Quasilocal Smarr relations for static black holes[1]

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- Black hole (BH) quantities obey laws that resemble thermodynamics, e.g. horizon area never decreases [BCH73].
- BH radiate and have an entropy (Bekenstein-Hawking)[Haw75]:

$$S_{BH} = \frac{A_H}{4},$$

- Thermodynamic aspects of gravity are not limited to BH, but they are manifested in any spacetime with horizons [Unr76][UW84].
  - This result applies to apparent horizons in Cosmology as well [GH77].

The Smarr relation

- It is the gravitational counterpart to the thermodynamic Euler equation, relating intensive and extensive variables.
  - For an ideal gas, for example, Euler relation reads,

$$U = TS - PV + \mu N.$$

- In the context of general relativity, for Kerr-Newman black holes [Sma73]:

$$M = 2TA_H + 2\Omega J + \Phi Q,$$

- The **Smarr Relation is a requirement that proposed thermodynamical variables must fulfill**.

The Problem

- Identification of thermodynamic quantities is based on analogies, **this can lead to ambiguities**:
  - For black holes, M can be internal energy [BCH73], or enthalpy [KRT09].
  - If a pressure is associated with Λ [KMT17], its conjugate variable (volume) is not clearly associated with a physical volume.
  - Some prescriptions lead to thermodynamically unstable potentials.
- We want to define thermodynamic variables appropriately for static black holes and study the associated **Smarr Relation.**

How can we identify thermodynamic variables?

- Conserved charges in asymptotically flat spacetimes[BCH73].
  - There are issues concerning thermodynamic stability.
- Possibility: variables defined on a finite region **(quasilocal)** [Sza09].
- Hamilton-Jacobi approach for Euclidean path integrals (Brown & York [BYJ93b]).
  - Thermodynamics can be connected to the canonical description of a system.
  - Provides directly a thermodynamic fundamental equation ($Entropy(state\ variables)$).

Hamilton-Jacobi method in Classical Mechanics

- This method identifies momentum and energy directly from the action.

$$S = \int dt \left[p\frac{dx}{dt} - H(x,p,t) \right] = \int_{\lambda'}^{\lambda''} d\lambda \left[p\dot{x} - \dot{t}H(x,p,t) \right]$$

- The variation of this action is

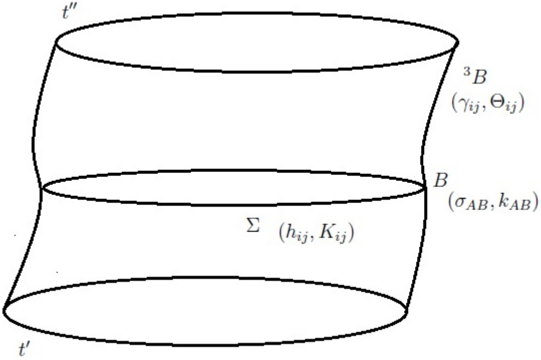
$$\delta S = (\text{e.o.m. terms}) + p\delta x|_{\lambda'}^{\lambda''} - H\delta t|_{\lambda'}^{\lambda''} .$$

- Therefore, for the action evaluated on classical solutions:

$$p = \frac{\partial S_{cl}}{\partial x}; E = H_{cl} = -\frac{\partial S_{cl}}{\partial t}$$

Brown-York Approach

- Spacetime: $M \simeq \Sigma \times I$, with I a closed interval in $\mathbb{R}$.



$$\gamma_{ij} dx^i dx^j = -N^2 dt^2 + \sigma_{AB}(d\theta^A + V^A dt)(d\theta^B + V^B dt),$$

Brown-York Approach

- Partition function is evaluated in terms of path integrals on spatially finite regions.

$$Z[\varphi,\pi] \propto \int \mathcal{D}[\varphi,\pi] \exp\left(-S^{(E)}_{grav}[\varphi,\pi]\right)$$

- Variation of the **on-shell action** defines the thermodynamic variables (Hamilton-Jacobi):

$$\delta S^{cl} = \int_{^3B} d^3z\sqrt{-\gamma}\pi^{ij}\delta\gamma_{ij} \text{ where } \pi^{ij} = \frac{2}{\sqrt{-\gamma}}\frac{\delta S_{grav}}{\delta\gamma_{ij}},$$

$$\delta S^{cl} = \int_{^3B} d^3z\sqrt{\sigma}\left(-\epsilon\delta N + j_a\delta V^a + \frac{N}{2}s^{sb}\delta\sigma_{ab}\right).$$

Brown-York Approach

- For a static spacetime ($V^a = 0$):

$$\epsilon = \frac{1}{8\pi} k \; ; j_a = 0,$$

$$s^{ab} = \frac{1}{8\pi} \left(k^{ab} + (n_\mu a^\mu - k)\sigma^{ab} \right)$$

- For black holes, these variables satisfy a "first law" [BYJ93a]:

$$\begin{aligned} \delta S[\epsilon, j, \sigma] \quad &\approx \delta \left(\frac{A_H}{4} \right) \\ &= \int_B d^2\theta \left[\beta \delta(\sqrt{\sigma}\epsilon) + \beta \left(\sqrt{\sigma} \frac{p^{ab}}{2} \right) \delta\sigma_{ab} \right], \end{aligned}$$

where p is defined in terms of time integrals of s^{ab} and N.

