THE MATTER SPECTRUM AT SECOND ORDER

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The universe has structure on large scale, and to understand this structure we must develop tools to study perturbations around the smooth background. Scott Dodelson (2021).

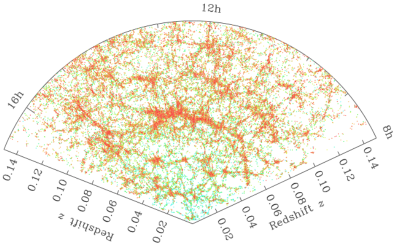

Figure 1: Dodelson(2021).

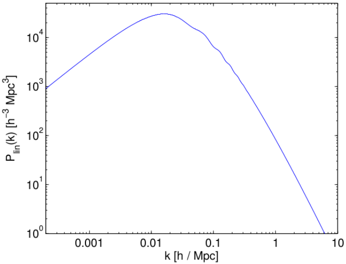

Figure 2: Skovbo(2011).

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In order to solve these questions we start with Eistein-Boltzmann equations

$$\nabla^2_{\mathbf{x}} \Phi_{\text{PER}}(\mathbf{x}, \tau) = \frac{3}{2} \mathcal{H}^2(\tau) \Omega_m(\tau) \delta(\mathbf{x}, \tau). \tag{1}$$

$$\frac{df}{d\tau} = \frac{\partial f}{\partial \tau} + \frac{\mathbf{p}}{mR(t)} \cdot \nabla_{\mathbf{x}} f - mR(t) \nabla_{\mathbf{x}} \Phi_{\text{PER}} \cdot \nabla_{\mathbf{p}} f = 0. \tag{2}$$

considering contrast density term $\rho(\mathbf{x}, \tau) \equiv \overline{\rho}(\tau) \left[1 + \delta(\mathbf{x}, \tau)\right]$.

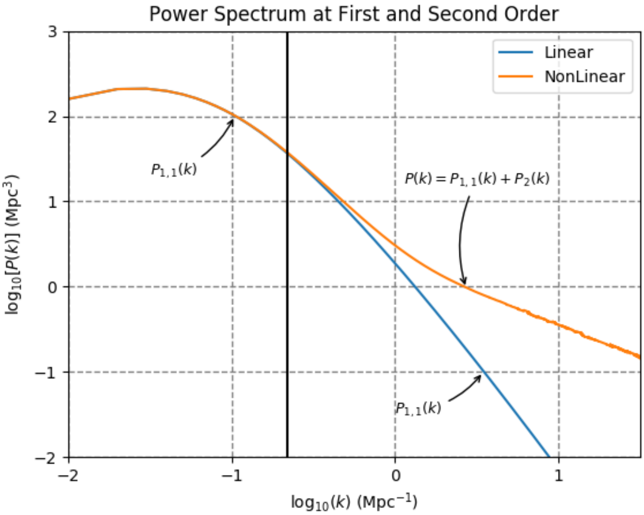

Figure 3: Power Spectrum at Second Order $P(k)$. (Fonseca & Castañeda.,2020)

Using Eulerian dynamics principles we get the movement equations for a Cold Dark Matter (CDM) fluid:

$$\nabla^2_{\mathbf{x}} \Phi_{\mathrm{PER}}(\mathbf{x}, \tau) = \frac{3}{2} \mathcal{H}^2(\tau) \Omega_m(\tau) \delta(\mathbf{x}, \tau), \tag{3}$$

$$\frac{\partial}{\partial \tau} \left[\delta(\mathbf{x}, \tau) \right] + \nabla_{\mathbf{x}} \cdot \left[\left[1 + \delta(\mathbf{x}, \tau) \right] \mathbf{u}(\mathbf{x}, \tau) \right] = 0. \tag{4}$$

$$\frac{\partial}{\partial \tau} \mathbf{u}(\mathbf{x}, \tau) + \mathcal{H}(\tau) \mathbf{u}(\mathbf{x}, \tau) + \left[\mathbf{u}(\mathbf{x}, \tau) \cdot \nabla_{\mathbf{x}} \right] \mathbf{u}(\mathbf{x}, \tau) = -\nabla_{\mathbf{x}} \Phi_{\mathrm{PER}} - \frac{\nabla_{\mathbf{x}} \cdot \sigma(\mathbf{x}, \tau)}{\rho(\mathbf{x}, \tau)}, \tag{5}$$

