambda = m_P^2 \Lambda$

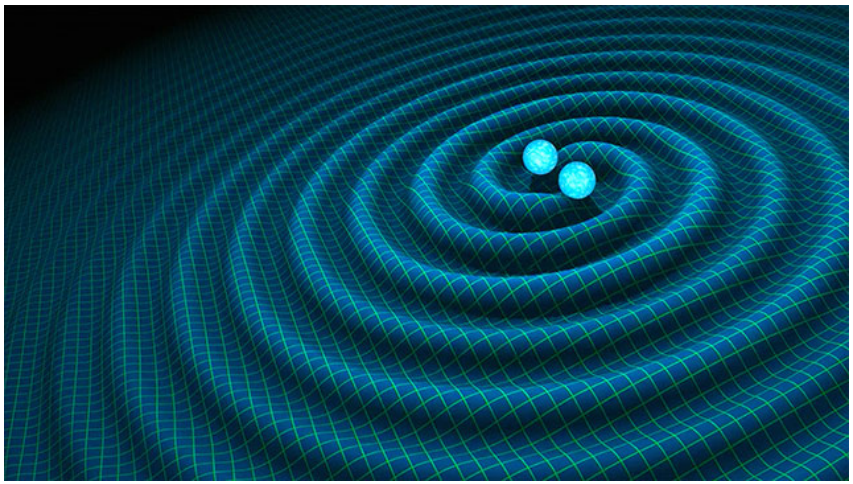
GWs have relative energy

The *local* energy-momentum current wrt v^μ

$$J^\mu = \left(T^\mu{}_\nu + t^\mu{}_\nu\right) v^\nu + \tfrac{1}{2} H^{\alpha\mu}{}_\nu \nabla_\alpha v^\nu .$$

We may have $E \neq 0$

- in the presence of matter.
- in the presence of $\underline{g}_{\mu\nu}$.
- in a non-inertial frame.
- wrt non-static clock.
- in nontrivial topology.



Otherwise we expect $E = 0$.

GWs have no absolute energy

Perturbations in flat space

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + (\delta_{ij} + h_{ij})\,\mathrm{d}x^i \mathrm{d}x^j \qquad h_{ij} \text{ transverse and traceless}$$

- EMT vanishes everywhere: $t^{\mu}{}_{\nu} = (\Box h^{i}{}_{j})\delta^{\mu}_{i}\delta^{j}_{\nu} = 0$
  - Energy: $E_0 = \oint \mathrm{d}^2 S \cancel{H^{i0}{}_{0}} n_i = 0$
  - Momenta: $E_i = \oint \mathrm{d}^2 S \dot{h}^{j}{}_{i} n_j \to 0$

PP-wave **exact** vacuum solution

$$\mathrm{d}s^2 = V(u,x,y)\mathrm{d}u^2 + 2\mathrm{d}u\mathrm{d}v + \mathrm{d}x^2 + \mathrm{d}y^2$$

- EMT vanishes everywhere: $t^{\mu}{}_{\nu} = -\frac{1}{2}\left(V_{,xx} + V_{,yy}\right)\delta^{\mu}_{v}\delta^{v}_{\nu} = 0$
  - Energy: $E_0 = \oint \mathrm{d}^2 S \sqrt{-g} \cancel{H^{i0}{}_{0}} n_i = 0$
  - Momenta: $E_i = \oint \mathrm{d}^2 S \sqrt{-g} H^{j0}{}_{i} n_j \to 0$

Entropy

Wald has shown that entropy too is a diffeomorphism charge

$$S = \frac{2\pi}{\kappa} \oint \mathrm{d}^2 S_{\mu\nu} \sqrt{-g} H^{\mu\nu}{}_{\alpha} \xi^{\alpha}$$

We have shown that it is also understood as a Lorentz charge

$$S = -2\pi \int \mathrm{d}^2 x \mathfrak{r}_{\alpha}{}^{\beta\mu\nu} \lambda^{\alpha}{}_{\beta} n^{\mu\nu}$$

