Stationary Worldline Power Distributions

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Third Colombian Meeting on High Energy Physics
December 2018

Stationary Worldlines

Linear

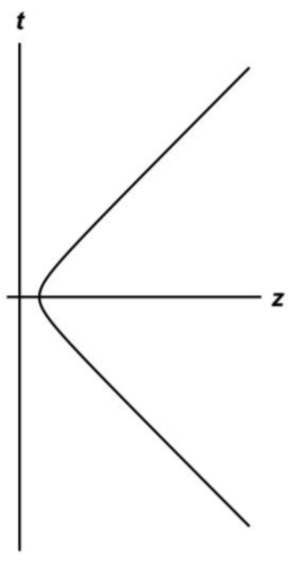
Circular

Stationary Worldlines

Linear

Circular

$$x^{\mu}(s) = \kappa^{-1} \left(\sinh(\kappa s), 0, 0, \cosh(\kappa s)\right).$$



Stationary Worldlines

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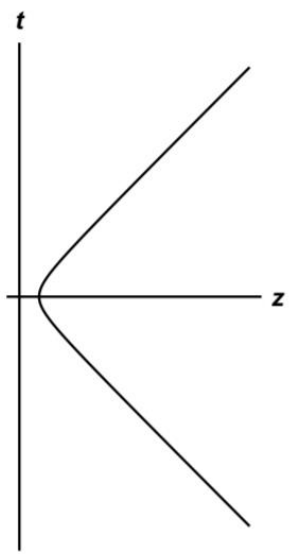
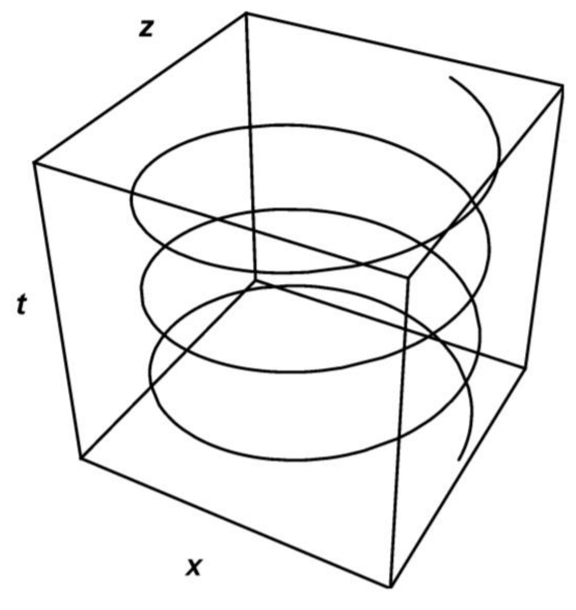


Circular

$$x^\mu(s) = \rho^{-2}\left(\tau\,\rho\,s, \kappa\cos\rho s, \kappa\sin\rho s, 0\right).$$



Stationary Worldlines

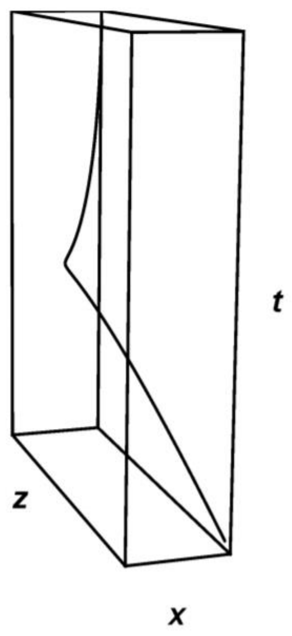
Cusp

Catenary

Stationary Worldlines

Cusp

$$x^{\mu}(s) = \left(s + \frac{1}{6}\kappa^2 s^3, \frac{1}{2}\kappa s^2, 0, \frac{1}{6}\kappa^2 s^3 \right).$$



Catenary

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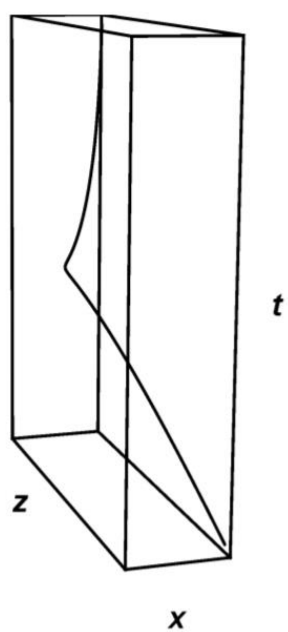
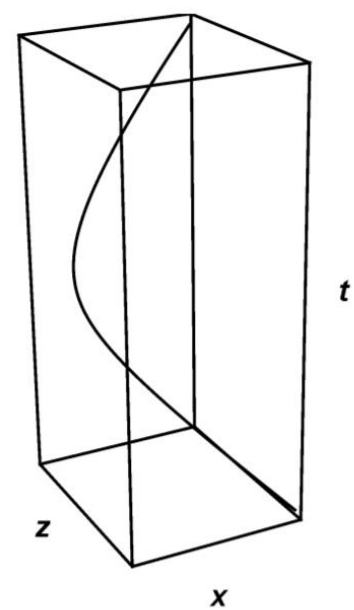


Catenary

$$x^\mu(s) = \sigma^{-2}\left(\kappa \sinh(s\sigma), \kappa \cosh(s\sigma), 0, s\tau\sigma\right).$$



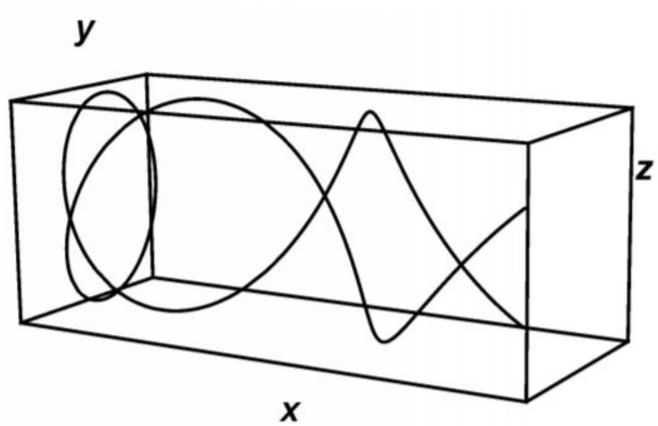
Hypertor

The Hypertor worldline is 3+1 dimensional, and so cannot be plotted on a 3D plot.

