Thermodynamics of $f(R)$ Theories of Gravity

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Abstract

Abstract

This work starts from a toy model for inflation in a class of modified theories of gravity in the metric formalism. Instead of the standard procedure — assuming a non-linear Lagrangian $f(R)$ in the Jordan frame — we start from a simple ϕ^2 potential in the Einstein frame and investigate the corresponding $f(R)$ in the former picture. Such approach yields plenty of new pieces of information, namely a self-terminating inflationary solution with a linear Lagrangian, a robust criterion for stability of such theories, the addition of an ad-hoc Cosmological Constant in the Einstein frame leads to a Thermodynamical interpretation of this physical system, which allows further insight on its (meta)stability and evolution.

Modifications to ϕ^2 Potential

We choose the inflationary quadratic potential and we will make the following modification:

$$V_{sqm}(\phi) = \frac{1}{2} m_\phi^2 \left(\phi - a\right)^2 + \Lambda, \tag{1}$$

From the Slow-Roll analysis we obtain the analytical solution

$$\phi_{sqm}(t) = a - 11\sqrt{2}\, M_P + \sqrt{\frac{2}{3}}\, m_\phi\, t \tag{2}$$

We solved the full equation system of movement for the scalar field for this case with initial conditions set from the analytic slow-roll above (2), namely:

$$\phi_{sqm}(0) \approx a - 15.5 M_P, \quad \dot{\phi}_{sqm}(0) = \sqrt{\frac{2}{3}}\, m_\phi \approx 0.81\, m_\phi. \tag{3}$$

Numerical solutions

Numerical solutions are shown in Figure 1.

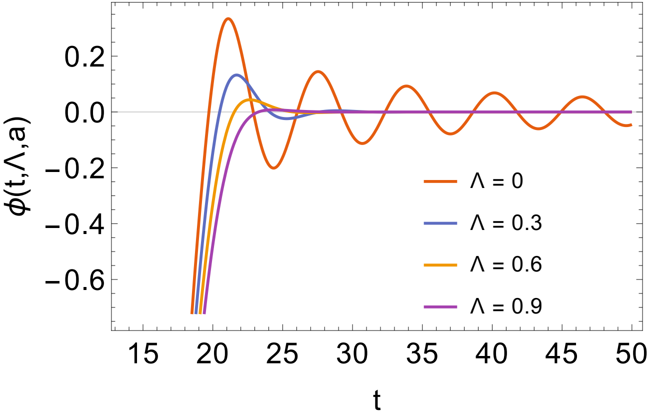


Figure: Numerical solutions potential (1), with $N = 60$ efolds and $a = 0$, using $m_\phi = 1$ and $M_P = 1$. Note that the curves are smoothed as Λ increases. Also, the curves shift up if $a > 0$ or shift down if $a < 0$.

Change of Frames

From Jordan to Einstein Frame

$$S_{met} = \frac{1}{2k} \int_V d^4x\, f(R) \sqrt{-g},$$

We can rewrite the function $f(R)$ using a Legendre transformation, defining the follow quantities:

$$R_p = R(p) \quad (4)$$
$$f^*(p) \equiv f[R(p)] \quad (5)$$
$$H(p) \equiv p\, R_p - f^*(p). \quad (6)$$

Now a new second-order Lagrangian $\hat{L}$ in the space of variables $\{g_{\mu\nu}, p\}$ by setting

$$\hat{L}(g, p) \equiv \sqrt{-g}\, (p\, R(g) - H(p))\,. \quad (7)$$

Change of Frames

Now, it is convenient to replace p by the new auxiliary field $\phi(p)$ defined by

$$p = e^{\beta\phi}, \quad \therefore \quad \beta \equiv \sqrt{2/3} \tag{8}$$

replacing (8) in the conformal transformation for R

$$\hat{R} = \Omega^{-2}\left\{R - 6g^{\alpha\gamma}\nabla_\alpha\nabla_\gamma \ln\Omega - 6g^{\alpha\gamma}\nabla_\alpha \ln\Omega \cdot \nabla_\gamma \ln\Omega\right\}. \tag{9}$$

The Lagrangian (5) then can be recast in a more familiar form:

$$\hat{L} = \sqrt{-\hat{g}}\left(\hat{R} - \hat{g}^{\mu\nu}\phi_{,\mu}\phi_{,\nu} - 2V(\phi)\right), \tag{10}$$

$$\therefore \quad V(\phi) \equiv \frac{1}{2p^2}\Big\{p(\phi)R[p(\phi)] - f[R(p(\phi))]\Big\}, \tag{11}$$

which is completely determined by the particular $f(R)$ chosen.

