Cosmic inflation in scalar-vector-tensor (SVT) theory of gravity

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General relativity (GR) and cosmology

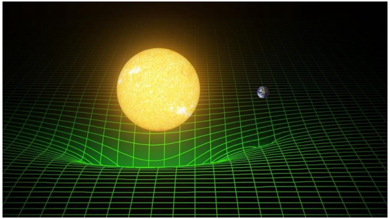


Figure 1: Curvature of spacetime. Credit: *LIGO/T. Pyle*

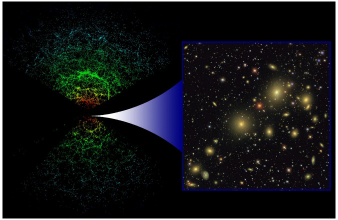


Figure 2: Large scale structure of universe. Credit: *Sloan Digital Sky Survey Team, NASA, NSF, DOE.*

Curvature $\Longleftrightarrow$ Matter/Energy

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = k^2 T_{\mu\nu} \,, \tag{1}$$

where $k^2 = M_{\mathrm{Pl}}^{-2}$.

$M_{\mathrm{Pl}} \rightarrow$ Reduced Planck mass.

Cosmology studies our universe, its origin and evolution. On large scales, the dominant interaction is gravity.

Cosmological principle

Homogeneity and isotropy

Weyl's postulate

Perfect fluid

Note: We use natural units, where $c = \hbar = 1$ such that $x^0 = t$. Also we use the signature $(- + + +)$.

FLRW metric and Friedmann equations

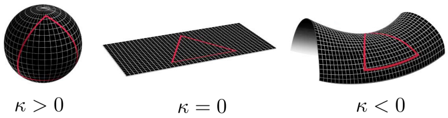
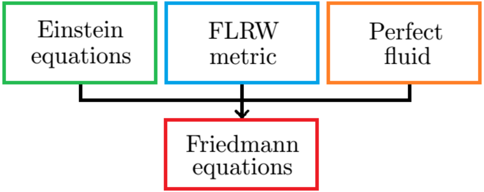


Figure 3: Spatial curvature. Credit: *NASA/WMAP Science Team.*



Cosmological principle leads to FLRW metric:

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + a^2(t)\gamma_{ij}\mathrm{d}x^i\mathrm{d}x^j \,, \tag{2}$$

where $\gamma_{ij} = \delta_{ij} + \kappa\dfrac{x_i x_j}{1-\kappa(x_k x^k)}$.

$a(t) \rightarrow$ Scale factor.

$\kappa \rightarrow$ Curvature parameter.

$$H^2 = \frac{k^2}{3}\rho - \frac{\kappa}{a^2} \tag{3}$$

$$\dot{H} + H^2 = -\frac{k^2}{6}(\rho + 3P) \tag{4}$$

$$\dot{\rho} + 3H(\rho + P) = 0 \tag{5}$$

where $H := \dot{a}/a$ is the Hubble parameter.

Shortcomings of standard model of cosmology

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The hot Big Bang model has been a successful theory which is consistent with observations. However, it cannot explain certain problems.

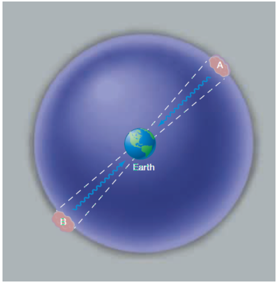


Figure 4: Horizon problem. [**McMillan C.**, *Astronomy*]

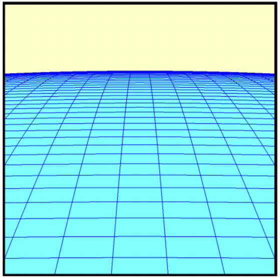


Figure 5: Flatness problem. [**Guth, A. H. y Kaiser D.**, *Science* **307**, 5711 (2005)]

Figure 6: Structure formation. Credit: © NASA's Goddard Space Flight Center

Cosmic inflation and inflaton scalar field

Inflation

It's defined as a stage of accelerated expansion of early universe which can explain (theoretically) the issues mentioned above.

$$\ddot{a} > 0 \quad \Longrightarrow \quad \rho + 3P < 0$$

Another way to express this condition is defining the slow-roll parameter

$$\epsilon := -\frac{\dot{H}}{H^2}\,, \tag{6}$$

such that the accelerated expansion is reached if $\epsilon < 1$. However, this violates the strong energy condition.