Spatial Projections

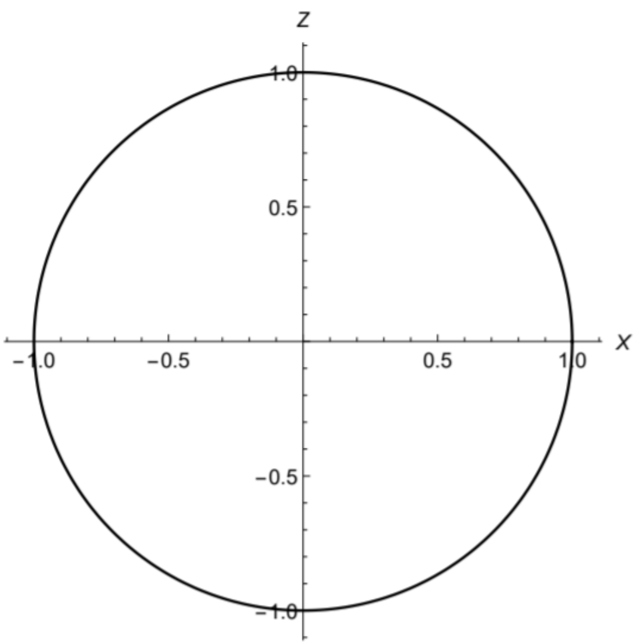
Helix

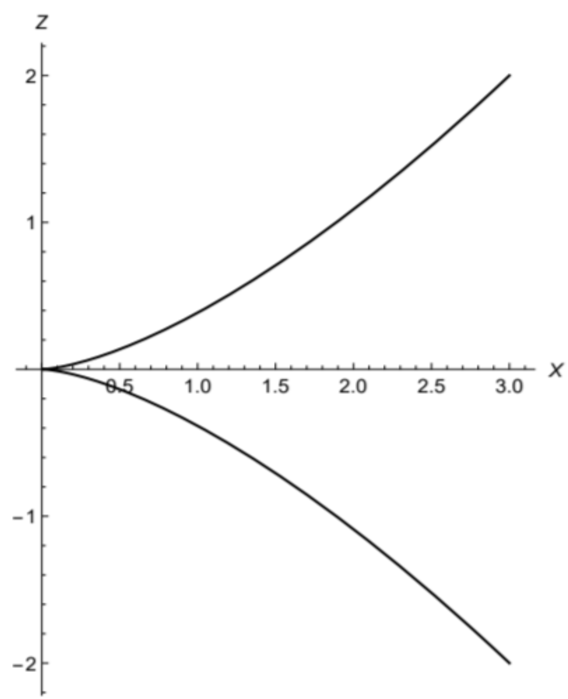
$$z^2 = \frac{\Delta^2}{R^2 R_+^2} \cosh^2 \left[\frac{R_+}{R_-} \cos^{-1} \left(y \frac{\Delta R R_-}{\kappa \tau} \right) \right] + \frac{\kappa^2 \tau^2}{\Delta^2 R^2 R_-^2} - x^2 - y^2$$

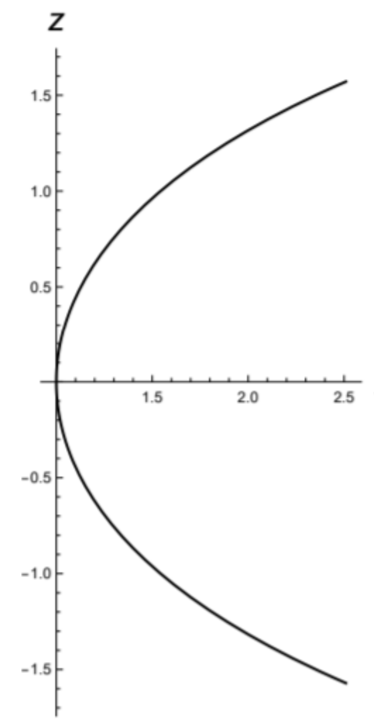


Spatial Projections

Circular	Cusp	Catenary
$z = \sqrt{\tau^2 - \kappa^2 - x^2}$	$z = \frac{\sqrt{2\kappa}}{3} x^{3/2}$	$z = \tau \cosh^{-1} \frac{x}{\kappa}$







Five Classes of Worldlines

- Nulltor
- Ultrator
- Parator
- Infrator
- Hypertor

$$\tau = 0$$

$$\tau > \kappa$$

$$\tau = \kappa$$

$$\tau < \kappa$$

$$\nu > 0$$

LARMOR RADIATION

$$P = \frac{2}{3} q^2 \alpha^2 \longrightarrow P = \frac{2}{3} q^2 \kappa^2$$

- four-acceleration

$$a^\mu = \frac{d^2 x^\mu}{d\tau^2} = \frac{dv^\mu}{d\tau}$$

- proper acceleration

$$\alpha^2 \equiv -a^\mu a_\mu = \kappa^2$$

- straight-line motion $\alpha^2 = \gamma^6 a^2$
- circular motion $\alpha^2 = \gamma^4 a^2$

ACCELERATION RATIO — COMPARISON

$$A^2_{\text{“world line”}} \equiv \frac{\alpha^2}{a^2}$$

$$A^2_{\text{line}} = \gamma^6$$

$$A^2_{\text{circ}} = \gamma^4$$

ACCELERATION RATIO — COMPARISON

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$$A^2_{\text{circ}} = \gamma^4$$

$$A^2_{\text{cusp}} = \frac{\gamma^6}{\gamma^2 - 2\gamma + 2}$$

$$A^2_{\text{cat}} = \frac{\gamma^6}{1+\gamma^2 v_R^2}$$

$$A^2_{\text{hyper}} = \frac{\gamma^4 \kappa^3 \tau}{R^2 \gamma^4_{\min} v_{\min} \left(R^2 v^2_{\min} + \frac{R^2_+}{\gamma^2} \right)} ,$$

$$\gamma_{\min} \equiv (1 - v^2_{\min})^{-1/2}, \qquad v_{\min} \equiv \frac{\kappa\tau}{\Delta^2} ,$$

$$\Delta^2 \equiv \frac{1}{2}(R^2 + \kappa^2 + \tau^2 + \nu^2) ,$$

$$R^2 \equiv R^2_+ + R^2_- ,$$

$$R^2_\pm \equiv \sqrt{a^2 + b^2} \pm a ,$$

$$a = \frac{1}{2}\left(\kappa^2 - \nu^2 - \tau^2\right) , \qquad b = \kappa\nu .$$

ANGULAR DISTRIBUTION

$$P = \int_0^{2\pi} \int_0^{\pi} \frac{dP(\theta, \phi)}{d\Omega} \sin\theta \, d\theta \, d\phi$$

$$\frac{dP}{d\Omega} = \frac{q^2}{4\pi} \frac{|\hat{\boldsymbol{\imath}} \times (\boldsymbol{u} \times \boldsymbol{a})|^2}{(\hat{\boldsymbol{\imath}} \cdot \boldsymbol{u})^5}$$

$$\hat{\boldsymbol{\mathscr{r}}} \times (\boldsymbol{u} \times \boldsymbol{a}) = (\hat{\boldsymbol{\mathscr{r}}} \cdot \boldsymbol{a})\boldsymbol{u} - (\hat{\boldsymbol{\mathscr{r}}} \cdot \boldsymbol{u})\boldsymbol{a}$$

$$|\hat{\boldsymbol{\mathscr{r}}} \times (\boldsymbol{u} \times \boldsymbol{a})|^2 = (\hat{\boldsymbol{\mathscr{r}}} \cdot \boldsymbol{a})^2 u^2 - 2(\boldsymbol{u} \cdot \boldsymbol{a})(\hat{\boldsymbol{\mathscr{r}}} \cdot \boldsymbol{a})(\hat{\boldsymbol{\mathscr{r}}} \cdot \boldsymbol{u}) + (\hat{\boldsymbol{\mathscr{r}}} \cdot \boldsymbol{u})^2 a^2$$

$$\boldsymbol{u} \equiv \hat{\boldsymbol{\mathscr{r}}} - \boldsymbol{v}$$

$$\hat{\boldsymbol{\mathscr{r}}} \times (\boldsymbol{u} \times \boldsymbol{a}) = (\hat{\boldsymbol{\mathscr{r}}} \cdot \boldsymbol{a})\boldsymbol{u} - (\hat{\boldsymbol{\mathscr{r}}} \cdot \boldsymbol{u})\boldsymbol{a}$$

