Cosmic Dynamo Equation Under Cosmological Perturbation Theory at First Order

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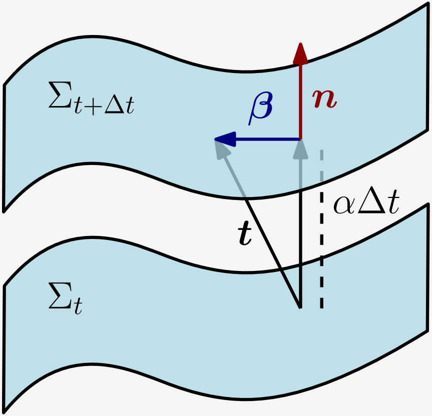


[Gourgoulhon, 2012, Alcubierre, 2008, Baumgarte and Shapiro, 2010, Shibata, 2015]

In the 3+1 formulation used in Numerical Relativity (NR) the spacetime is splitted in a spatial part and a temporal part. A foliation of spacetime is made and there is a spatial projection over hypersurfaces and along the normal vector to the hypersurfaces. Let us denote Σ_t the hypersurfaces of the foliation and $\boldsymbol{n}$ the normal vector to the hypersurfaces.

Let α be the lapse function, $\boldsymbol{\beta}$ be the shift vector and $\boldsymbol{\gamma}$ the metric tensor of the hypersurfaces. The extrinsic curvature, change of the normal vector along the hypersurfaces, is denoted as $\boldsymbol{K}$. The Eulerian observers are the observers with 4-velocity $\boldsymbol{n}$.

Maxwell equation in NR

The electric and magnetic field, $\boldsymbol{E}$ and $\boldsymbol{B}$ respectively, are tangent to the hypersurfaces Σ_t. The charge density ρ_e and the electric current $\boldsymbol{J}$ observed by an Eulerian observer are projection along the normal vector and the hypersurface respectively. Using the Faraday tensor is posible to obtain the Maxwell equations in $3+1$ formalism

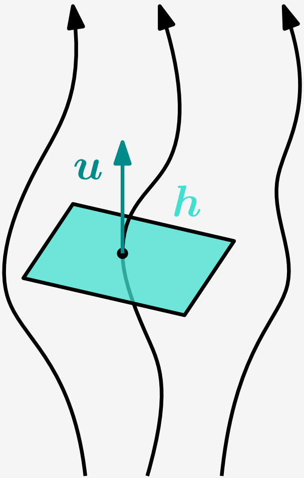
$$D_i B^i = 0, \tag{1}$$

$$D_i E^i = \mu_0 \rho_e, \tag{2}$$

$$\left(\partial_t - \mathcal{L}_\beta\right) B^i - \alpha K B^i + \epsilon^{ijk} D_j \left(\alpha E_k\right) = 0, \tag{3}$$

$$\left(\partial_t - \mathcal{L}_\beta\right) E^i - \alpha K E^i - \epsilon^{ijk} D_j \left(\alpha B_k\right) = -\mu_0 \alpha J^i, \tag{4}$$

where $\boldsymbol{D}$ is the covariant derivative of the induced metric $\boldsymbol{\gamma}$, $\mathcal{L}$ is the Lie derivative and K is the trace of the extrinsic curvature tensor.
[Gourgoulhon, 2012, Baumgarte and Shapiro, 2010, Shibata, 2015]



[Ellis et al., 2012, Ellis, 2009, Roy, 2014, Park, 2018]

For a perfect fluid, if we have an observer moving along the fluid, Lagrangian observer, we can split the spacetime in this case in a spatial hypersurface given by $\boldsymbol{h}$ and a projection along the fluid velocity $\boldsymbol{u}$, this splitting is known as the $1+3$ formalism.

The characteristic quantities that characterize the kinematic features of the fluid are given by the decomposition of the covariant derivative of $\boldsymbol{u}$

$$\nabla_\alpha u_\mu = \sigma_{\mu\alpha} + \omega_{\mu\alpha} - \dot{u}_\mu u_\alpha + \frac{1}{3}\Theta h_{\mu\alpha}. \quad (5)$$

where $\sigma_{\mu\alpha}$ is the shear, $\omega_{\mu\alpha}$ is the vorticity and Θ is the expansion rate.