In general, we can characterize the CDM fluid throught

$$\nabla_{\mathbf{x}} \cdot \mathbf{u}(\mathbf{x},\tau) \equiv \theta(\mathbf{x},\tau); \qquad \mathbf{w}(\mathbf{x},\tau) \equiv \nabla_{\mathbf{x}} \times \mathbf{u}(\mathbf{x},\tau), \tag{6}$$

Therefore

Linear Regime

$$\frac{\partial^2}{\partial\tau^2}\delta(\mathbf{x},\tau) + \mathcal{H}(\tau)\frac{\partial}{\partial\tau}\delta(\mathbf{x},\tau) - \frac{3}{2}\mathcal{H}^2(\tau)\Omega_m(\tau)\delta(\mathbf{x},\tau) = 0, \tag{7}$$

$$\frac{d^2}{d\tau^2}D(\tau) + \mathcal{H}(\tau)\frac{d}{d\tau}D(\tau) - \frac{3}{2}\mathcal{H}^2(\tau)\Omega_m(\tau)D(\tau) = 0, \tag{8}$$

$$(z+1)P(z)\frac{d^2}{dz}D(z) + Q(z)\frac{d}{dz}D(z) - \frac{3}{2}\Omega_{m,0}(z+1)^2 D(z) = 0. \tag{9}$$

with solution Heat y Edwars (1977)

$$D^{(+)}(z) = CP^{1/2}(z)\int_z^\infty \frac{s+1}{P^{3/2}(s)}ds. \tag{10}$$

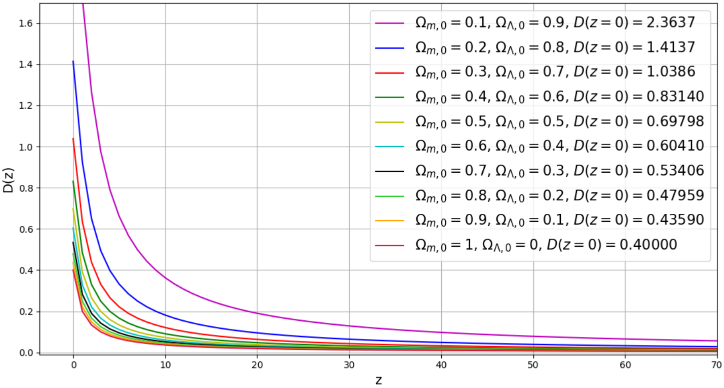

Figure 4: Growth Factor-Redshift (Fonseca & Castañeda, 2020).

Now, we have the equations system with all terms Scoccimarro (2001)

$$\delta(\mathbf{x},\tau) = \sum_{n=1}^{\infty} \delta^n(\mathbf{x},\tau); \qquad \theta(\mathbf{x},\tau) = \sum_{n=1}^{\infty} \theta^n(\mathbf{x},\tau), \tag{11}$$

and its Fourier space representation

$$\frac{\partial}{\partial\tau}\tilde{\delta}(\mathbf{k},\tau) + \tilde{\theta}(\mathbf{k},\tau) = -\int_{\mathbf{k}_1}\int_{\mathbf{k}_2} d^3\mathbf{k}_2 d^3\mathbf{k}_1 \delta^D(\mathbf{k}-\mathbf{k}_1-\mathbf{k}_2)\alpha(\mathbf{k}_1,\mathbf{k}_2)\tilde{\delta}(\mathbf{k}_1,\tau)\tilde{\theta}(\mathbf{k}_2,\tau), \tag{12}$$

with the function $\alpha(\mathbf{k}_1,\mathbf{k}_2) = 1 + (\mathbf{k}_1\cdot\mathbf{k}_2)/k_2^2$. And

$$\frac{\partial}{\partial\tau}\tilde{\theta}(\mathbf{k},\tau) + \mathcal{H}(\tau)\tilde{\theta}(\mathbf{k},\tau) + \frac{3}{2}\mathcal{H}^2(\tau)\Omega_m(\tau)\tilde{\delta}(\mathbf{k},\tau) = -\int_{\mathbf{k}_1}\int_{\mathbf{k}_2} d^3\mathbf{k}_2 d^3\mathbf{k}_1 \delta^D(\mathbf{k}-\mathbf{k}_1-\mathbf{k}_2)\beta(\mathbf{k}_1,\mathbf{k}_2)\tilde{\theta}(\mathbf{k}_1,\tau)\tilde{\theta}(\mathbf{k}_2,\tau), \tag{13}$$

$\beta(\mathbf{k}_1,\mathbf{k}_2) = |\mathbf{k}_1+\mathbf{k}_2|^2(\mathbf{k}_1\cdot\mathbf{k}_2)/2k_1^2k_2^2$.