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$$\boldsymbol{u} \equiv \hat{\boldsymbol{\mathscr{r}}} - \boldsymbol{v} \qquad \hat{\boldsymbol{\mathscr{r}}} \equiv \frac{\boldsymbol{r} - \boldsymbol{r}'}{|\boldsymbol{r} - \boldsymbol{r}'|} = \hat{\boldsymbol{r}} \equiv \sin\theta\cos\phi\,\hat{\boldsymbol{x}} + \sin\theta\sin\phi\,\hat{\boldsymbol{y}} + \cos\theta\,\hat{\boldsymbol{z}}$$

$$\hat{\boldsymbol{\mathscr{r}}} \times (\boldsymbol{u} \times \boldsymbol{a}) = (\hat{\boldsymbol{\mathscr{r}}} \cdot \boldsymbol{a})\boldsymbol{u} - (\hat{\boldsymbol{\mathscr{r}}} \cdot \boldsymbol{u})\boldsymbol{a}$$

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$$\hat{\boldsymbol{\mathscr{r}}} \cdot \boldsymbol{a} = a_x \sin\theta\cos\phi + a_y \sin\theta\sin\phi + a_z\cos\theta,$$

$$u^2 = (\hat{\boldsymbol{\mathscr{r}}} - \boldsymbol{v})^2 = 1 - 2(v_x \sin\theta\cos\phi + v_y\sin\theta\sin\phi + v_z\cos\theta) + v^2,$$

$$\boldsymbol{u} \cdot \boldsymbol{a} = (\hat{\boldsymbol{\mathscr{r}}} - \boldsymbol{v}) \cdot \boldsymbol{a} = a_x \sin\theta\cos\phi + a_y\sin\theta\sin\phi + a_z\cos\theta - v_x a_x - v_y a_y - v_z a_z,$$

$$\hat{\boldsymbol{\mathscr{r}}} \cdot \boldsymbol{u} = \hat{\boldsymbol{\mathscr{r}}}^2 - \hat{\boldsymbol{\mathscr{r}}} \cdot \boldsymbol{v} = 1 - v_x\sin\theta\cos\phi - v_y\sin\theta\sin\phi - v_z\cos\theta,$$

$$a^2 = a_x^2 + a_y^2 + a_z^2, \quad v^2 = v_x^2 + v_y^2 + v_z^2.$$

ANGULAR DISTRIBUTION; LINE / NULLTOR

- Rectilinear

$$\frac{dP(\theta,\phi)}{d\Omega} \equiv \frac{2}{3} q^2 \gamma^6 a^2 I_{\mathrm{line}}$$

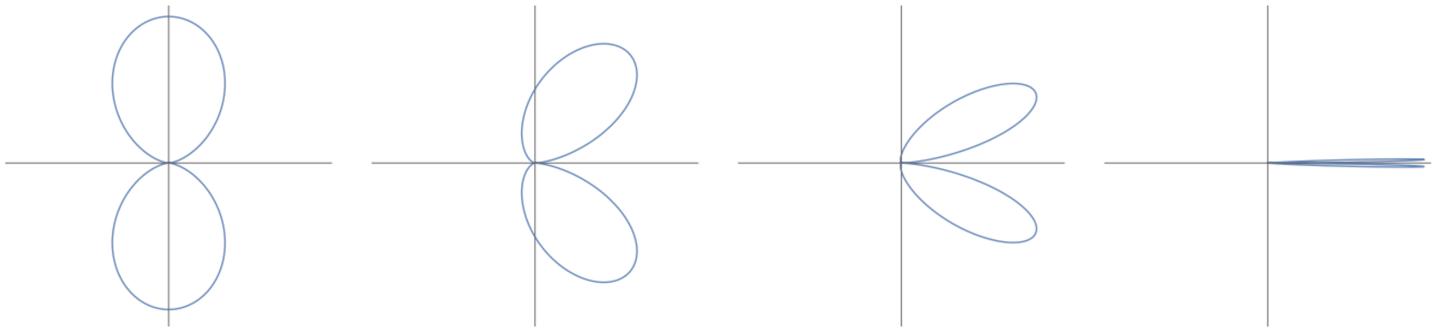
$$I_{\mathrm{line}} \equiv \frac{3}{8\pi\gamma^6} \frac{\sin^2\theta}{(1-\beta\cos\theta)^5}$$

$$\int_0^{2\pi}\int_0^{\pi} I_{\mathrm{line}} \sin\theta d\theta d\phi = 1 \qquad P_{\mathrm{line}} = \frac{2}{3} q^2 \gamma^6 a^2$$

linear motion

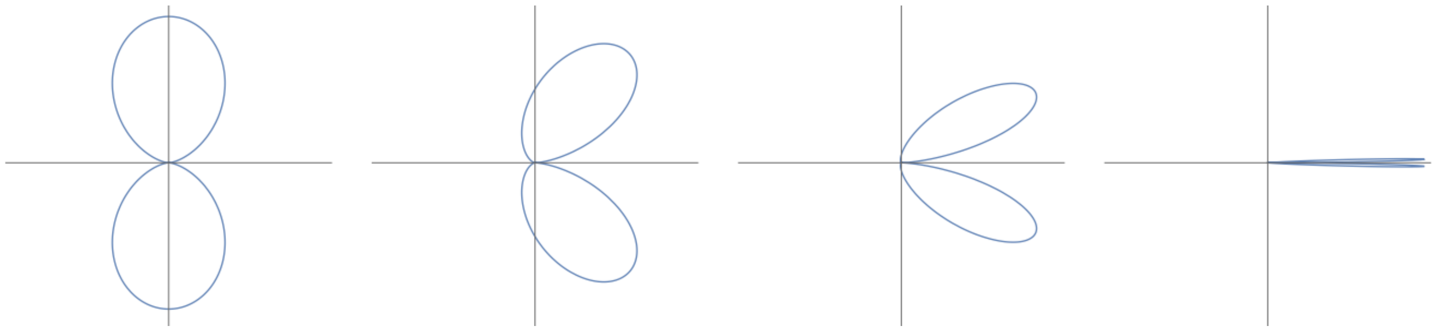
circular motion

linear motion

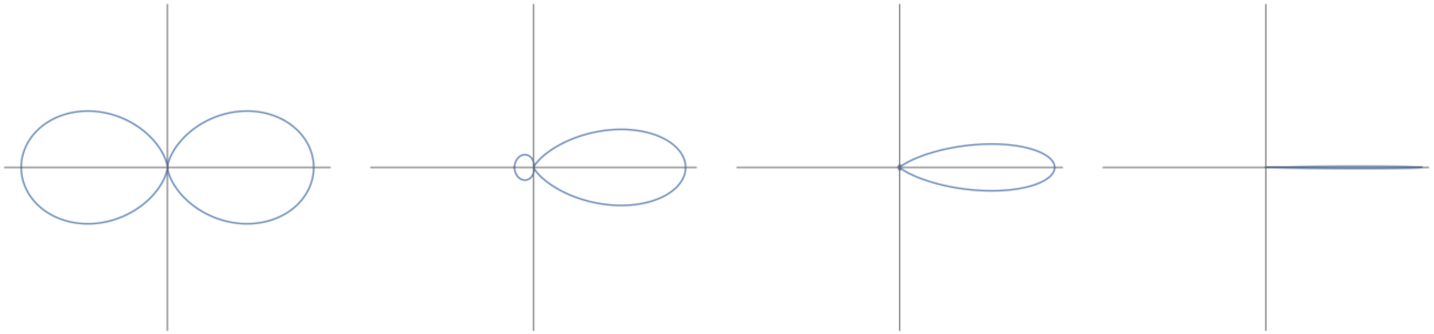


circular motion

linear motion



circular motion



ANGULAR DISTRIBUTION; CIRCLE / ULTRATOR

- Synchrotron

$$\frac{dP(\theta,\phi)}{d\Omega} \equiv \frac{2}{3} q^2 \gamma^4 a^2 I_{\text{circ}}$$

$$I_{\text{circ}} \equiv \frac{3}{8\pi} \frac{(1-\beta\cos\theta)^2 - \left(1-\beta^2\right)\sin^2\theta\cos^2\phi}{\gamma^4(1-\beta\cos\theta)^5}$$

$$\int_0^{2\pi}\int_0^{\pi} I_{\text{circ}} \sin\theta d\theta d\phi = 1 \qquad P_{\text{circ}} = \frac{2}{3} q^2 \gamma^4 a^2$$