The Inverse Problem

From Einstein to Jordan frame

Parametric solution for the $f(R)$ function that depends completely on the potential $V_E(\phi)$.[1]

$$f(\phi) = \mathrm{e}^{2\beta\,\phi}\left[2V_E(\phi) + 2\beta^{-1}\frac{dV_E(\phi)}{d\phi}\right], \tag{12}$$

$$R(\phi) = \mathrm{e}^{\beta\,\phi}\left[4V_E(\phi) + 2\beta^{-1}\frac{dV_E(\phi)}{d\phi}\right]. \tag{13}$$

[1] Guido Magnano and Leszek M. Sokolowski. Phys. Rev.D50 (1994), pp. 5039–5059. arXiv:gr-qc/9312008 [gr-qc].[1]

The Inverse Problem

From Einstein to Jordan frame

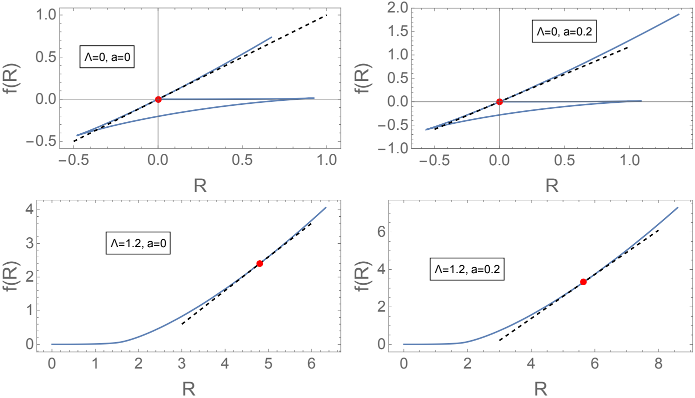
We generalize the potential $V_E(\phi)$, to include a shift in the ϕ-vacuum value and a cosmological constant

$$V_E(\phi) = \frac{1}{2} m_\phi^2 (\phi - a)^2 + \Lambda. \tag{14}$$

We then obtain the corresponding parametric form of $f(R)$:

$$f(\phi) = e^{2\beta\phi}\left[m_\phi^2 (a - \phi)\left(a - \phi - \frac{2}{\beta}\right) + 2\Lambda\right] \tag{15}$$

$$R(\phi) = 2e^{\beta\phi}\left[m_\phi^2 (a - \phi)\left(a - \phi - \frac{1}{\beta}\right) + 2\Lambda\right], \tag{16}$$



If $\Lambda < \Lambda_c$ (to be defined later on), the curve features a 3-branch structure. In particular, on the final branch, one recovers GR only if $f' = \exp(\beta a) = 1$, i.e, if $a = 0$. Regardless of a, the system does reach a de Sitter state, the modified Lagrangian given by Eqs. (15) and (16) can be written as the linear function $f(R) = \exp(\beta a)R - 2\Lambda_J$.

Results

Thermodynamics Interpretation

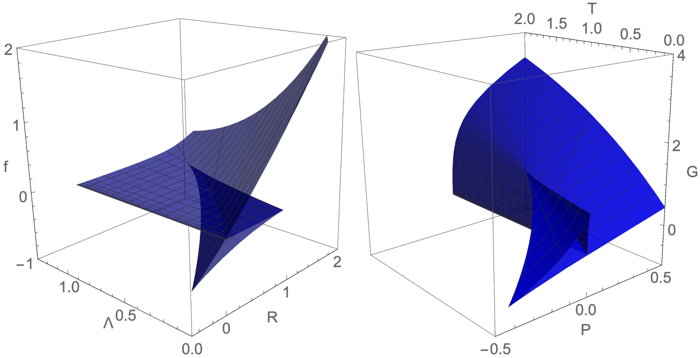


Figure: Plots of $f(R,\Lambda)$, given by Eqs. (15, 16), and $G(P,T)$, given by Eqs. (23) and (24) with $\beta=\sqrt{2/3}$ and $a=0$.