The solution for density field movement equations $\delta^n(\mathbf{k})$ are in $n \geqslant 2$ case:

$$\delta^n(\mathbf{k}) = \int d^3\mathbf{q}_1 \cdots \int d^3\mathbf{q}_n \delta^D(\mathbf{k} - \mathbf{q}_1 - \cdots - \mathbf{q}_n) \times F_n(\mathbf{q}_1, ..., \mathbf{q}_n)\delta^1(\mathbf{q}_1)\cdots\delta^1(\mathbf{q}_n), \quad (14)$$

and the solution give peculiar velocities field $\theta^n(\mathbf{k})$:

$$\theta^n(\mathbf{k}) = \int d^3\mathbf{q}_1 \cdots \int d^3\mathbf{q}_n \delta^D(\mathbf{k} - \mathbf{q}_1 - \cdots - \mathbf{q}_n) \times G_n(\mathbf{q}_1, ..., \mathbf{q}_n)\delta^1(\mathbf{q}_1)\cdots\delta^1(\mathbf{q}_n). \quad (15)$$

In agreement to Goroff (1986), Makino (1992), Scoccimarro (2001) F_n and G_n are:

$$F_n(\mathbf{q}_1, ..., \mathbf{q}_n) = \sum_{m=1}^{n-1} \frac{G_m(\mathbf{q}_1, ..., \mathbf{q}_m)}{(2n+3)(n-1)} \Big[(1+2n)\alpha(\mathbf{k}_1, \mathbf{k}_2) \times F_{n-m}(\mathbf{q}_{m+1}, ..., \mathbf{q}_n) + 2\beta(\mathbf{k}_1, \mathbf{k}_2) G_{n-m}(\mathbf{q}_{m+1}, ..., \mathbf{q}_n)\Big], \quad (16)$$

$$G_n(\mathbf{q}_1, ..., \mathbf{q}_n) = \sum_{m=1}^{n-1} \frac{G_m(\mathbf{q}_1, ..., \mathbf{q}_m)}{(2n+3)(n-1)} \Big[3\alpha(\mathbf{k}_1, \mathbf{k}_2) F_{n-m}(\mathbf{q}_{m+1}, ..., \mathbf{q}_n) + 2n\beta(\mathbf{k}_1, \mathbf{k}_2) G_{n-m}(\mathbf{q}_{m+1}, ..., \mathbf{q}_n)\Big], \quad (17)$$

where $\mathbf{k}_1 \equiv \mathbf{q}_1 + \cdots + \mathbf{q}_m$, $\mathbf{k}_2 \equiv \mathbf{q}_{m+1} + \cdots + \mathbf{q}_n$, $\mathbf{k} = \mathbf{k}_1 + \mathbf{k}_2$, and $F_1 = G_1 \equiv 1$.

Correlation Function and Power Spectrum

From these solutions and using correlation function:

$$\left\langle \delta(\mathbf{k}_1,\tau)\delta(\mathbf{k}_2,\tau) \right\rangle = \delta^D(\mathbf{k}_1+\mathbf{k}_2)P(k_2),\ P(k) := \int \frac{d^3\mathbf{r}}{(2\pi)^3} e^{-i\mathbf{k}\cdot\mathbf{r}}\xi(r), \quad (18)$$

and stantard loop correction (for only one loop) Figure 5:

$$P(k) = R^2(\tau)P_{1,1}(k) + R^4(\tau)\left[P_{2,2}(k) + 2P_{1,3}(k)\right]. \quad (19)$$

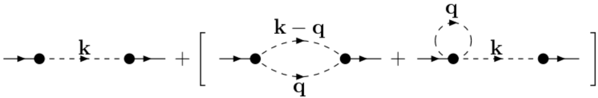

Figure 5: One loop correction Scoccimarro (1996).