Maxwell equations for Lagrangian observers

Similar to the 3+1 formalism, the electric and magnetic field, ***e*** and ***b*** respectively, are tanget to the hypersurfaces given by ***h***. The charge density ρ_u and the electric current $\boldsymbol{J}_u$ observed by an Eulerian observer are projection along the normal vector and the hypersurface respectively. Using the Faraday tensor is posible to obtain the Maxwell equations in $1+3$ formalism

$$\bar{\nabla}_\alpha b^\alpha = 2\omega^\alpha e_\alpha, \tag{6}$$

$$\bar{\nabla}_\beta e^\beta = 2\omega^\delta b_\delta - \rho_u, \tag{7}$$

$$h^\gamma_\alpha \dot{b}^\alpha = \left(\sigma^\gamma_\beta + \omega^\gamma_\beta - \frac{2}{3}\Theta\delta^\gamma_\beta\right) b^\beta - \epsilon^{\gamma\mu\nu\beta} u_\mu \nabla_\beta e_\nu + h^\gamma_\alpha \epsilon^{\mu\nu\alpha\beta} \dot{u}_\mu u_\beta e_\nu, \tag{8}$$

$$h^\gamma_\alpha \dot{e}^\alpha = \left(\sigma^\gamma_\beta + \omega^\gamma_\beta - \frac{2}{3}\Theta\delta^\gamma_\beta\right) e^\beta - \epsilon^{\gamma\mu\nu\beta} u_\mu \nabla_\beta b_\nu + h^\gamma_\alpha \epsilon^{\mu\nu\alpha\beta} \dot{u}_\mu u_\beta b_\nu - J^\gamma_u, \tag{9}$$

where $(\dot{\cdots}) = u^\alpha \nabla_\alpha (\cdots)$ and $\bar{\nabla}$ is the projection of the covariant derivative under ***h***.

[Ellis et al., 2012, Ellis, 2009, Subramanian, 2016, Marklund and Clarkson, 2005]

Background and perturbed universe

Here we will apply cosmological perturbations, therefore we need to fix a background solution. The universe will be describe by a spatially flat Friedman-Lemaître-Robertson-Walker (FLRW) solution

$$ds^2 = a^2(\eta)\left(-d\eta^2 + \delta_{ij}dx^i dx^j\right), \tag{10}$$

where $a(\eta)$ is the scale factor and η the conformal time. At first order the line element is given by

$$ds^2 = a^2(\eta)\left[-(1+2\Psi)\, d\eta^2 - 2\omega_i dx^i d\eta + (1+2\Phi)\,\delta_{ij}dx^i dx^j + \chi_{ij}dx^i dx^j\right], \tag{11}$$

where (Ψ, Φ) are scalar, (ω_i) vector and (χ_{ij}) tensor perturbations.
[Hortua et al., 2013, Durrer, 2008, Bruni et al., 1997]

Using the Newtonian gauge

$$ds^2 = a^2(\eta)\left[-(1+2\Psi)\,d\eta^2 + (1-2\Phi)\,\delta_{ij}dx^i dx^j\right]. \tag{12}$$

The background perfect fluid space-time is given by

$$T_{\mu\nu} = (\rho + P)\,u_\mu u_\nu + P g_{\mu\nu}. \tag{13}$$

The field equations perturbed at first order are given by [Macpherson et al., 2016]

$$\nabla^2\Phi - 3\mathcal{H}\left(\Phi' + \mathcal{H}\Psi\right) = 4\pi\rho\delta a^2, \qquad \mathcal{H}\partial_i\Psi + \partial_i\Phi' = -4\pi\rho a^2\delta_{ij}v^j_{(1)}, \tag{14}$$