Spherically Symmetric case: Schwarzschild

- In spherically symmetric static spacetimes we consider the metric

$$ds^2 = -N(r)^2 dt^2 + h(r)^2 dr^2 + r^2 d\Omega^2.$$

- Σ are hypersurfaces $t = constant$, whereas ${}^3B$ is defined by $r = constant$. **Ignoring matter terms** in the action:

$$\epsilon = \frac{1}{4\pi}\left(\frac{1}{r} - \frac{1}{rh}\right)\Bigg|_{r=R},$$

$$p \equiv \frac{1}{2}\sigma_{ab}s^{ab} = \frac{1}{8\pi}\left(\frac{N'}{Nh} + \frac{1}{rh} - \frac{1}{r}\right)\Bigg|_{r=R}.$$

Spherically Symmetric case: Schwarzschild

- Defining the quasilocal energy as $E = \int_B d^2\theta \sqrt{\sigma}\epsilon$, we find

$$T\delta S = \delta E + p\delta A.$$

- The scaling behavior of these variables leads to a **Quasilocal Smarr Relation**:

$$2TS = E + 2pA.$$

- With the obtained quasilocal quantities, it is found that:

$$2TS = \frac{1}{N}\left[\frac{1}{2}R^2\frac{(N^2)'}{Nh}\right] = \frac{1}{N}E_{Komar}(R),$$

which can be verified by replacing N and h for the Schwarzschild metric.

Reissner-Nördstrom Spacetime

- In this case we have a matter Lagrangian:

$$\mathcal{L}_M = -\frac{1}{16\pi} F_{\mu\nu} F^{\mu\nu},$$

- **Boundary terms of the Hamiltonian must be considered.** This is a general feature of this approach in presence of matter.
- Variation of the on-shell action must be supplemented by a Noether charge analysis to obtain the first law [Cre96]:

$$\delta S = \int_B d^2\theta \sqrt{\sigma} \beta \left(\delta\epsilon + s^{AB} \delta\sigma_{AB} + \Phi\delta\Xi \right),$$

where Φ is the electrostatic potential, and

$$\Xi = \frac{1}{4\pi} r_b E^b,$$

Reissner-Nördstrom Spacetime

- The electrostatic potential in this case is given by,

$$\Phi(r) = \frac{Q}{N(r)} \left(\frac{1}{r} - \frac{1}{r_H} \right),$$

- Integration of the first law on the spherical quasilocal surface or radius r leads to

$$T\delta S = \delta E + P\delta A + \Phi\delta Q.$$

- Therefore, the **Quasilocal Smarr relation for Reissner-Nordstrom black holes is**:

$$2TS = E + 2PA + \Phi Q.$$

Reissner-Nördstrom Spacetime

- The thermodynamic variables in this case are written as:

$$
\begin{aligned}
E &= -r\frac{1}{h(r)}, \\
P &= \frac{1}{8\pi h(r)}\left(\frac{1}{r} + \frac{d}{dr}[\log N(r)]\right), \\
T &= \frac{1}{N(r)}\frac{2 - 2M/r_H}{4\pi r_H}, \\
A &= 4\pi r^2, \\
S &= \frac{A_H}{4} = \pi r_H^2.
\end{aligned}
$$

Reissner-Nördstrom Spacetime

- Inserting these variables into the Quasilocal Smarr Relation, **together with non-thermodynamic information: the definition of r_H and that** $N(r)h(r) = 1$, leads to

$$\frac{2M}{r^2} - \frac{2Q^2}{r^3} = \frac{d}{dr}[N(r)^2].$$

- This equation can be integrated trivially to give

$$N^2(r) = 1 - \frac{2M}{r} + \frac{Q^2}{r^2},$$

- Therefore, **the Quasilocal Smarr Relation can be regarded as a realization of Einstein equations.**

Remarks

- The resulting Smarr Relations are **independent of the explicit metric**, and could be regarded as a constraint between any possible thermodynamic variables.
- To recover Einstein equations, **some non-thermodynamic information** must be supplied.
- In cosmological settings, we will need to take into account **boundary terms associated to sources**, together with the Noether charge construction, to identify the microcanonical action.
- Some sources are unable to satisfy the resulting Smarr relation, this fact **could provide thermodynamic criteria to filter out models** (such approach could be a quasilocal equivalent to [BJNAN15]).

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