ANGULAR DISTRIBUTION

$$\frac{dP(\theta,\phi)}{d\Omega} \equiv \frac{2}{3} q^2 a^2 A_{\text{cusp}}^2 I_{\text{cusp}}$$

- Cusp:

$$\frac{dP(\theta,\phi)}{d\Omega} \equiv \frac{2}{3} \frac{q^2 \gamma^6 a^2 I_{\text{cusp}}}{\gamma^2 - 2\gamma + 2}$$

$$I_{\text{cusp}} \equiv \frac{\lambda_1 + \lambda_2 \cos\phi + \lambda_3 \cos^2\phi}{(\cos\phi + \lambda_4)^5}$$

$$\int_0^{2\pi} \int_0^{\pi} I_{\text{cusp}} \sin\theta d\theta d\phi = 1$$

$$P_{\text{cusp}} = \frac{2}{3} q^2 \kappa^2$$

ANGULAR DISTRIBUTION; CUSP / PARATOR

- θ — factors

$$\lambda_0 = -\frac{3}{8\pi\gamma\sqrt{2\gamma_1}^5 \sin^5\theta}$$

$$\lambda_1 = \lambda_0\left[2+\gamma\gamma_2+\gamma_1\cos\theta\left(\gamma_3\cos\theta-2\gamma_2\right)\right]$$

$$\lambda_2 = -2\lambda_0\sqrt{2\gamma_1}\sin\theta\left(\gamma_1-\gamma_2\cos\theta\right)$$

$$\lambda_3 = 2\lambda_0\gamma_{3/2}\sin^2\theta$$

$$\lambda_4 = \frac{\gamma_1\cos\theta-\gamma}{\sqrt{2\gamma_1}\sin\theta}$$

$$\gamma_n \equiv \gamma - n \text{ for } n = 1, 2, 3 \text{ and } 3/2$$

ANGULAR DISTRIBUTION

$$\frac{dP(\theta,\phi)}{d\Omega} \equiv \frac{2}{3} q^2 a^2 A_{\mathrm{cat}}^2 I_{\mathrm{cat}}$$

- Cat:

$$\frac{dP(\theta,\phi)}{d\Omega} \equiv \frac{2}{3} \frac{q^2 \gamma^6 a^2 I_{\mathrm{cat}}}{1+\gamma^2 v_R^2}$$

$$I_{\mathrm{cat}} \equiv \frac{3}{8\pi R(\theta,\phi)^5} \left[C_2\, F(\theta,\phi)^2 + 2\, C_1 F(\theta,\phi) G(\theta,\phi) - \frac{1}{\gamma^2} G(\theta,\phi)^2 \right]$$

$$\int_0^{2\pi} \int_0^{\pi} I_{\mathrm{cat}} \sin\theta d\theta d\phi = 1$$

$$P_{\mathrm{cat}} = \frac{2}{3} q^2 \kappa^2$$

ANGULAR DISTRIB. CATENARY / INFRATOR

$$
\begin{aligned}
G(\theta,\phi) &= \frac{\sin(\theta)\,F(\theta,\phi)}{R(\theta,\phi)}\left(\cos(\phi) - v_R\,\sin(\phi)\,\sinh(\alpha)\right)\ , \\
F(\theta,\phi) &= (1 - v_R^2)\,\mathrm{sech}^3(\alpha)\,R(\theta,\phi)\ , \\
R(\theta,\phi) &= 1 - v_R\,\sin(\theta)\,\sin(\phi)\,\mathrm{sech}(\alpha) + \sin(\theta)\,\cos(\phi)\,\tanh(\alpha)\ ,
\end{aligned}
$$

$$
\begin{aligned}
C_1 &= (1 - v_R^2)\tanh(\alpha)\ , \\
C_2 &= v_R^2\sinh^2(\alpha) + 1\ ,
\end{aligned}
$$

$$
\begin{aligned}
\alpha &= \frac{1}{2}\ln(2\,\xi\,\eta - 1)\ , \\
\eta &= \sqrt{\xi^2 - 1} + \xi\ , \\
\xi &= \gamma\sqrt{1 - v_R^2}\ .
\end{aligned}
$$

ANGULAR DISTRIBUTION; HELIX / HYPERTOR

- Hypertor

$$I_{\text{hyper}} \equiv \frac{3}{8\pi\kappa^2 Q^5(\theta,\phi)}\left[C_1 Q^2(\theta,\phi) + 2\,C_2 Q(\theta,\phi)P(\theta,\phi) - \frac{1}{\gamma^2}P^2(\theta,\phi)\right]$$

$$\int_0^{2\pi}\int_0^{\pi} I_{\text{hyper}}\sin\theta d\theta d\phi = 1$$

$$P_{\text{hyper}} = \frac{2}{3}q^2\kappa^2$$

ANGULAR DISTRIBUTION; HYPERTOR

$$
\begin{aligned}
Q(\theta, \Phi) &= 1 - A_1 \cos(\theta) - A_2 \sin(\theta) \cos(\phi) + A_3 \sin(\theta) \sin(\phi) \ , \\
P(\theta, \Phi) &= F_2 \cos(\theta) + B_2 \sin(\theta) \cos(\phi) + F_1 \sin(\theta) \sin(\phi) \ ,
\end{aligned}
$$

$$
\begin{aligned}
A_1 &= v_{\mathrm{min}} b \cos(\Omega_\gamma) \ , \\
A_2 &= \sqrt{1 - b^2} \ , \\
A_3 &= v_{\mathrm{min}} b \sin(\Omega_\gamma) \ ,
\end{aligned}
$$

ANGULAR DISTRIBUTION; HYPERTOR

$$
\begin{aligned}
B_1 &= b(A_2 A_3 D_1 - A_1 D_2) \ , \\
B_2 &= D_1 b^3 \ , \\
B_3 &= b(A_3 D_2 + A_2 A_1 D_1) \ ,
\end{aligned}
$$

$$
\begin{aligned}
C_1 &= B_2^2 + B_1^2 + B_3^2 \ , \\
C_2 &= -A_2 B_2 + A_3 B_1 + A_1 B_3 \ ,
\end{aligned}
$$