$$\Phi'' + \mathcal{H}\left(\Psi' + 2\Phi'\right) = \frac{1}{2}\nabla^2\left(\Phi - \Psi\right), \qquad 0 = \left[\partial_i\partial_j - \frac{1}{3}\delta_{ij}\nabla^2\right]\left(\Phi - \Psi\right), \tag{15}$$

where $\mathcal{H} = a'/a$ and $\delta = -1 + \rho^{(1)}/\rho$. From $\nabla_\alpha T^\alpha_\mu = 0$

$$\delta' = 3\Phi' - \partial_i v^i_{(1)}, \qquad \mathcal{H}v^i_{(1)} + \left(v^i_{(1)}\right)' = -\partial^i\Psi. \tag{16}$$

Perturbed NR quantities

Perturbed quantities for NR [Durrer and Straumann, 1988]

$$\alpha = a(\eta)(1+\psi) \tag{17}$$

$$\beta_i = a^2(\eta)\,\omega_i \tag{18}$$

$$\gamma_{ij} = a^2(\eta)\left(\delta_{ij} - 2\phi\delta_{ij} + \chi_{ij}\right) \tag{19}$$

$$K = -3\mathcal{H}a^{-1} + a^{-1}\left[3\phi' + 3\mathcal{H}\psi + \delta^{ij}\partial_{(i}\omega_{j)}\right] \tag{20}$$

$$\Gamma^k_{ij} = -2\delta^k_{(i}\partial_{j)}\phi + \partial_{(i}\chi^k_{j)} + \delta_{ij}\delta^{kl}\partial_l\phi - \frac{1}{2}\delta^{kl}\partial_l\chi_{ij} \tag{21}$$

The perturbations of the electromagnetic fields seen by an Eulerian observer are

$$B^i = \frac{1}{a^2}\left(B^i_{(0)} + B^i_{(1)}\right), \qquad E^i = \frac{1}{a^2}\left(E^i_{(0)} + E^i_{(1)}\right), \tag{22}$$

$$B_i = a^2\left(B_i^{(0)} + B_i^{(1)}\right), \qquad E_i = a^2\left(E_i^{(0)} + E_i^{(1)}\right), \tag{23}$$

and for the 4-current

$$j^\mu = \frac{1}{a}\left(j^\mu_{(0)} + j^\mu_{(1)}\right) \qquad j^\mu = \left(\rho_e, J^i\right) \tag{24}$$

$$\boxed{D_i B^i = 0,} \tag{25}$$

Background

$$\partial_i B^i_{(0)} = 0, \tag{26}$$

First order perturbation

$$\partial_i B^i_{(1)} + \frac{1}{2} B^k_{(0)} \left[-2\delta^i_k \partial_i \phi + \partial_i \chi^i_k - 4\partial_k \phi - \delta^{il} \partial_l \chi_{ik} \right] = 0. \tag{27}$$

$$\boxed{\left(\partial_t - \mathcal{L}_\beta\right) B^i - \alpha K B^i + \epsilon^{ijk} D_j \left(\alpha E_k\right) = 0,} \tag{28}$$

Background

$$\left(a^{-2} B^i_{(0)}\right)' + 3\mathcal{H}^{-2} B^i_{(0)} + \epsilon^{ijk} a^3 \partial_j E^{(0)}_k = 0, \tag{29}$$

First order perturbation

$$\left(a^{-2} B^i_{(1)}\right)' - \mathcal{H} a^{-1} B^i_{(0)} \left(3\phi' + \delta^{kj} \partial_{(j}\, \omega_{\, k)}\right) - 3\mathcal{H} a^{-2} B^i_{(1)} + a^{-2} B^k_{(0)} \partial_k \omega^i +$$
$$a^{-2} \omega^k \partial_k B^k_{(0)} + a^3 \epsilon^{ijk} \left[\partial_j \left(E^{(1)}_k + \psi E^{(0)}_k \right) - E^{(0)}_l \left(\delta_{jk} \delta^{lm} \partial_m \phi - \frac{1}{2} \delta^{lm} \partial_m \chi_{jk} \right) \right] = 0. \tag{30}$$