With help developed by Makino (1992)

$$P_2(k) = R^2(\tau)P_{1,1}(k) + R^4(\tau)\left[2\int d^3\mathbf{q}P_{1,1}(q)P_{1,1}(|\mathbf{k}-\mathbf{q}|)\right.$$
$$\left.\times\left[F_2^{(s)}(\mathbf{q},\mathbf{k}-\mathbf{q})\right]^2 + 6P_{1,1}(k)\int d^3\mathbf{q}P_{1,1}(q)F_3^{(s)}(\mathbf{q},-\mathbf{q},\mathbf{k})\right], \quad (20)$$

where $P_{2,2}(k)$ is:

$$P_{2,2}(k) = \frac{k^3}{98(2\pi)^2}\int_0^\infty\int_{-1}^1 drdxP_{1,1}(kr)P_{1,1}\left[k(1+r^2-2rx)^{1/2}\right]$$
$$\times\frac{(3r+7x-10rx^2)^2}{(1+r^2-2rx)^2}, \quad (21)$$

and the contribution $P_{1,3}(k)$ is described by:

$$2P_{1,3}(k) = \frac{k^3}{252(2\pi)^2}P_{1,1}(k)\int_0^\infty drP_{1,1}(kr)\left[\frac{12}{r^2} - 158 + 100r^2 - 42r^4\right.$$
$$\left.+\frac{3}{r^3}(r^2-1)^3(7r^2+2)\ln\left|\frac{1+r}{1-r}\right|\right], \quad (22)$$

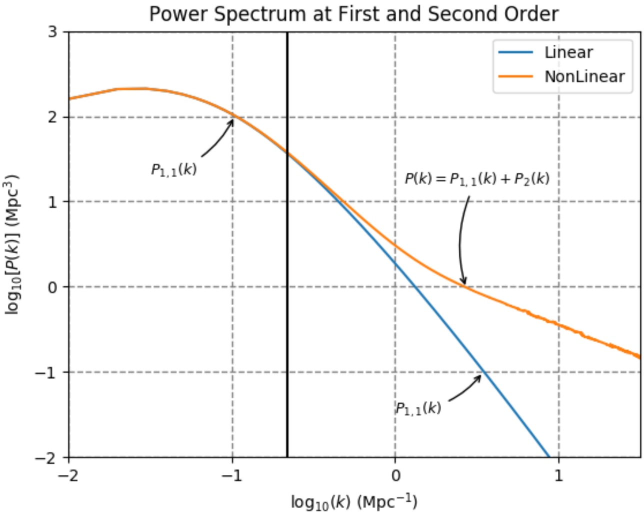

Figure 6: Power Spectrum at Second Order $P(k)$. (Fonseca & Castañeda.,2020)

CDM and Baryonic Matter

Now we propose the solution using baryonic matter Shoji y Komatsu (2009)

$$\frac{\partial}{\partial\tau}\left[\delta_{\mathrm{CDM}}(\mathbf{x},\tau)\right] + \nabla_{\mathbf{x}}\cdot\left[\left[1+\delta_{\mathrm{CDM}}(\mathbf{x},\tau)\right]\mathbf{u}_{\mathrm{CDM}}(\mathbf{x},\tau)\right] = 0, \quad (23)$$

$$\frac{\partial}{\partial\tau}\left[\delta_{\mathrm{B}}(\mathbf{x},\tau)\right] + \nabla_{\mathbf{x}}\cdot\left[\left[1+\delta_{\mathrm{B}}(\mathbf{x},\tau)\right]\mathbf{u}_{\mathrm{B}}(\mathbf{x},\tau)\right] = 0, \quad (24)$$

$$\frac{\partial}{\partial\tau}\mathbf{u}_{\mathrm{CDM}}(\mathbf{x},\tau) + \mathcal{H}(\tau)\mathbf{u}_{\mathrm{CDM}}(\mathbf{x},\tau) + \left[\mathbf{u}_{\mathrm{CDM}}(\mathbf{x},\tau)\cdot\nabla_{\mathbf{x}}\right]\mathbf{u}_{\mathrm{CDM}}(\mathbf{x},\tau) = -\nabla_{\mathbf{x}}\Phi_{\mathrm{PER}}, \quad (25)$$