[**Guth A.**, *Phys. Rev. D* **23**, 347 (1981)]
[**Linde A.**, *Phys Lett B* **108**, 6 (1982)]

For modelling this situation, a single scalar field $\phi(t)$ is introduced. The energy and pressure densities of ϕ are:

$$\rho = \frac{1}{2}\dot{\phi}^2 + V(\phi)\,, \quad P = \frac{1}{2}\dot{\phi}^2 - V(\phi)\,, \tag{7}$$

where V is the potential energy. From the condition $\epsilon < 1$ is shown that $\dot{\phi}^2/2 < V$. In order to keep this condition, is necessary that the parameter

$$\eta := -\frac{\ddot{\phi}}{H\dot{\phi}} \tag{8}$$

is smaller than 1. Thus, inflation is reached if $\{\epsilon, |\eta|\} < 1$.

Modifications of GR

GR works really good

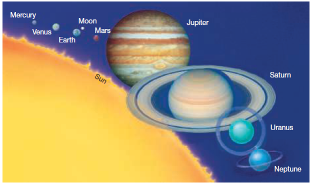


Figure 7: Planetary and stellar scales.
[**McMillan C.**, *Astronomy*]

GR fails

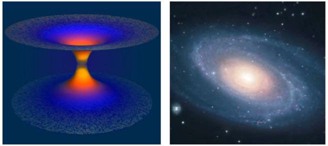
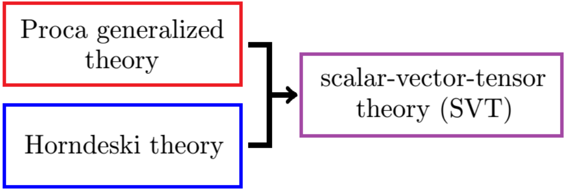


Figure 8: Quantum and extragalactic scales.
[**McMillan C.**, *Astronomy*]

D. Lovelock postulated an important theorem in theoretical physics. An alternative to modify GR is adding extra degrees freedom.

Besides metric tensor $g_{\mu\nu}$, it's possible to add a scalar field ϕ as extra degree freedom. In analogous way, we can add a vector field A_μ.



[**Lovelock D.**, *J Math Phys* **12**, 498 (1971)]
[**Lovelock D.**, *J Math Phys* **13**, 874 (1972)]
[**Heisenberg L.**, *JCAP* **2018**, 10 (2018)]

The SVT model

This model is a particular case of general SVT theory showed in [Heisenberg L., JCAP 2018, 10 (2018)]. The action is given by:

$$S_{\mathrm{SVT}} = \int \mathrm{d}^4x\sqrt{-g}\Big[\frac{M_{\mathrm{Pl}}^2}{2}R - \frac{1}{2}\nabla_\mu\phi\,\nabla^\mu\phi - V(\phi) - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{2}\beta_A M^2 A_\mu A^\mu - \frac{1}{2}\beta_m M A^\mu\nabla_\mu\phi + \Big(\beta_G A^\mu A^\nu + f(\phi)\nabla^\mu A^\nu\Big)G_{\mu\nu}\Big], \tag{9}$$

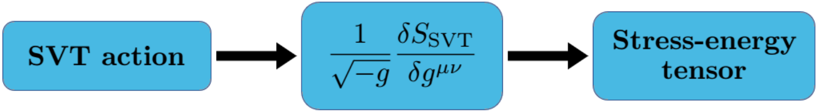
where:

$V(\phi)$: Potential of scalar field,
M : Constant with mass dimension,
$\beta_{m,A,G}$: Coupling constants,

$f(\phi)$: Scalar coupling function,
$F_{\mu\nu} = \nabla_\mu A_\nu - \nabla_\nu A_\mu$.

The objective of this SVT model is to study the inflationary dynamics and carry out and stability analysis.