$$\boxed{D_i E^i = \mu_0 \rho_e,} \tag{31}$$

Background

$$\partial_i E^i_{(0)} = a\mu_0 \rho_e, \tag{32}$$

First order perturbation

$$\partial_i E^i_{(1)} - E^j_{(0)} \left[\delta^l_j \partial_l \phi - \delta^i_j \partial_i \phi - 3\partial_j \phi + \frac{1}{2} \left(\partial_i \chi^i_j - \delta^{il} \partial_l \chi_{ij} \right) \right] = a\mu_0 \rho^{(1)}_e . \tag{33}$$

$$\boxed{\left(\partial_t - \mathcal{L}_\beta\right) E^i - \alpha K E^i - \epsilon^{ijk} D_j \left(\alpha B_k\right) = -\mu_0 \alpha J^i,} \tag{34}$$

Background

$$\left(a^{-2} E^i_{(0)}\right)' + 3\mathcal{H} a^{-1} E^i_{(0)} + a^3 \epsilon^{ijk} \partial_j B^{(0)}_k = -\mu_0 J^i_{(0)}, \tag{35}$$

First order perturbation

$$\left(a^{-2} E^i_{(1)}\right)' - \mathcal{H} a^{-1} E^i_{(0)} \left(3\phi' + \delta^{kj} \partial_{(j}\, \omega_{\, k)}\right) - 3\mathcal{H} a^{-2} E^i_{(1)} + a^{-2} E^k_{(0)} \partial_k \omega^i + a^{-2} \omega^k \partial_k E^k_{(0)}$$
$$+ a^3 \epsilon^{ijk} \left[\partial_j \left(B^{(1)}_k + \psi B^{(0)}_k \right) - B^{(0)}_l \left(\delta_{jk} \delta^{lm} \partial_m \phi - \frac{1}{2} \delta^{lm} \partial_m \chi_{jk} \right) \right] = -\mu_0 \left(J^i_{(1)} + \psi J^i_{(0)} \right) . \tag{36}$$

Perturbed kinematic quantities

The perturbations of the electromagnetic field for the Lagrangian obverse are

$$b^i = a^{-2}\left(b^i_{(0)} + b^i_{(1)}\right), \qquad e^i = a^{-2}\left(e^i_{(0)} + e^i_{(1)}\right), \tag{37}$$

$$b_i = a^2\left(b_i^{(0)} + b_i^{(1)}\right), \qquad e_i = a^2\left(e_i^{(0)} + e_i^{(1)}\right), \tag{38}$$

and for the 4-current $j^\mu = a^{-1}\left(j^\mu_{(0)} + j^\mu_{(1)}\right)$ and the velocity $\boldsymbol{u}$

$$j^\mu = \left(\rho_u, J^i_u\right), \qquad u^\mu = a^{-1}\left(1-\psi, v^i\right), \qquad u_\mu = a\left(-1-\psi, \omega_i + v_i\right). \tag{39}$$

from the decomposition of $\nabla_\beta u_\alpha$

$$\omega_{ij} = a\left(\partial_{[i}\,\omega^{(1)}_{j]} + \partial_{[i}\,v^{(1)}_{j]}\right), \tag{40}$$

$$\sigma_{ij} = a\left(\partial_{(i}\,\omega^{(1)}_{j)} + \partial_{(i}\,v^{(1)}_{j)}\right) - \frac{a^2}{3}\left[\left(-H\psi^{(1)} - \phi^{(1)}\right)\delta_{ij} + H\left(-2\phi^{(1)}\delta_{ij} + \chi^{(1)}_{ij}\right)\right], \tag{41}$$

$$\Theta = a^{-1}\left(\partial_j v^j - 3\phi\right). \tag{42}$$

For the background solution, zero order, we have isotropy and homogeneity, then, taking a spatial average