The Stability Criteria from Thermodynamics Analogy

For now, let us associate the Cosmological Constant Λ to an effective temperature $T \equiv \Lambda$. We define a new pair of coordinates $\{-G, P\}$ as a rotation of the original one $\{f, R\}$:

$$\begin{pmatrix} -G \\ P \end{pmatrix} \equiv \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} f \\ R \end{pmatrix}. \tag{17}$$

which yields

$$\begin{aligned} G(\phi, T) = {} & e^{\beta\phi} \sin(\theta) \left(\frac{2(\phi - a)(\beta(\phi - a) + 1)}{\beta} + 4T \right) + \\ & - e^{2\beta\phi} \cos(\theta) \left(\frac{2(\phi - a)}{\beta} + (\phi - a)^2 + 2T \right) \end{aligned} \tag{18}$$

$$\begin{aligned} P(\phi, T) = {} & e^{2\beta\phi} \sin(\theta) \left(\frac{2(\phi - a)}{\beta} + (\phi - a)^2 + 2T \right) + \\ & + e^{\beta\phi} \cos(\theta) \left(\frac{2(\phi - a)(\beta(\phi - a) - 1)}{\beta} + 4T \right) \end{aligned} \tag{19}$$

The effective volume V is the variable "canonically conjugated" to the effective pressure P, i.e, since

$$dG(P,T) = V \cdot dP - S \cdot dT, \tag{20}$$

one can define an effective volume

$$V \equiv \left.\frac{\partial G}{\partial P}\right|_T = \left.\frac{\partial G/\partial \phi}{\partial P/\partial \phi}\right|_T = \frac{1 - e^{\beta\phi}\cot(\theta)}{e^{\beta\phi} + \cot(\theta)}, \tag{21}$$

which can be inverted and yield

$$\phi = \frac{1}{\beta}\log\left(\frac{1 - V\cot(\theta)}{\cot(\theta) + V}\right). \tag{22}$$

In order to define the exact correspondence, $\theta = \theta_* \equiv \pi/2$

$$G = e^{\beta\phi}\left(\frac{2(\phi - a)[\beta(\phi - a) + 1]}{\beta} + 4T\right) \tag{23}$$

$$P = e^{2\beta\phi}\left(\frac{2(\phi - a)}{\beta} + (\phi - a)^2 + 2T\right) \tag{24}$$

$$V = \exp(-\beta\phi) \quad \Leftrightarrow \quad \phi = -\frac{1}{\beta}\log(V). \tag{25}$$

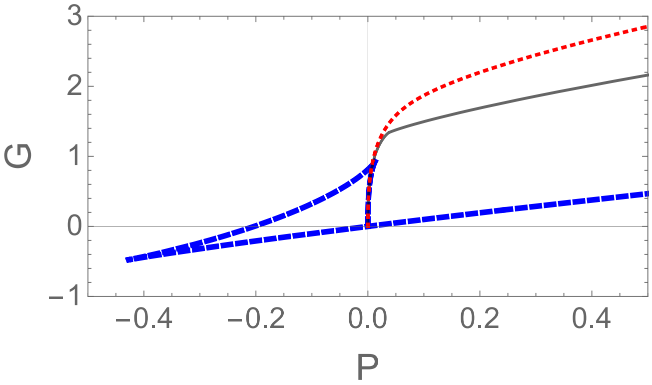


Figure: Plot of the Gibbs Potential G as a function of the pressure P, for $\beta = \sqrt{2/3}$, $\theta = \theta_*$ and $T = 0$ (dashed blue), $T = T_c = 15/16$ (solid black) and $T = 1.5$ (dotted red).

One can also calculate the Helmholtz energy

$$F(T,V) \equiv G - P \cdot V = \frac{e^{2\beta\phi}\csc(\theta)\left[(a-\phi)^2 + 2T\right]}{e^{\beta\phi} + \cot(\theta)} \tag{26}$$

$$= \frac{1}{V}\left[\left(a + \frac{1}{\beta}\log V\right)^2 + 2T\right] \qquad \text{if} \quad \theta = \pi/2 \tag{27}$$

from which one can define the entropy as

$$S(T,V) \equiv -\left.\frac{\partial F}{\partial T}\right|_V = -\frac{2\sin(\theta)(V\cot(\theta) - 1)^2}{\cot(\theta) + V} \tag{28}$$

$$= -\frac{2}{V} \qquad \text{if} \quad \theta = \pi/2 \tag{29}$$

$C_V \equiv T \cdot \partial S/\partial T|_V = 0\,\forall T$. Such feature is not unusual: it has been already found in studies of thermodynamics and phase transitions of black holes [2].