ANGULAR DISTRIBUTION; HYPERTOR

$$D_1 = \frac{R\,R_+}{\Delta}\ ,$$
$$D_2 = \frac{R\,R_-}{\Delta}\ ,$$

$$\Omega_\gamma = \frac{R_-}{R_+}\cosh^{-1}(\frac{1}{b})\ , \quad b = \frac{\gamma_{\rm min}}{\gamma}\ , \quad v_{\rm min} = \frac{\kappa\,\tau}{\Delta^2}\ , \quad \gamma_{\rm min} = \frac{1}{\sqrt{1 - v_{\rm min}^2}}\ ,$$

$$R = \sqrt{R_+^2 + R_-^2}\ ,$$

$$R_+^2 = \frac{1}{2}(\kappa^2 - \nu^2 - \tau^2) + \sqrt{\kappa^2\,\tau^2 + \frac{1}{4}(\kappa^2 - \nu^2 - \tau^2)^2}\ ,$$
$$R_-^2 = -\frac{1}{2}(\kappa^2 - \nu^2 - \tau^2) + \sqrt{\kappa^2\,\tau^2 + \frac{1}{4}(\kappa^2 - \nu^2 - \tau^2)^2}\ .$$

cusp motion

catenary motion

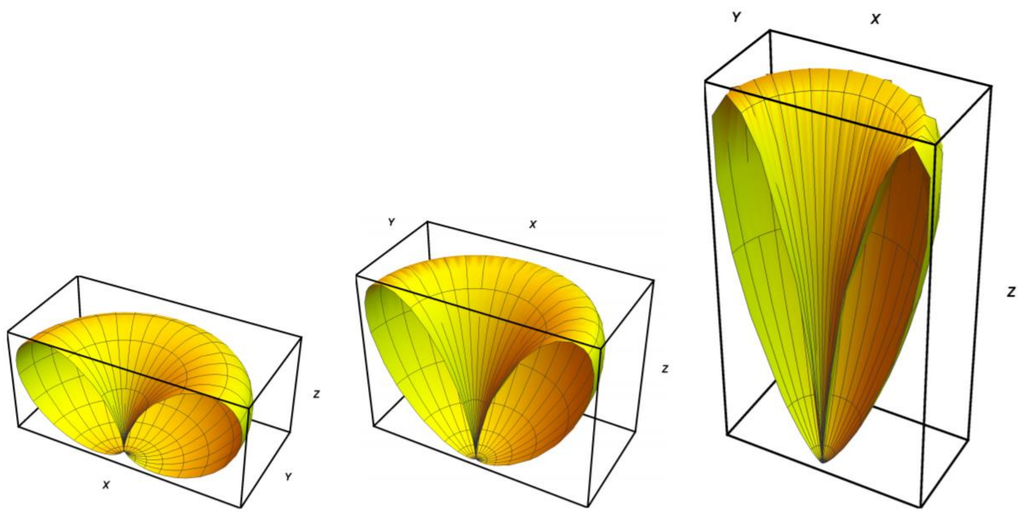
cusp motion

catenary motion

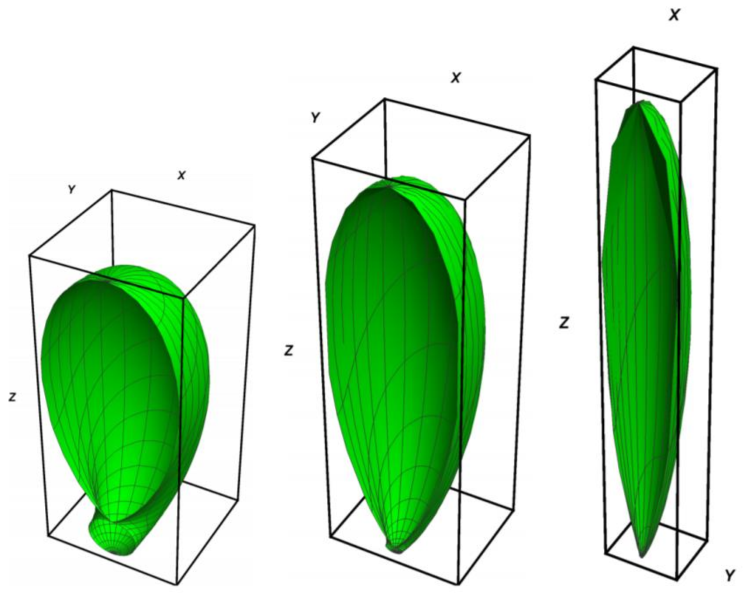
cusp motion

catenary motion

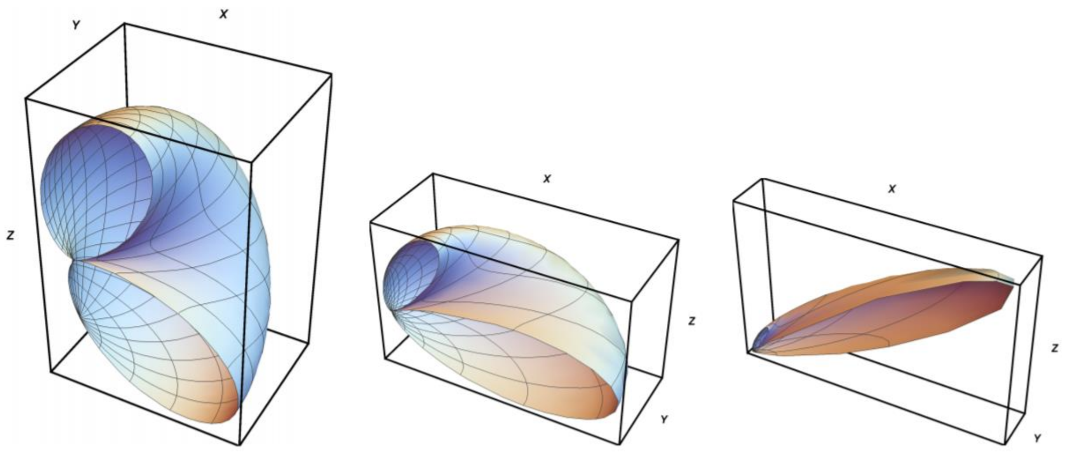
linear motion



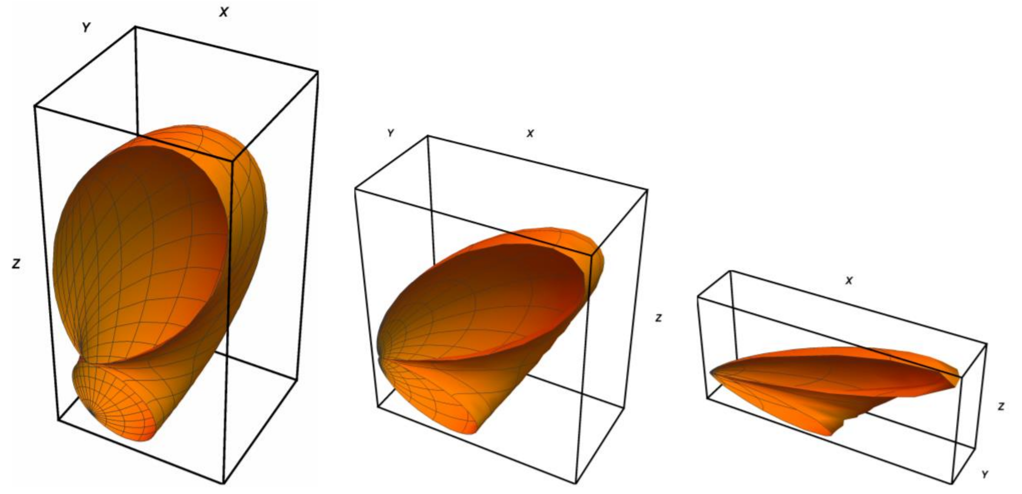
circular motion



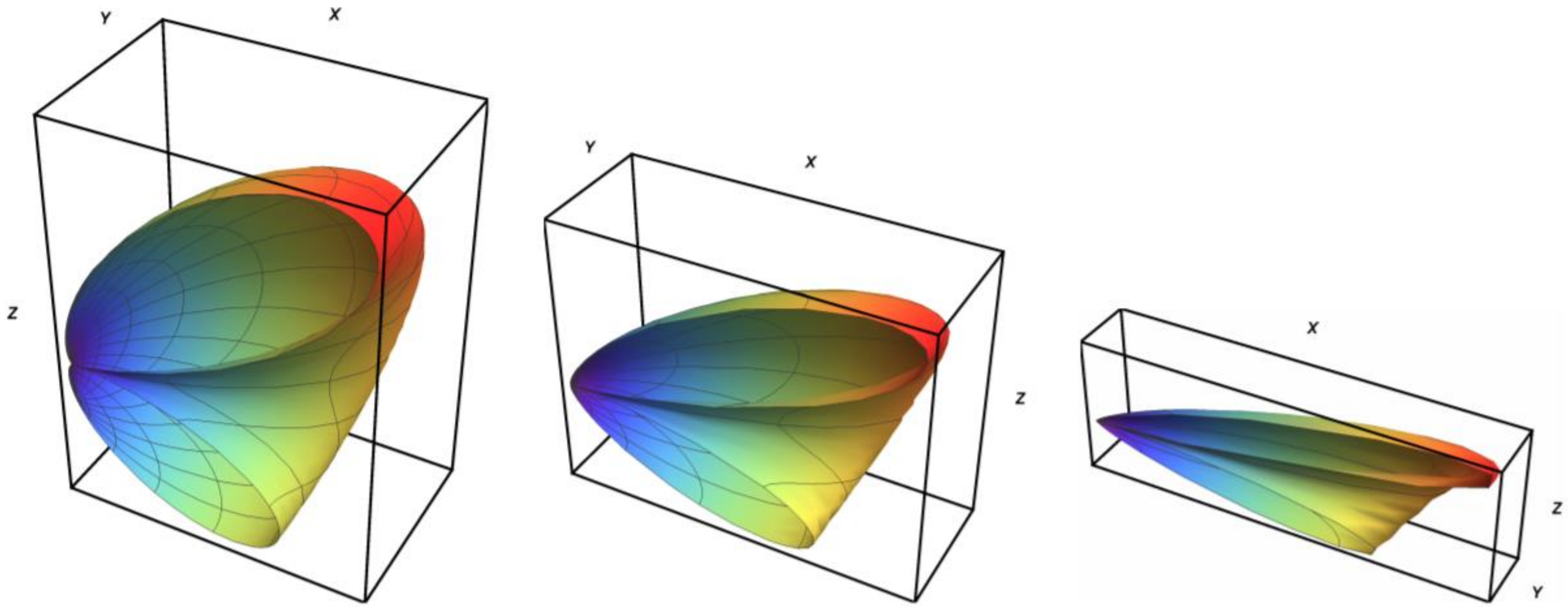
cusp motion



catenary motion



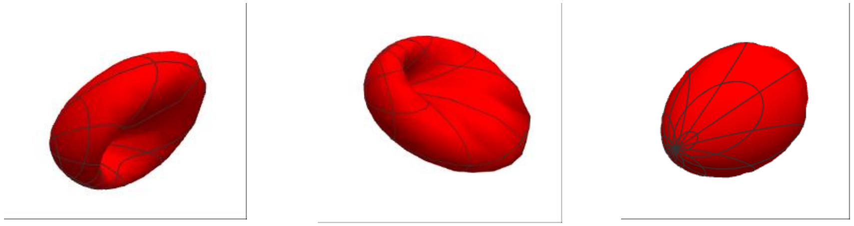
Hypertor motion

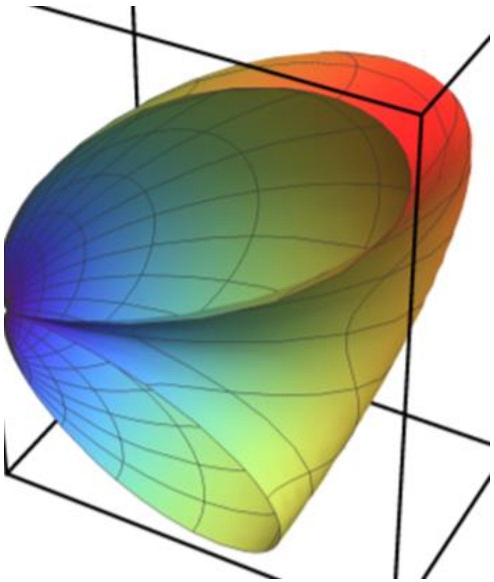


Summary

- Power
- Dynamics
- Direction

$$A^2 = \frac{\gamma^6}{\gamma^2 - 2\gamma + 2}$$