$$b^i_{(0)} = e^i_{(0)} = J^i_{(0)} = \rho^{(0)}_u = 0. \tag{43}$$

This allows to rewrite the Maxwell equations for the Lagrangian fields as follow

$$\partial_j b^j_{(1)} = 0, \qquad \partial_j e^j_{(1)} = -a\rho^u_{(1)}, \tag{44}$$

$$\left(b^i_{(1)}\right)' - 2\mathcal{H} b^i_{(1)} - \epsilon^{ij}{}_k a\partial_j e^k_{(1)} = 0, \qquad \left(e^i_{(1)}\right)' - 2\mathcal{H} e^i_{(1)} - \epsilon^{ij}{}_k a\partial_j b^k_{(1)} = a^2 J^i_{u(1)}. \tag{45}$$

For the Ohm's law

$$J^i_{u(1)} = \sigma_{cond}\left[e^i_{(1)} + \epsilon^i{}_{jk}\left(\omega^j + v^j\right) b^k_{(0)}\right]. \tag{46}$$

The cosmic dynamo equations are the following [Hortua et al., 2013, Hortua, 2011, Marklund and Clarkson, 2005]

$$\left(b^i_{(1)}\right)' - 2\mathcal{H} b^i_{(1)} - \epsilon^{ij}{}_k a\partial_j\left[\epsilon^{kl}{}_m\left(\omega^l + v^l\right) b^m_{(0)}\right] - \epsilon^{ij}{}_k\partial_j\left\{\eta_{dif}\, a^{-1}\left[\left(e^k_{(1)}\right)' - 2\mathcal{H} e^k_{(1)} - \epsilon^{kl}{}_m a\partial_l b^m_{(1)}\right]\right\} = 0, \tag{47}$$

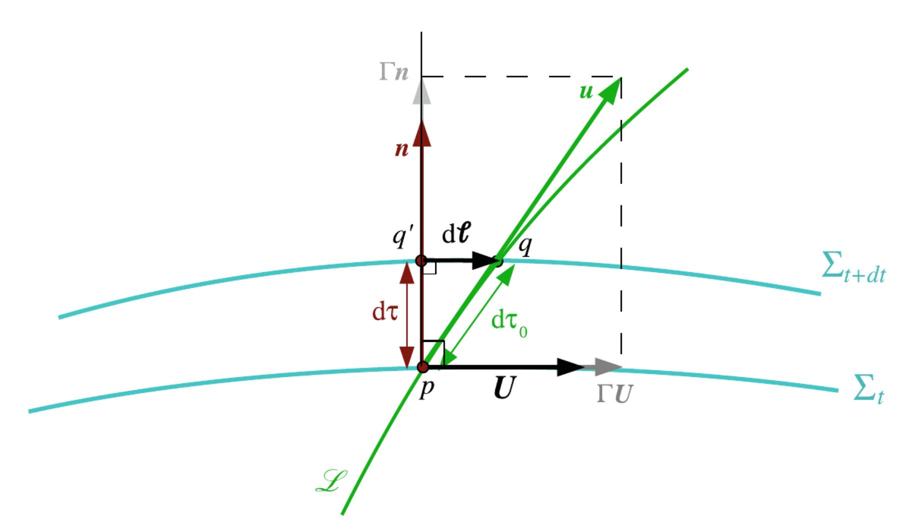


Figure: The normal vector and the 4-velocity vector do not always have to match, then the hypersurfaces given by $\boldsymbol{h}$ and γ neather .Taken from [Gourgoulhon, 2012]

Transformation between 1+3 and 3+1 formalism

The relation between Lagrangian (1+3) and Eulerian (3+1) observers is given by [Bona et al., 2009, Bucciantini and Del Zanna, 2012]

$$e^{\mu} = -WE^{\nu} - (E^{\nu} u_{\nu})\, n^{\mu} + \epsilon^{\delta\gamma\mu\nu} B_{\gamma} n_{\delta} u_{\nu}, \tag{48}$$

$$b^{\mu} = WB^{\mu} + (B^{\nu} u_{\nu})\, n^{\mu} + \epsilon^{\delta\gamma\mu\nu} E_{\gamma} n_{\delta} u_{\nu}. \tag{49}$$