Gibbons and Hawking determined it from the Euclidean action

$$S = \frac{1}{2} \int \mathrm{d}^4 x \sqrt{-g} Q$$

- Each expression gives $S = 2\pi m_P^2 A$ for the BH
- ~~ambiguities~~, ~~regulating terms~~, ~~boundary terms~~, ~~infinities~~

Summary of part II

The field excitation is sourced by material and inertial currents.

$$\nabla_\alpha \left(\sqrt{-g} H^{\alpha\mu}{}_\nu\right) = \sqrt{-g}\left(T^\mu{}_\nu + t^\mu{}_\nu\right) \overset{\text{inertial frame}}{=} \sqrt{-g}\, T^\mu{}_\nu \,.$$

The (quasi)**local** energetics and thermodynamics are **uniquely** determined by the integrals of the excitation.

Nontrivial contributions are due to spacetime topology.

Integral of the zero energy density = nonzero energy charge!

The reference matrix

- A flat $R^{\alpha}{}_{\beta\mu\nu} = 0$ connection must be given by a gauge transformation $\Lambda^{\alpha}{}_{\beta} \in \mathrm{GL}(4)$, as $\Gamma^{\alpha}{}_{\mu\nu} = (\Lambda^{-1})^{\alpha}{}_{\rho}\partial_{\mu}\Lambda^{\rho}{}_{\beta}$.
- The fundamental field of $G_{||}R$ is rather the 16 components of the **reference matrix** $\Lambda^{\alpha}{}_{\beta}$ than the 64 connection coefficients
- Thus, we consider $L_{||}(g, \Lambda) = X^{\alpha}{}_{\sigma\mu}X^{\sigma}{}_{\nu\alpha} - X^{\alpha}{}_{\sigma\alpha}X^{\sigma}{}_{\mu\nu}$.

The 2 fields, the metric tensor and the reference matrix

induce respectively the 2 geometrical structures, the metric-compatible $\left\{{\alpha \atop \mu\nu}\right\}$ and the frame-compatible $\Gamma^{\alpha}{}_{\mu\nu}$.

Cosmological symmetry

Homogeneity and isotropy is described by

- ISO(3): spatially flat
- SO(3,1): open universe
- SO(4): closed universe

The metric

$$\mathrm{d}s^2 = -n^2(t)\mathrm{d}t^2 + a^2(t)\left(\delta_{ij} + \frac{x_i x_j}{\xi^2}\right)\mathrm{d}x^i\mathrm{d}x^j$$

The reference matrix

- Standard case: residual symmetries generated by the unbroken Diffs
- For the $\hat{\Lambda}$: the frame can be adjusted to repair some broken Diffs

Diff×GL(4) → ISO(3,1)

- Frame transformation: $g^{\mu\nu} \to g_{\mu\nu}\,, \quad \delta_\lambda \Lambda^\mu{}_\nu \to A^\mu{}_\alpha \Lambda^\alpha{}_\nu$
- Diff: $\mathcal{L}_\xi g^{\mu\nu} = 2\nabla^{(\mu}\xi^{\nu)} + \left(Q_\alpha{}^{\mu\nu} - 2T^{(\mu\nu)}{}_\alpha\right)\xi^\alpha\,, \quad \mathcal{L}_\xi \Lambda^\mu{}_\nu = \left(\nabla_\nu \xi^\alpha + T^\alpha{}_{\beta\nu}\xi^\beta\right)$

$$\hat{\Lambda} = \left(\begin{array}{c|c} \mu(t) & 0 \\ \hline 0 & \sigma(t)\mathbb{1} \end{array}\right)$$

Homogeneity is trivial.

Consider **synchronisation** and **dilations** s.t. $\mu = \mu(t), \sigma = \sigma(t)$.

Isotropy is not. Let's Diff by $\xi^i = R^i{}_j x^j$ and GL by $A^i{}_j = O^i{}_j$:

$\Lambda^i{}_j(\vec{x}) \to A^i{}_\alpha R_j{}^k \Lambda^\alpha{}_k(\hat{R}\cdot\vec{x}) = \sigma O^i{}_k R_j{}^l \delta^k_l = \sigma(\hat{O}\cdot\hat{R}^{-1})^i{}_j = \Lambda^i{}_j(\vec{x})$, where the last step holds if $\hat{O} = \hat{R}$.