Field equations and stress-energy tensor



Varying the action (9) respect to $g^{\mu\nu}$, we get the Einstein field equations:

$$R_{\mu\nu}-\frac{1}{2}Rg_{\mu\nu}=k^2\left[T^{(\phi)}_{\mu\nu}+T^{(A)}_{\mu\nu}+T^{(A\phi)}_{\mu\nu}\right], \tag{10}$$

where:

$$T^{(\phi)}_{\mu\nu}=\nabla_\mu\phi\,\nabla_\nu\phi-g_{\mu\nu}\left(\frac{1}{2}\nabla_\alpha\phi\,\nabla^\alpha\phi+V(\phi)\right), \tag{11}$$

$$\begin{aligned}T^{(A)}_{\mu\nu}=&\,F_{\mu\alpha}F_\nu{}^\alpha+\beta_A M^2A_\mu A_\nu+\beta_G\Big[R_{\mu\nu}A_\alpha A^\alpha-4G_{(\mu\alpha}A_{\nu)}A^\alpha-RA_\mu A_\nu+2\nabla_\alpha\nabla_{(\mu}\left(A_{\nu)}A^\alpha\right)\\&-\Box\left(A_\mu A_\nu\right)-\nabla_\mu\nabla_\nu\left(A_\alpha A^\alpha\right)\Big]-g_{\mu\nu}\Big[\frac{1}{4}F_{\alpha\beta}F^{\alpha\beta}+\frac{1}{2}\beta_A M^2A_\alpha A^\alpha+\beta_G\Big(\nabla_\alpha\nabla_\beta\left(A^\beta A^\alpha\right)\\&-\Box\big(A_\alpha A^\alpha\big)-G_{\alpha\beta}A^\alpha A^\beta\Big)\Big],\end{aligned} \tag{12}$$

Field equations and stress-energy tensor

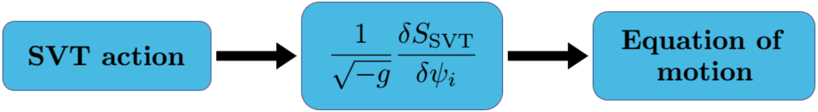
$$
\begin{aligned}
T_{\mu\nu}^{(A\phi)} &= \beta_m M A_{(\mu}\nabla_{\nu)}\phi + f(\phi)\left[R_{\mu\nu}\nabla_\alpha A^\alpha - R\nabla_{(\mu}A_{\nu)} - 2G_{(\mu\alpha}\nabla_{\nu)}A^\alpha\right] + 2G_{(\mu\alpha}A_{\nu)}\nabla^\alpha f(\phi) \\
&- \nabla_\alpha\left[f(\phi)G_{\mu\nu}A^\alpha\right] - \nabla_\mu\nabla_\nu\left[f(\phi)\nabla_\alpha A^\alpha\right] + \nabla_\alpha\nabla_{(\mu}\left[f(\phi)\nabla^\alpha A_{\nu)}\right] + \nabla_\alpha\nabla_{(\mu}\left[f(\phi)\nabla_{\nu)}A^\alpha\right] \\
&- \Box\left[f(\phi)\nabla_{(\mu}A_{\nu)}\right] - g_{\mu\nu}\left[\frac{1}{2}\beta_m M A^\alpha\nabla_\alpha\phi - f(\phi)G_{\alpha\beta}\nabla^\alpha A^\beta + \nabla_\alpha\nabla_\beta\left[f(\phi)\nabla^\beta A^\alpha\right]\right. \\
&\left. - \Box\left[f(\phi)\nabla_\alpha A^\alpha\right]\right].
\end{aligned}
\tag{13}
$$

$\Box(\cdot) := \nabla_\mu\nabla^\mu(\cdot)$ is the covariant D'Alembertian operator.

$T_{\mu\nu}^{(\phi)} \rightarrow$ Contribution due to ϕ, $T_{\mu\nu}^{(A)} \rightarrow$ Contribution due to A_μ,

$T_{\mu\nu}^{(A\phi)} \rightarrow$ Contribution due to interaction between A_μ y ϕ.

Equations of motion for ϕ and A_μ



Varying the action (9) respect to the fields ϕ and A_μ, we get, respectively, the equations of motion for each field:

$$\Box\phi - \frac{\partial V}{\partial \phi} + \frac{1}{2}\beta_m M \nabla_\mu A^\mu + G_{\mu\nu}\nabla^\mu A^\nu \frac{\partial f}{\partial \phi} = 0\,, \tag{14}$$

$$\nabla^\nu F_{\mu\nu} + \frac{1}{2}\beta_m M \nabla_\mu \phi + \beta_A M^2 A_\mu - \Big(2\beta_G A^\nu - \nabla^\nu f(\phi)\Big) G_{\mu\nu} = 0\,. \tag{15}$$

Notice that the equation for ϕ is the Klein-Gordon equation with extra terms due to the new couplings in the action (9).