Projecting along the normal vector and over the hypersurfaces

$$e^{\mu} n_{\mu} = E^{\nu} u_{\nu}, \qquad \gamma_{\mu\nu} e^{\nu} = -WE_{\mu} + \epsilon_{\mu}^{\delta\gamma\alpha} B_{\gamma} n_{\delta} u_{\alpha}, \tag{50}$$

$$b^{\mu} n_{\mu} = -B^{\nu} u_{\nu}, \qquad \gamma_{\mu\nu} b^{\nu} = WB_{\mu} + \epsilon_{\mu}^{\delta\gamma\alpha} E_{\gamma} n_{\delta} u_{\alpha}. \tag{51}$$

For the 4-current

$$\boldsymbol{j} = \rho_u \boldsymbol{u} + \boldsymbol{J}_u = \rho \boldsymbol{n} + \boldsymbol{J}, \tag{52}$$

taking $\rho_u \boldsymbol{u} + \boldsymbol{J}_u$ and projecting along the normal vector and over the hypersurfaces

$$\rho = -W\rho_u + J_u^{\mu} n_{\mu}, \qquad J_{\mu} = \rho_u \left(\gamma_{\mu\nu} u^{\nu}\right) + \gamma_{\mu\nu} J_u^{\nu}. \tag{53}$$

For the 1+3 formalism

$$h_{\alpha\beta} = g_{\alpha\beta} + u_\alpha u_\beta, \tag{54}$$

and for 3+1

$$\gamma_{\alpha\beta} = g_{\alpha\beta} + n_\alpha n_\beta. \tag{55}$$

We are working under an induced coordinate system and with first order perturbations. For

$$u_\alpha u_\beta \text{ and } n_\alpha n_\beta \sim (\text{Background}) + (\text{Second order terms}), \tag{56}$$

as a consequence

$$h_{ij} = \gamma_{ij}. \tag{57}$$

This implies that $\boldsymbol{n}$ and $\boldsymbol{u}$ are colineal, we can see it also from the Lorentz factor

$$W = -n^\mu u_\mu = 1, \tag{58}$$

but we have to take into account that they are different.

The Lagrangian fields, ***e*** and ***b***, and the Eulerian fields, ***E*** and ***B***, lies on the same hypersurface, therefore

$$e_i = -E_i + \epsilon_{ijk} B^j u^k, \tag{59}$$

$$b_i = B_i - \epsilon_{ijk} E^j u^k. \tag{60}$$

Because isotropy and homogeneity

$$e_i^{(1)} = -E_i^{(1)} + a^{-3} \epsilon_{ijk} B^j_{(0)} v^k, \tag{61}$$

$$b_i^{(1)} = B_i^{(1)} + a^{-3} \epsilon_{ijk} E^j_{(0)} v^k = B_i^{(1)}. \tag{62}$$

Finally, the cosmic dynamo equation is

$$\left(B^i_{(1)}\right)' - 2\mathcal{H} B^i_{(1)} = -\epsilon^{ij}{}_k a\partial_j \left[\epsilon^{kl}{}_m \left(\omega^l + v^l\right) B^m_{(0)}\right] + \epsilon^{ij}{}_k \partial_j \left\{\eta_{dif} a^{-1} \left[\left(-E_k^{(1)} + a^{-3} \epsilon^k{}_{lm} B^l_{(0)} v^m\right)' + 2\mathcal{H}\left(E_k^{(1)} - a^{-3} \epsilon^k{}_{lm} v^m B^l_{(0)}\right) - \epsilon^{kl}{}_m B^m_{(1)}\right]\right\}. \tag{63}$$