$$\frac{\partial}{\partial\tau}\mathbf{u}_{\mathrm{B}}(\mathbf{x},\tau) + \mathcal{H}(\tau)\mathbf{u}_{\mathrm{B}}(\mathbf{x},\tau) + \left[\mathbf{u}_{\mathrm{B}}(\mathbf{x},\tau)\cdot\nabla_{\mathbf{x}}\right]\mathbf{u}_{\mathrm{B}}(\mathbf{x},\tau) = -\nabla_{\mathbf{x}}\Phi_{\mathrm{PER}} - \frac{\nabla_{\mathbf{x}}\cdot\sigma(\mathbf{x},\tau)}{\rho_{\mathrm{B}}(\mathbf{x},\tau)}. \quad (26)$$

$$\sigma_{ij} = -P\delta_{ij} + \eta\left[\nabla_i u_j + \nabla_j u_i - \frac{2}{3}\delta_{ij}\nabla\cdot\mathbf{u}\right] + \xi\delta_{ij}\nabla\cdot\mathbf{u}. \quad (27)$$

$$\nabla_{\mathbf{x}}^2\Phi_{\mathrm{PER}}(\mathbf{x},\tau) = \frac{3}{2}\mathcal{H}^2(\tau)\delta(\mathbf{x},\tau) = \frac{6}{\tau^2}\delta(\mathbf{x},\tau), \quad (28)$$

Finally,

$$\frac{\partial}{\partial\tau}\tilde{\delta}_{\mathrm{B}}(\mathbf{k},\tau)+\tilde{\theta}_{\mathrm{B}}(\mathbf{k},\tau) = -\int_{\mathbf{k}_1}\int_{\mathbf{k}_2} d^3\mathbf{k}_2 d^3\mathbf{k}_1 \delta^D(\mathbf{k}-\mathbf{k}_1-\mathbf{k}_2)\alpha(\mathbf{k}_1,\mathbf{k}_2)\tilde{\delta}_{\mathrm{B}}(\mathbf{k}_1,\tau)\tilde{\theta}_{\mathrm{B}}(\mathbf{k}_2,\tau), \quad (29)$$

$$\frac{\partial}{\partial\tau}\tilde{\theta}_{\mathrm{B}}(\mathbf{k},\tau)+\mathcal{H}(\tau)\tilde{\theta}_{\mathrm{B}}(\mathbf{k},\tau)+\frac{3}{2}\mathcal{H}^2(\tau)\Omega_m(\tau)\tilde{\delta}_{\mathrm{B}}(\mathbf{k},\tau) = -\int_{\mathbf{k}_1}\int_{\mathbf{k}_2} d^3\mathbf{k}_2 d^3\mathbf{k}_1 \delta^D(\mathbf{k}-\mathbf{k}_1-\mathbf{k}_2)\beta(\mathbf{k}_1,\mathbf{k}_2)\tilde{\theta}_{\mathrm{B}}(\mathbf{k}_1,\tau)\tilde{\theta}_{\mathrm{B}}(\mathbf{k}_2,\tau) - C_s^2 k^2\left[\tilde{\delta}_{\mathrm{B}}(\mathbf{k},\tau)-\frac{1}{k^2}\int_{\mathbf{k}_1}\int_{\mathbf{k}_2} d^3\mathbf{k}_2 d^3\mathbf{k}_1 \delta^D(\mathbf{k}-\mathbf{k}_1-\mathbf{k}_2)\mathbf{k}_2\cdot(\mathbf{k}_1+\mathbf{k}_2)\tilde{\delta}_{\mathrm{B}}(\mathbf{k}_1,\tau)\tilde{\delta}_{\mathrm{B}}(\mathbf{k}_2,\tau)\right]. \quad (30)$$

1. We reconstruct all perturbation theory at firts order. We found that growth factor is growing on independent of cosmological model, in agreement with cosmological parameters reported in the literature, this factor could be normalized to unity.
2. An important achievement for this work was to get the power spectrum at second order using a semianalitical tools. These approximations there are not widely developed in literature.

Outlook

Renormalized perturbation theory seems to be crucial for future work, to the hope that it holds the key for crucial improvements using methods that permits include baryonic matter on theoretical models, at low computational cost.

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