Nontrivial case

- We may also realise the homogeneity in a nontrivial fashion
- The key point: ISO(3) $\in$ GL(4)
- generalisation of the previous realisation by energy scale λ:

$$\hat{\Lambda} = \left(\begin{array}{c|c} \mu(t) & -\sigma(t)\lambda\vec{x} \\ \hline 0 & \sigma(t)\mathbb{1} \end{array}\right)$$

Consider the GL element $\hat{A} = \left(\begin{array}{c|c} 1 & \vec{y} \\ \hline 0 & \hat{O} \end{array}\right)$ and the translation $\vec{x} \to \vec{x} + \vec{\xi}$.

$$\begin{aligned} \Lambda^0{}_i = -\sigma\lambda x_i \quad &\to \quad A^0{}_\beta R_i{}^j \Lambda^\beta{}_j (\hat{R}\cdot\vec{x} + \vec{\xi}) \\ &= \quad R_i{}^j \left[-\sigma\lambda\left(R_j{}^k x_k + \xi_j\right)\right] + y_m R_i{}^j \sigma \delta^m{}_j \\ &= \quad -\sigma\lambda x_i - \sigma R_i{}^j \left[\lambda\xi_j - y_j\right], \quad \text{so we should set} \quad \vec{y} = \lambda\vec{\xi}. \end{aligned}$$

Another nontrivial case

$$\hat{\Lambda} = \left(\begin{array}{c|c} \mu(t) & 0 \\ \hline \mu(t)\lambda\vec{x} & \sigma(t)\mathbb{1} \end{array}\right), \quad \hat{\Lambda}^{-1} = \left(\begin{array}{c|c} \mu^{-1}(t) & 0 \\ \hline -\sigma^{-1}(t)\lambda\vec{x} & \sigma^{-1}(t)\mathbb{1} \end{array}\right).$$

Dual representation

This nontrivial realisation is dual to the previous one. The matrices are related by $\hat{\Lambda}^{\star} = (\hat{\Lambda}^{-1})^{T}$, and the respective affine connections are related by $\Gamma^{\star}_{\mu} = -\Gamma^{T}$. However, we will see that these are not equivalent physicswise.

Spatial curvature $k = \pm\ell^2$

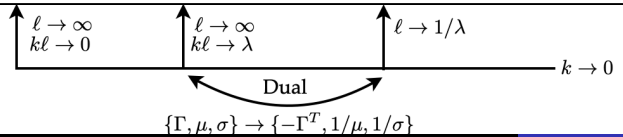
$$\hat{\Lambda} = \left(\begin{array}{c|c} \mu\chi & \frac{-k\sigma\ell\vec{x}}{\chi} \\ \hline \mu\ell^{-1}\vec{x} & \sigma\mathbb{1} \end{array}\right), \quad \text{where} \quad \xi = \sqrt{1-kr^2}\,.$$

The symmetry generators can represented as

Diffs: $J_i = \varepsilon_{ij}{}^k x^j \partial_k$, $P_i = \chi\partial_i$, GL: $T_i = \left(\begin{array}{c|c} 0 & k\ell\vec{e}_i \\ \hline -\frac{1}{\ell}\vec{e}_i & \hat{0} \end{array}\right)$

- Trivial flat configuration. The trivial case I is obtained by taking $k \to 0$, $\ell \to \infty$ with $k\ell \to 0$.
- Non-trivial configuration I. The non-trivial case II is recovered by taking the limit $k \to 0$, $\ell \to \infty$ while keeping $k\ell \equiv \lambda$ fixed.
- Non-trivial configuration II. The dual non-trivial case III is recovered by simply taking $k \to 0$ while keeping ℓ finite. We then identify $\lambda = \ell^{-1}$.