$$\langle \theta^2 \rangle = \frac{2}{\gamma}$$

Direction Measure at High Speeds

$$\langle \theta^2 \rangle_{\text{line}} = \frac{1}{\gamma^2} + \mathcal{O}\left(\frac{1}{\gamma}\right)^3$$

$$\langle \theta^2 \rangle_{\text{circ}} = \frac{1}{\gamma^2} + \mathcal{O}\left(\frac{1}{\gamma}\right)^3$$

$$\langle \theta^2 \rangle_{\text{cusp}} = \frac{2}{\gamma} + \mathcal{O}\left(\frac{1}{\gamma}\right)^2$$

Using $\alpha^2 = \gamma^6 a^2 - \gamma^6 (\boldsymbol{v} \times \boldsymbol{a})^2$, $(\gamma + 1)^2 \omega_T^2 = \gamma^4 (\boldsymbol{v} \times \boldsymbol{a})^2$, and $\alpha = \kappa$, one finds:

$$\gamma^2 (\gamma + 1)^2 \omega_T^2 = \gamma^6 a^2 - \kappa^2.$$

Nulltor

$$\omega_T = 0.$$

Ultrator

$$\omega_T = \frac{\kappa \beta}{\gamma + 1} \approx \frac{\kappa \beta}{2}.$$

Parator

$$\omega_T = \frac{\kappa}{\gamma} \left(\frac{\gamma - 1}{\gamma + 1} \right) \approx \frac{\kappa \beta^2}{4}.$$

Infrator

$$\omega_T = \frac{\tau}{\gamma + 1} \approx \frac{\tau}{2} = \frac{\kappa \beta_R}{2}.$$

Maximum Intensity

$$f \equiv \frac{dP/d\Omega|^{\text{ultra-rel}}_{\theta=\theta_{\text{max}}}}{dP/d\Omega|^{\text{rest}}_{\theta=\theta_{\text{max}}}}$$