The internal energy $U(T,V)$ is given by its standard definition:

$$U \equiv G - P\cdot V + T\cdot S = \frac{(a-\phi)^2 e^{2\beta\phi}\csc(\theta)}{e^{\beta\phi}+\cot(\theta)} \tag{30}$$

$$= \frac{1}{V}\left(a + \frac{1}{\beta}\log V\right)^2 \qquad \text{if} \quad \theta = \pi/2, \quad \forall\, \mathbf{T}. \tag{31}$$

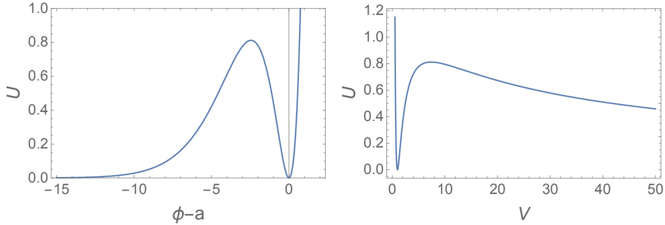


Figure: Plot of the U with $\theta = \theta_* = \pi/2$ as a function of $\phi - a$ (left panel) and of the volume V with $a = 0$ (right panel). We recall that $V = \exp(-\beta\phi)$.

Equations (24) and (25) yield the equation of state for our vdW-like "efective gas", i.e, an expression that relates P, V and T:

$$P = \frac{\beta\left(a^2\beta - 2a + 2\beta T\right) + (2a\beta - 2 + \log V)\log V}{\beta^2 V^2}. \tag{32}$$

Here one obtains $P \propto TV^{-2}$ in the high-temperature limit, instead of the standard ideal-gas behavior $P \propto TV^{-1}$.

We can define the **binodal** and **spinodal** curves, that indicate, respectively, the regions of **metastability** and **instability** of the system — see Fig. 5. The *critical point* $\{P_c, T_c, V_c\}$, defined at the crossing of those curves, indicates the end of the coexistence line.

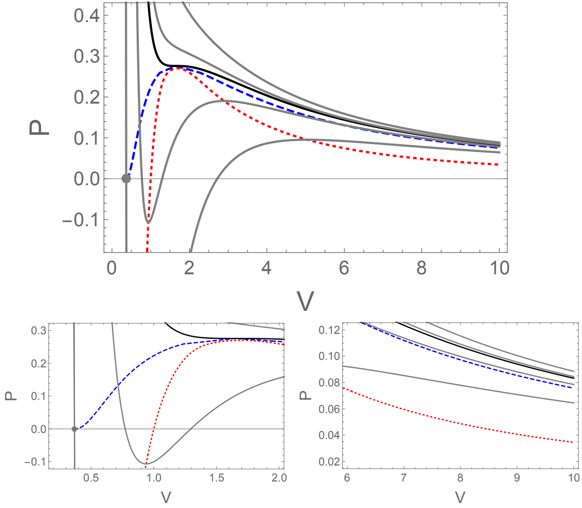


Figure: Plot of the effective pressure P as a function of the effective volume V, $T = T_c \equiv 15/16$ (solid thick black). The **binodal** curve is plotted in dashed blue. The **spinodal** curve is plotted in dotted red.

The entropy as a function of pressure and temperature provides another very important piece of information. $S(P, T)$ is depicted in Fig. 6, which also shows the spinodal and binodal curves. The region where the entropy is multi-valued is known in Catastrophe Theory [3] as a cusp and indicates the existence of a first-order phase transition and unstable configurations.

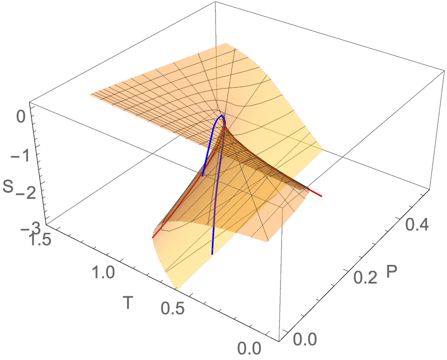


Figure: Surface given by $S(P, T)$ for $a = a_*$. The spinodal and binodal curves are indicated in red (horizontal cusp shape) and blue (vertical "$\subset$" shape), respectively.

From $S(P, T)$ we can get the specific heat at constant pressure, $C_P \equiv T \cdot \partial S/\partial T|_P$, shown in Fig. 7. We obtain the expected behavior for temperatures around the coexistence curve, for pressures both below (finite jump) and above (smooth behavior) the critical value P_c. We also obtain the usual divergence at the critical point $\{T_c, P_c\}$ (solid black line in Fig. 7) as given by $C_P|_{P_c} \sim [(T_c - T)/T_c]^{\alpha}$, with $\alpha \approx 1.00$.