		Spatially flat			Spatially curved	
		Trivial	Non-Trivial		Regular	Exceptional $k\ell^2 = 1$
			I	II		
Frame	Λ	$\left(\begin{array}{c\|c} \mu & 0 \\ \hline 0 & \sigma\delta^i_j \end{array}\right)$	$\left(\begin{array}{c\|c} \mu & -\sigma\lambda x_j \\ \hline 0 & \sigma\delta^i_j \end{array}\right)$	$\left(\begin{array}{c\|c} \mu & 0 \\ \hline \mu\lambda x^i & \sigma\delta^i_j \end{array}\right)$	$\left(\begin{array}{c\|c} \mu\chi & \frac{-k\sigma\ell x_j}{\chi} \\ \hline \mu\ell^{-1}x^i & \sigma\delta^i_j \end{array}\right)$	$\left(\begin{array}{c\|c} \mu & 0 \\ \hline 0 & \sigma\left(\chi\delta^i_j + \frac{x^i x_j}{\ell^2\chi} - k\ell\varepsilon^i{}_{jk}x^k\right) \end{array}\right)$
	Λ^{-1}	$\left(\begin{array}{c\|c} \frac{1}{\mu} & 0 \\ \hline 0 & \frac{1}{\sigma}\delta^i_j \end{array}\right)$	$\left(\begin{array}{c\|c} \frac{1}{\mu} & \frac{\lambda}{\mu}x_j \\ \hline 0 & \frac{1}{\sigma}\delta^i_j \end{array}\right)$	$\left(\begin{array}{c\|c} \frac{1}{\mu} & 0 \\ \hline -\frac{\lambda}{\sigma}x^i & \frac{1}{\sigma}\delta^i_j \end{array}\right)$	$\left(\begin{array}{c\|c} \frac{\chi}{\mu} & \frac{k\ell}{\mu}x_j \\ \hline -\frac{\chi}{\ell\sigma}x^i & \frac{1}{\sigma}(\delta^i_j - kx^i x_j) \end{array}\right)$	$\left(\begin{array}{c\|c} \frac{1}{\mu} & 0 \\ \hline 0 & \frac{1}{\sigma}\left(\chi\delta^i_j + \frac{1}{\ell}\varepsilon^i{}_{jk}x^k\right) \end{array}\right)$
Parallel connection	$\Gamma^0{}_{00}$	$\partial_0 \log \mu$				
	$\Gamma^i{}_{0j}$	$\partial_0 \log \sigma\delta^i_j$				
	$\Gamma^i{}_{j0}$	0	0	$\frac{\mu}{\sigma}\lambda\delta^i_j$	$\frac{\mu}{\ell\sigma}\delta^i_j$	0
	$\Gamma^0{}_{ij}$	0	$-\frac{\sigma}{\mu}\lambda\delta_{ij}$	0	$-k\ell\frac{\sigma}{\mu}\gamma_{ij}$	0
	$\Gamma^i{}_{jk}$	0	0	0	$kx^i\gamma_{jk}$	$kx^i\gamma_{jk} + \frac{1}{\ell}\epsilon^i{}_{jk}$
Non-metricity	Q_1	$\frac{1}{n}\partial_0 \log \frac{n}{\mu}$				
	Q_2	$\frac{1}{n}\partial_0 \log \frac{\sigma}{a}$				
	Q_3	0	$\frac{\lambda}{a^2}\frac{n\sigma}{\mu}$	$\frac{\lambda\mu}{\sigma n}$	$\frac{\mu}{\ell\sigma n} + \frac{k\ell}{a^2}\frac{n\sigma}{\mu}$	0
Torsion	T_1	$\frac{1}{n}\partial_0 \log \sigma$	$\frac{1}{n}\partial_0 \log \sigma$	$\frac{1}{n}\partial_0 \log \sigma - \frac{\lambda\mu}{n\sigma}$	$\frac{1}{n}\partial_0 \log \sigma - \frac{\mu}{\ell n\sigma}$	$\frac{1}{n}\partial_0 \log \sigma$
	T_2	0				$\frac{1}{\ell a}$



Isotropic and homogeneous action

$$\sqrt{-g}L = 3m_P^2 n a^3 \left(\frac{k}{a^2} - H^2\right) + \frac{3}{2} m_P^2 \partial_0 \left(\frac{a^3 \mu}{\ell n \sigma} - \frac{k \ell a n \sigma}{\mu}\right)$$

The frame d.o.f. enter as a surface term:

- The trivial case: no surface term
- The nontrivial case I: the 1st surface piece contributes
- The nontrivial case II: the 2nd surface piece contributes