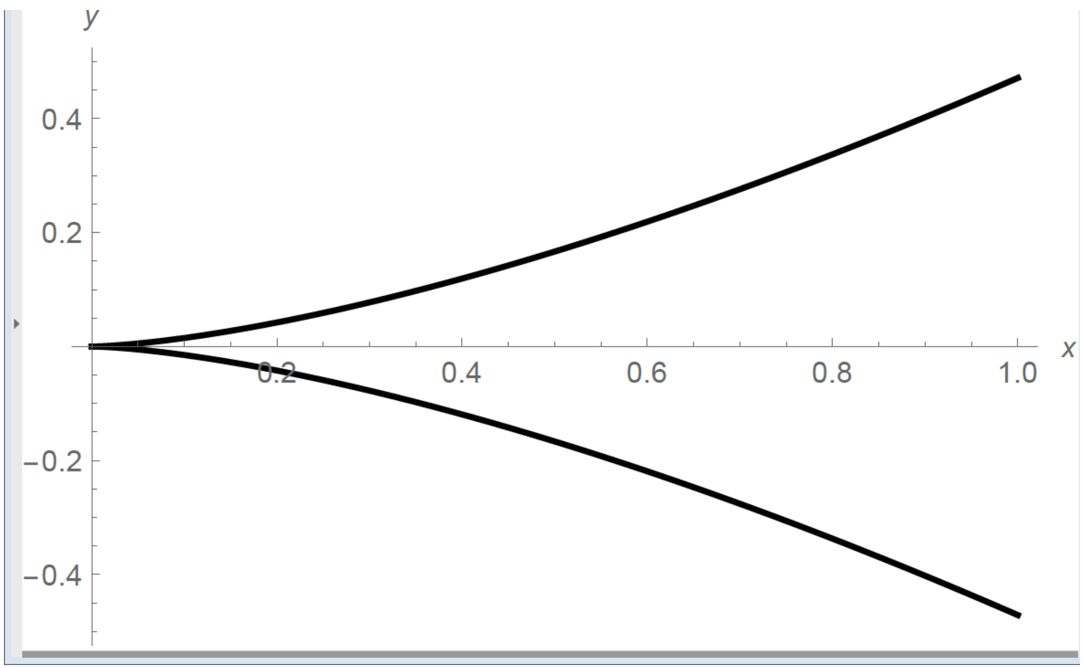
$$f_{\text{line}} = \frac{8192}{3125}\gamma^8 = 2.62\gamma^8$$

$$f_{\text{circ}} = 8\gamma^6$$

$$f_{\text{cusp}} = 8\gamma^6$$

The Cusp Worldline

$$x^\mu(\tau) = \left(\tau + \frac{1}{6}\kappa^2\tau^3, \frac{1}{2}\kappa\tau^2, \frac{1}{6}\kappa^2\tau^3, 0\right)$$



$$y = \frac{\sqrt{2\kappa}}{3}x^{3/2}$$

The Cusp Worldline

- Acceleration

$$\alpha = \kappa$$

- Angular Velocity

$$\omega = \kappa$$

The Cusp Worldline

- 2+1 dimensional motion.
  - Simplified like circular or catenary.
- One parameter only.
  - Unlike circular or catenary.
- Exact spectra.
  - Similar to rectilinear or inertial world lines.

Exact Vacuum Spectra

$$\frac{1}{e^{2\pi\omega/\kappa} - 1} \approx e^{-2\pi\omega/\kappa} \qquad \omega/\kappa \gg 1$$

$$e^{-\sqrt{12}\omega/\kappa}$$

$$T = \kappa/2\pi = 0.159\kappa$$

$$T = \kappa/\sqrt{12} = 0.289\kappa$$

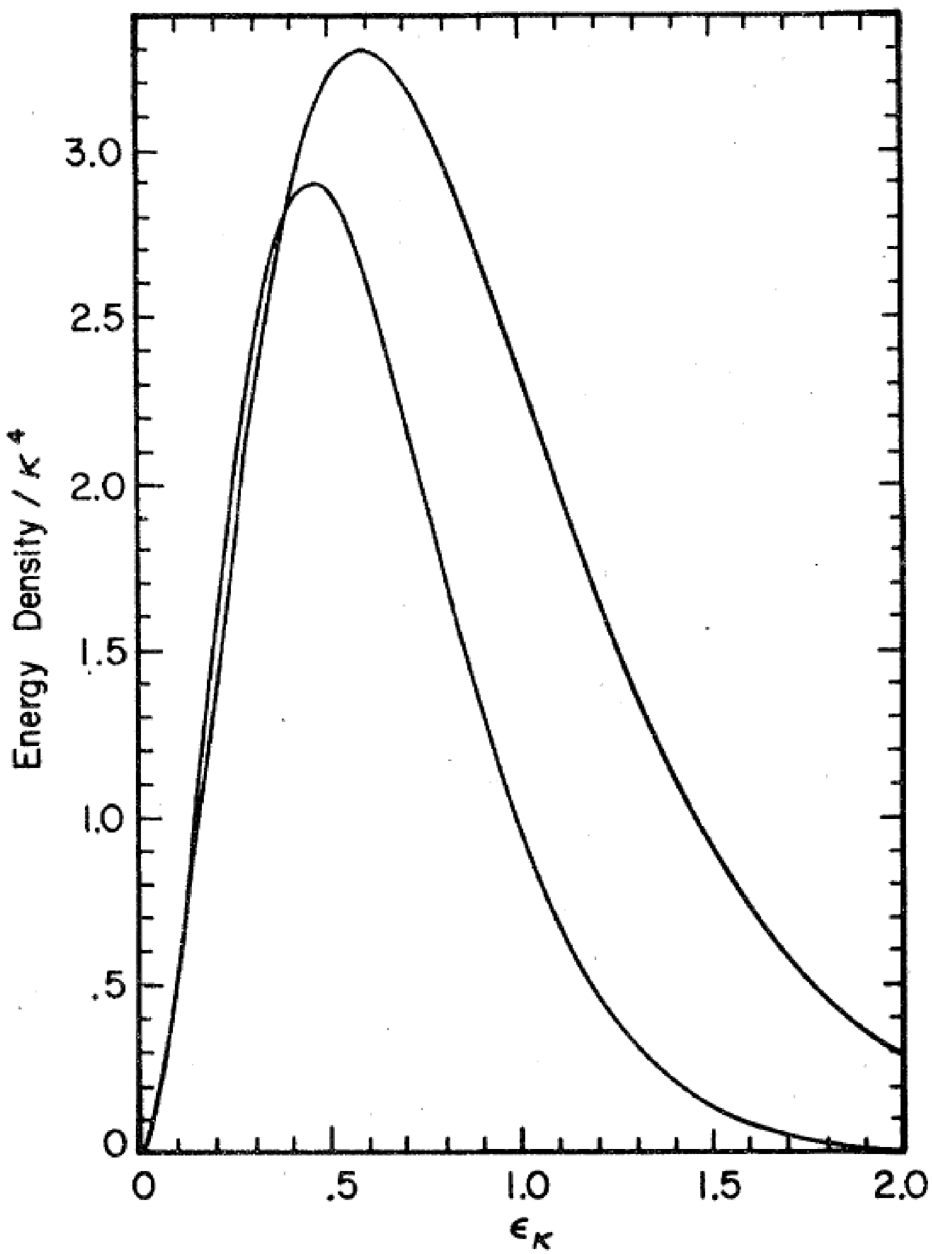


FIG. 1. Spectra for detectors on world lines with $\tau = \upsilon = 0$ (lower curve) and $\kappa = \tau$, $\upsilon = 0$ (upper curve).