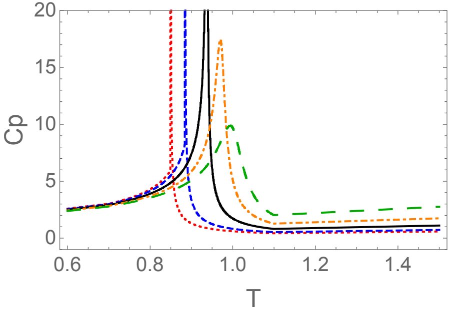


Figure: Behavior of the specific heat at constant pressure C_P as a function of the temperature T close to its transition value ($T_c = 15/16 \approx 0.94$ if $P = P_c$).

vdW fluid

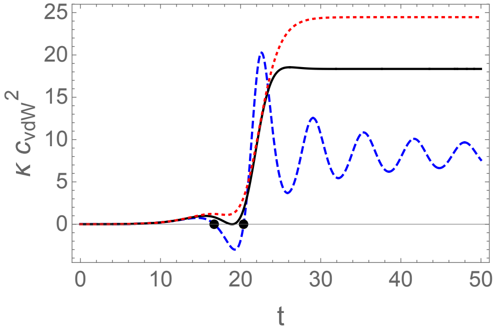


Figure: Plot of $\kappa \cdot c^2_{\rm vdW} \equiv \kappa \cdot \dot{P}/\dot{\rho}$ for the effective vdW gas, for $a = a_*$ and $T = 0$ (dashed blue), $T = 15/16$ (black) and $T = 1.5$ (dotted red), as functions of time. The black dots indicate when $\dot{R}(t) = 0$, between which $f''(R) < 0$.

Two important pieces of information are available only from the vdW gas and *not* from either the curvature fluid or the ϕ field.

1. **The sound speed squared:** defined as $c^2_{\rm vdW} \equiv \dot{P}/\dot{\rho} = -(V^2/\kappa)\dot{P}/\dot{V}$ (where we define $\kappa > 0$ by $\rho =: \kappa/V$) and plotted in Fig. 8. We can see that $c^2_{\rm vdW} < 0$ *only* between the first two extrema of $R(t)$, i.e, in the second branch, when $f'' < 0$, as expected from the usual *perturbative* argument on stability of $f(R)$ theories [4]. Obviously, for $T > T_c$, the second branch is suppressed and one obtains $c^2_{\rm vdW} > 0\, \forall t$. With an imaginary sound speed, fluctuations grow exponentially fast, but, during the spinodal decomposition process, only a given range of wavelength do so [5]. This is similar to a feature that has already been proposed in the preheating scenario [6].
2. **The sudden change in the entropy:** from $S(\phi \to -\infty) = 0$ to $S(\phi = a) = -2e^{2\beta a}$, marking the release of latent heat, just as expected in an ordinary first-order phase transition, which has already been pointed out by the C_P behavior. The relation with (p)reheating will also be the subject of future work.

Conclusions and Outlooks

- We study the conformal transformations between the modified gravity theories $f(R)$ and Einstein's gravity, knowing the function $f(R)$ in the Jordan frame from an inflationary potential in the Einstein frame. We choose the case of the potential ϕ^2 as a toy model, a multivalued function with catastrophic behavior was found for the parametric equations (12) and (13).
- We explored this viable stability criteria for the modified gravity theories $f(R)$ that contain more complete information than the standard stability criterion based on the second derivative of $f(R)$:
    - The Stability criteria from thermodynamics analogy.

Conclusions and Outlooks

- It is important to highlight the role played by introducing the cosmological constant in the Einstein frame where it is not a dynamic quantity, while its transformation in the Jordan frame introduces various states of stability and instability (phase transition) in the system. On the other hand, the cosmological constant allowed the study of the stability criterion based on the thermodynamic analogy in which we identify it as the effective temperature of the system.
- We are currently investigating other physical consequences of the thermodynamic approach, that may indicate that either the potential we assume is a simple toy model, as expected, or that there might be some compatibility with observable quantities from inflation, for instance. In particular, during the spinodal decomposition process, only a given range of wavelength is exponentially amplified [5]. A similar feature has already been proposed in the preheating scenario [6].

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