An inertial frame is given by a solution to

$$t^{\mu}{}_{\nu} = -3m_P^2 \left[(Q_2 - T_1)^2 + T_2^2\right] u^{\mu} u_{\nu} - m_P^2 \left[Q_1 Q_3 + (Q_2 - T_1)(3Q_2 - Q_3 - T_1) + T_2^2\right] h^{\mu}{}_{\nu} = 0\,.$$

Inertial frame

$$\mu = \mu_0 n\,, \quad \sigma = \frac{\mu_0}{\ell H} \quad \Rightarrow \quad \hat{\Lambda} = \mu_0 \left(\begin{array}{c|c} n\chi & -\frac{k}{\chi H} x_j \\ \hline n\ell^{-1} x^i & (\ell H)^{-1} \delta^i_j \end{array} \right) .$$

- Trivial case: $k \to 0$, $\ell \to \infty$ with $k\ell \to 0$. Fails since $H \to 0$.
- Nontrivial case I: $k \to 0$, $\ell \to \infty$, $k\ell \equiv \lambda$. Fails since $H \to 0$.
- Nontrivial case II: $k \to 0$, $\lambda \equiv \ell^{-1}$. The limit is regular.

The field excitation

$$H_\alpha{}^{\mu\nu} = \tfrac{1}{2} \mathcal{S} h_\alpha{}^{[\mu} u^{\nu]} \quad \text{where} \quad \mathcal{S} = m_P^2 H (1 - \Omega_k)\,.$$

The field equation

$$T^\mu{}_\nu = 3H\mathcal{S} u^\mu u_\nu - \left[\mathcal{S}' + \left(\log \frac{a^3}{\sigma} \right)' \mathcal{S} \right] h^\mu{}_\nu\,.$$

Non-inertial frames

- Smooth-boundary frame: $L \overset{\text{b}}{=} 0$.
- Symmetric teleparallel frame: $T^{\alpha}{}_{\mu\nu} = 0$.
- Metric teleparallel frame: $Q_{\alpha}{}^{\mu\nu} = 0$. Fictitious anti-energy which is radiation-like.
- Coincident frame: $\Gamma^{\alpha}{}_{\mu\nu} = 0$. This is like the previous case, but with stiff anti-energy.
- Energy-free frame: $H_{\alpha}{}^{\mu\nu} = 0$.
- Material (pseudo)frame: $t^{\mu}{}_{\nu} = 2S_{\alpha}{}^{\mu\beta}\Gamma^{\alpha}{}_{\beta\nu}$.
- Hypothetical constant-rank frame for modified gravity.

Conclusion

- $L_{\parallel}(g,\hat{\Lambda}) = g^{\mu\nu}\left(X^{\alpha}{}_{\sigma\mu}X^{\sigma}{}_{\nu\alpha} - X^{\alpha}{}_{\sigma\alpha}X^{\sigma}{}_{\mu\nu}\right)$.
  - Done: covariant version of the original theory
  - Claim: also its *relativistic* version

Gravitational excitation is sourced by material ~~and inertial~~ currents

$$\underbrace{\nabla_{\alpha}\left(\sqrt{-g}H^{\alpha\mu}{}_{\nu}\right)}_{\text{Noether's 2nd theorem}} = \underbrace{\sqrt{-g}\left(T^{\mu}{}_{\nu} + \cancel{t^{\mu}{}_{\nu}}\right)}_{\text{Noether's 1st theorem}} \underbrace{= \sqrt{-g}T^{\mu}{}_{\nu}}_{\text{inertial frame}} .$$

- Towards QM
  - Renormalisable?
  - $t^{\mu}{}_{\nu}$? t?

Thank you.

Appendix: Lorentz gauge theory of gravity & spacetime

In a nutshell, the novelity is that $g = \mathbf{D}\phi^a \otimes \mathbf{D}\phi_a$.

$I = -i \int \phi_a \Box\Box \phi^a$

- the pregeometric phase: $\phi^a = 0$
- the geometric phase: dynamics of GR in the presence of dust

Towards unification

- The shadow-charged EoM is consistent for both SM&GR
- Incorporated SO(4) into SO(10) would seem natural