Exact Vacuum Spectra

- Inertial

$$S(\epsilon) = \frac{\kappa^2 \epsilon_\kappa^2}{4\pi^3} \int_{-\infty}^{\infty} \frac{e^{-i\kappa\epsilon_\kappa}}{W(\tau,0)} d\tau,$$

$$W(\tau,0) = [x_\mu(\tau) - x_\mu(0)][x^\mu(\tau) - x^\mu(0)]$$

$$W_{\text{inertial}} = -\tau^2$$

$$S_{\text{inertial}}(\epsilon) = -\frac{\kappa^2 \epsilon^2}{4\pi^3} \int_{-\infty}^{\infty} e^{-i\kappa\epsilon} \tau^{-2} d\tau = \frac{\kappa^2 \epsilon^2}{4\pi^3} \pi\kappa\epsilon = \frac{\kappa^3 \epsilon^3}{4\pi^2}$$

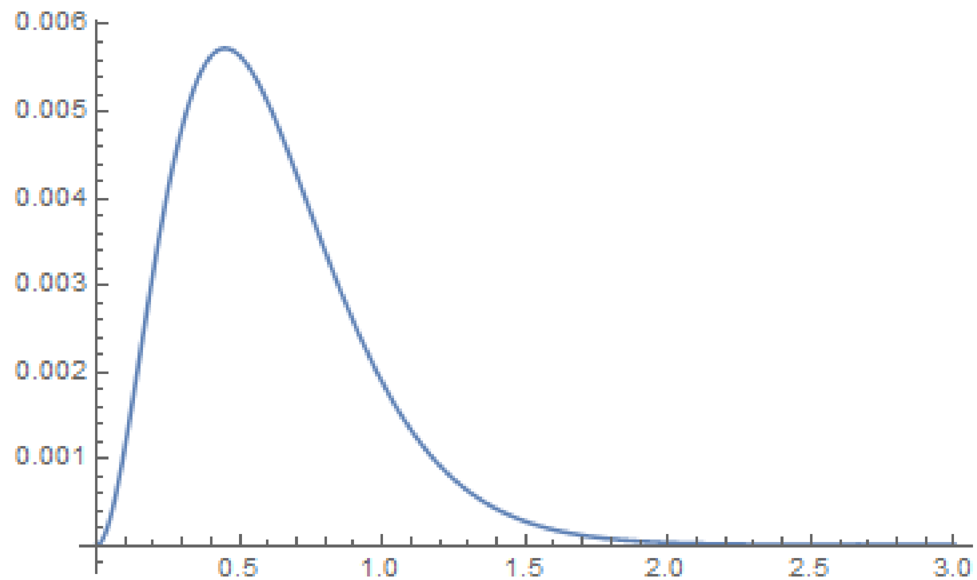
Exact Vacuum Spectra

- Rectilinear

$$W_{\text{Planck}} = \kappa^{-2}(2 - 2\cosh\kappa\tau)$$

$$S_{\text{line}}(\epsilon) = \frac{\kappa^2\epsilon^2}{4\pi^3}\int_{-\infty}^{\infty} e^{-i\kappa\epsilon}\frac{\kappa^2}{2-2\cosh\kappa\tau}d\tau = \frac{\kappa^3\epsilon^3\coth(\pi\epsilon)}{4\pi^2} + \frac{\kappa^3\epsilon^2}{4\pi^3}$$

$$S_{\text{line}} = \frac{\kappa^3\epsilon^3}{2\pi^2\left(e^{2\pi\epsilon}-1\right)}$$



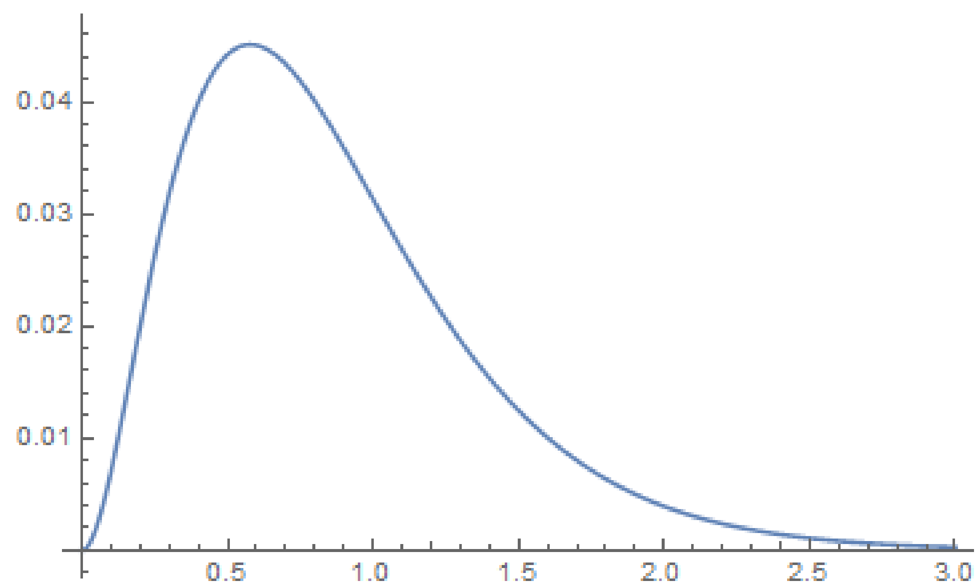
Exact Vacuum Spectra

- Cusp

$$W_{\text{cusp}} = (-\tau^2 - \frac{\kappa^2\tau^2}{12})^{-1}$$

$$S_{\text{cusp}} = \frac{\kappa^2\epsilon^2}{4\pi^3}\int_{-\infty}^{\infty} e^{-i\kappa\epsilon}(-\tau^2 - \frac{\kappa^2\tau^2}{12})d\tau = \frac{\kappa^3\epsilon^3}{4\pi^2} + \frac{\kappa^3\epsilon^2 e^{-2\sqrt{3}\epsilon}}{8\sqrt{3}\pi^2}$$

$$S_{\text{cusp}} = \frac{\kappa^3\epsilon^2 e^{-2\sqrt{3}\epsilon}}{8\sqrt{3}\pi^2}$$



Direction Measure at Low Speeds

$$\langle \sin^2\theta \rangle_{\text{line}} = -\frac{\left(\beta^2-1\right)\left(\beta\left(5\beta^2-3\right)+3\left(\beta^2-1\right)^2\tanh^{-1}\beta\right)}{2\beta^5} = \frac{4}{5}+\mathcal{O}(\beta)$$

$$\langle \sin^2\theta \rangle_{\text{circ}} = -\frac{\left(\beta^2-1\right)\left(\beta\left(\beta^2+3\right)+3\left(\beta^4-1\right)\tanh^{-1}\beta\right)}{4\beta^5} = \frac{3}{5}+\mathcal{O}(\beta)$$

$$\langle \sin^2\theta \rangle_{\text{cusp}} = \begin{array}{l} \frac{1}{8\beta^5\left(1+\sqrt{1-\beta^2}\right)^2}\left(117\beta-339\beta^3+351\beta^5-129\beta^7\right. \\ +\beta\sqrt{1-\beta^2}\left(-93+212\beta^2-135\beta^4+32\beta^6\right)-3\left(1-\beta^2\right)^2 \\ \left.\left[39-48\beta^2+13\beta^4+\sqrt{1-\beta^2}\left(-31+19\beta^2\right)\right]\tanh^{-1}\beta\right) \end{array} = \frac{3}{5}+\mathcal{O}(\beta)$$

Stationary Worldlines

- Inertial $\kappa = \tau = \nu = 0$
- Linear $\tau = \nu = 0$
- Circular $|\kappa| < |\tau| \quad \nu = 0$
- Cusp $|\kappa| = |\tau| \quad \nu = 0$
- Catenary $|\kappa| > |\tau| \quad \nu = 0$
- Helicoid $\nu \neq 0$