Flat FLRW geometry

Now we introduce flat FLRW metric (setting $\kappa = 0$ in (2)), given by the line-element:

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + a^2(t)\delta_{ij}\mathrm{d}x^i\mathrm{d}x^j \,. \tag{16}$$

Furthermore, we set the scalar and vector fields as

$$\phi = \phi(t), \qquad A_\mu = \Big(A_0(t), 0, 0, 0\Big) \tag{17}$$

in order to preserve homogeneity and isotropy. Thus, the Einstein tensor components are:

$$\begin{aligned} G_{00} &= 3H^2, \\ G_{ij} &= -\Big(2\dot{H} + 3H^2\Big)a^2\delta_{ij} \,, \\ G_{i0} &= G_{0i} = 0 \,. \end{aligned} \tag{18}$$

Flat FLRW geometry

In this geometry, sum of the stress-energy tensors shown in (11)-(13) becomes in:

$$T_{00} = \frac{1}{2}\dot{\phi}^2 + V(\phi) + \frac{1}{2}\beta_A M^2 A_0^2 + 9\beta_G H^2 A_0^2 + \frac{1}{2}\beta_m M A_0 \dot{\phi} - 9H^2 A_0 \dot{\phi}\frac{\partial f}{\partial \phi} = \rho\,, \tag{19}$$

$$\begin{aligned} T_{ij} = \Bigg[& \frac{1}{2}\dot{\phi}^2 - V(\phi) + \frac{1}{2}\beta_A M^2 A_0^2 - 3\beta_G H^2 A_0^2 - 2\beta_G \dot{H} A_0^2 - 4\beta_G H A_0 \dot{A}_0 \\ & + \frac{1}{2}\beta_m M A_0 \dot{\phi} + 3H^2 A_0 \dot{\phi}\frac{\partial f}{\partial \phi} + 2H\dot{A}_0\dot{\phi}\frac{\partial f}{\partial \phi} + 2\dot{H} A_0 \dot{\phi}\frac{\partial f}{\partial \phi} \\ & + 2HA_0\ddot{\phi}\frac{\partial f}{\partial \phi} + 2HA_0\dot{\phi}^2\frac{\partial^2 f}{\partial \phi^2}\Bigg] a^2\delta_{ij} = Pa^2\delta_{ij}\,. \end{aligned} \tag{20}$$

Now, we can identify the energy and pressure densities (ρ, P). Then, the Einstein equations (1) becomes in:

$$3M_{\rm Pl}^2 H^2 = \rho\,, \quad -\left(2\dot{H} + 3H^2\right) M_{\rm Pl}^2 = P\,. \tag{21}$$

Flat FLRW geometry

In a similar way, introducing the flat FLRW metric (16) and the fields (17) in the equations of motion for ϕ and A_μ, we get respectively:

$$\ddot{\phi} + 3H\dot{\phi} + \frac{\partial V}{\partial \phi} + \frac{1}{2}\beta_m M\left(\dot{A}_0 + 3HA_0\right) - 3H\left(3H^2 A_0 + H\dot{A}_0 + 2\dot{H}A_0\right)\frac{\partial f}{\partial \phi} = 0\,, \tag{22}$$

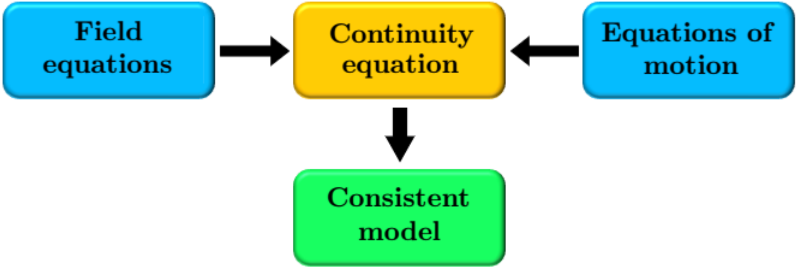
$$\left[\beta_A M^2 + 6\beta_G H^2\right]A_0 - \left[3H^2\frac{\partial f}{\partial \phi} - \frac{1}{2}\beta_m M\right]\dot{\phi} = 0\,. \tag{23}$$

Notice that:

- The equation for ϕ is the known equation for the single inflaton field with extra terms due to the new couplings considered in the action of the model.
- The equation for A_0 establish a direct connection between $\dot{\phi}$ and A_0.
- Both equations, as expected, are not higher than second order.

Consistency of the SVT model

We verify that our model is consistent with conservation of energy-momentum. Also this is useful for checking if equations are correct.