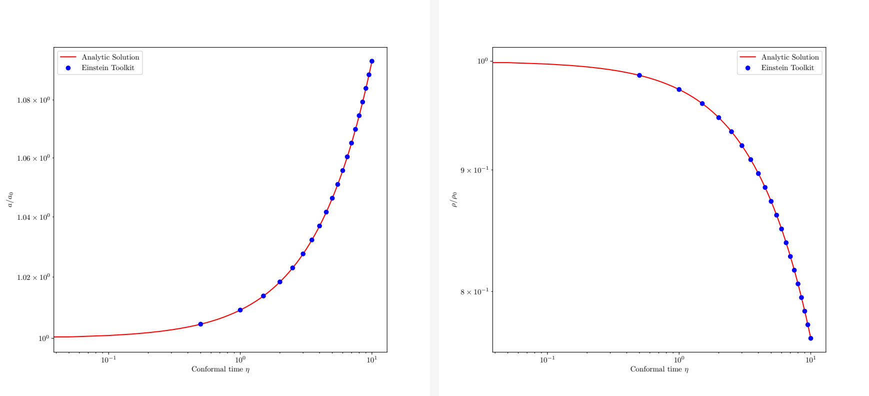


Figure: Preliminary test results for the background universe without a magnetic field. The results where obtain using `Einstein Toolkit` and `FLRWSolver` [Macpherson et al., 2016]

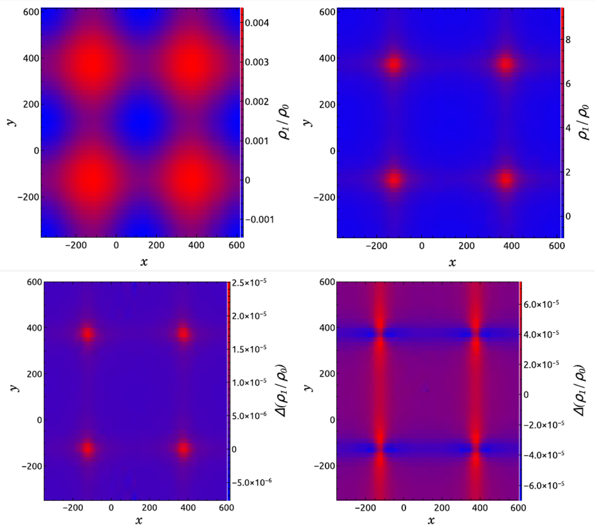


Figure: Evolution of NR equations only including scalar perturbations, and scalar and primordial tensor perturbations in the simulations. Taken from [Wang, 2018].

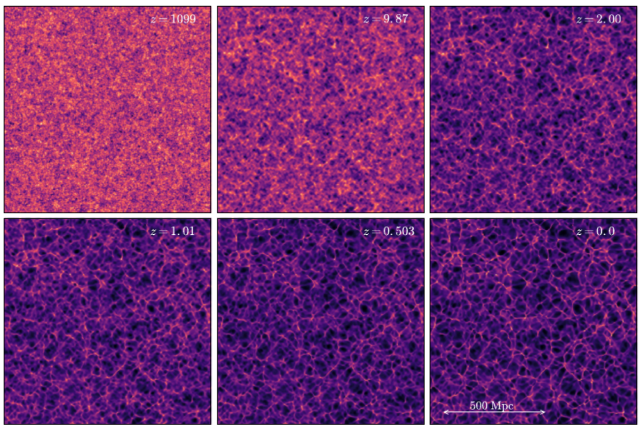


Figure: Evolution of a fully general-relativistic cosmic web using `Einstein Toolkit` with 1 Gpc domain. Taken from [Macpherson et al., 2019].

Conclusions and future perspectives

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- From the dynamo equations obtain for the observer moving along with the fluid, the dynamo equations for the Eulerian observer, transforming the fields from Lagrangian to Eulerian perspective, were obtain following [Bucciantini and Del Zanna, 2012].

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- From the dynamo equations obtain for the observer moving along with the fluid, the dynamo equations for the Eulerian observer, transforming the fields from Lagrangian to Eulerian perspective, were obtain following [Bucciantini and Del Zanna, 2012].
- Next step is to implement computationally the dynamo equation to be able to obtain the values of the magnetic field along the cosmological history.

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