Consistency of the SVT model

Substituting $\dot{\rho}$, ρ y P in (5), we get:

$$\dot{\rho}+3H(\rho+p)=\dot{\phi}\left[\ddot{\phi}+3H\dot{\phi}+\frac{\partial V}{\partial\phi}+\frac{1}{2}\beta_m M\left(\dot{A}_0+3HA_0\right)-3H\left(3H^2A_0+H\dot{A}_0+2\dot{H}A_0\right)\frac{\partial f}{\partial\phi}\right]$$
$$+3HA_0\left[\left(\beta_A M^2+6\beta_G H^2\right)A_0-\left(3H^2\frac{\partial f}{\partial\phi}-\frac{1}{2}\beta_m M\right)\dot{\phi}\right]+A_0\left[\frac{1}{2}\beta_m M\ddot{\phi}\right.\quad, \tag{24}$$
$$\left.+\beta_A M^2\dot{A}_0+12\beta_G H\dot{H}A_0+6\beta_G H^2\dot{A}_0-6H\dot{H}\dot{\phi}\frac{\partial f}{\partial\phi}-3H^2\ddot{\phi}\frac{\partial f}{\partial\phi}-3H^2\dot{\phi}^2\frac{\partial^2 f}{\partial\phi^2}\right].$$

but, it can be showed that:

$$\frac{1}{2}\beta_m M\ddot{\phi}+\beta_A M^2\dot{A}_0+12\beta_G H\dot{H}A_0+6\beta_G H^2\dot{A}_0-6H\dot{H}\dot{\phi}\frac{\partial f}{\partial\phi}-3H^2\ddot{\phi}\frac{\partial f}{\partial\phi}$$
$$-3H^2\dot{\phi}^2\frac{\partial^2 f}{\partial\phi^2}=\frac{d}{dt}\left[\left(\beta_A M^2+6\beta_G H^2\right)A_0-\left(3H^2\frac{\partial f}{\partial\phi}-\frac{1}{2}\beta_m M\right)\dot{\phi}\right] \tag{25}$$

Consistency of the SVT model

thus

$$\dot{\rho}+3H(\rho+P)=\dot{\phi}\left[\ddot{\phi}+3H\dot{\phi}+\frac{\partial V}{\partial\phi}+\frac{1}{2}\beta_m M\left(\dot{A}_0+3HA_0\right)-3H\left(3H^2A_0+H\dot{A}_0+2\dot{H}A_0\right)\frac{\partial f}{\partial\phi}\right]$$
$$+3HA_0\left[\frac{1}{2}\beta_m M\dot{\phi}+\beta_A M^2A_0+3H^2\left(2\beta_G A_0-\dot{\phi}\frac{\partial f}{\partial\phi}\right)\right] \qquad (26)$$
$$+A_0\frac{d}{dt}\left[\frac{1}{2}\beta_m M\dot{\phi}+\beta_A M^2A_0+3H^2\left(2\beta_G A_0-\dot{\phi}\frac{\partial f}{\partial\phi}\right)\right],$$

$$\dot{\rho}+3H(\rho+P)=0. \qquad (27)$$

Hence, this show us that the SVT model is consistent with conservation of energy and momentum.

Conclusions and expectations

So far, we can safely conclude that the equations obtained in this model are consistent with the principle of conservation of moment energy. Furthermore, as expected, they are second order equations, which frees Ostrogradski's model of instabilities.

From the current development of this work, we hope to successfully achieve the following:

- Define slow-roll parameters for the scalar coupling function $f(\phi)$ in order to simplify the equations showed above.
- Study the inflationary dynamics with a particular potential $V(\phi)$. In addition, taking $f(\phi) = \omega/\phi^n$ where $n \in \mathbb{Z}$, find the values of n such that cosmic inflation is reached.
- Do a stability analysis of this SVT model through cosmological perturbation theory.

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$$\left(\frac{\dot{R}}{R}\right)^2 + \frac{kc^2}{R^2} = \frac{8\pi}{3}G\rho + \frac{\Lambda}{3}$$

$$\frac{\ddot{R}}{R} = -\frac{4\pi G}{3}\left(\rho + 3\frac{P}{c^2}\right) + \frac{\Lambda}{3}$$

$$\dot{\rho} = -3\left(\frac{\dot{R}}{R}\right)\left(\rho + \frac{P}{c^2}\right)$$

Thanks!