

The Path Integral over Relativistic Worldlines

B. Koch

with E. Muñoz and I. Reyes

based on:

Phys. Rev. D96 (2017) no.8, 085011

and arXiv:1706.05388.

**NON PERTURBATIVE
ASPECTS OF QFT AND
LOEWE'S 65 FEST**

5 - 7 DECEMBER 2017

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Content

- PI of the RPP, Status
- Local Symmetry: Velocity Rotations
- Constructing the PI of the RPP
- Conclusion

The Path Integral

Propagator

The Path Integral

Propagator

•
A

•
B

The Path Integral

Propagator



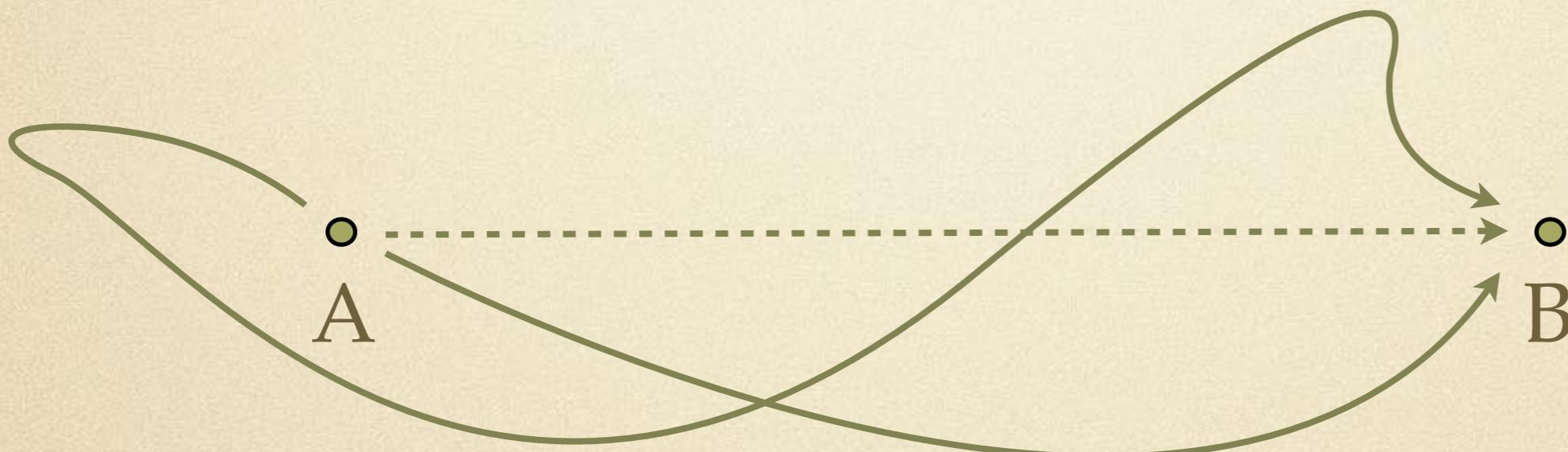
The Path Integral

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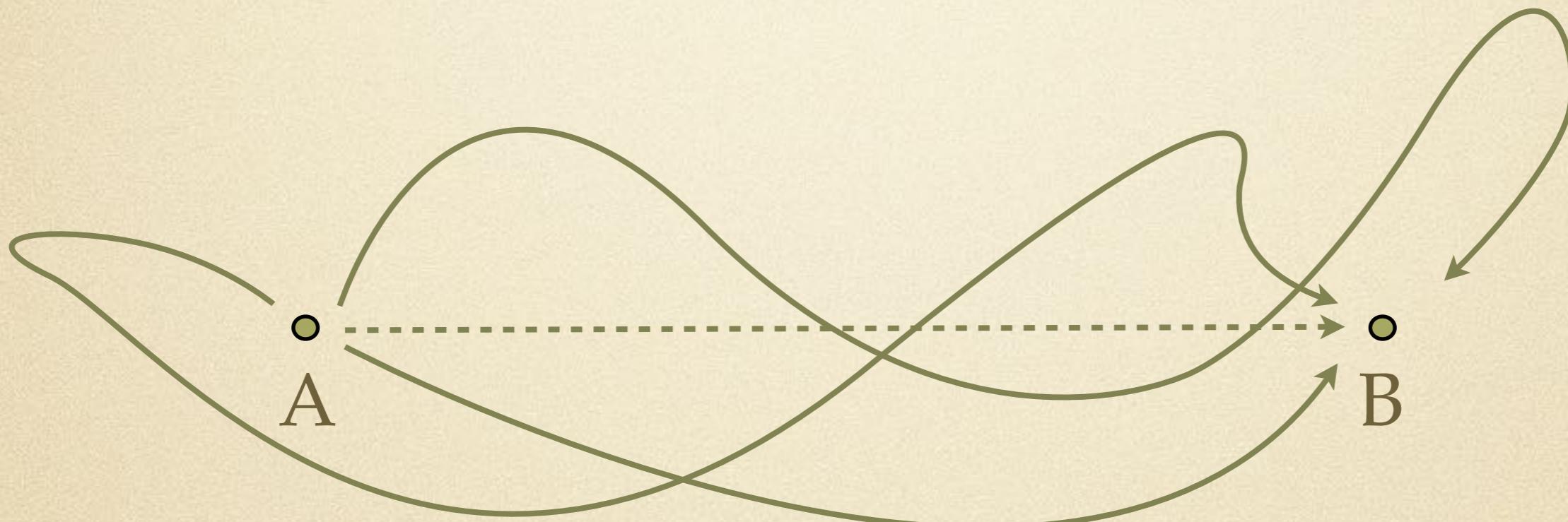
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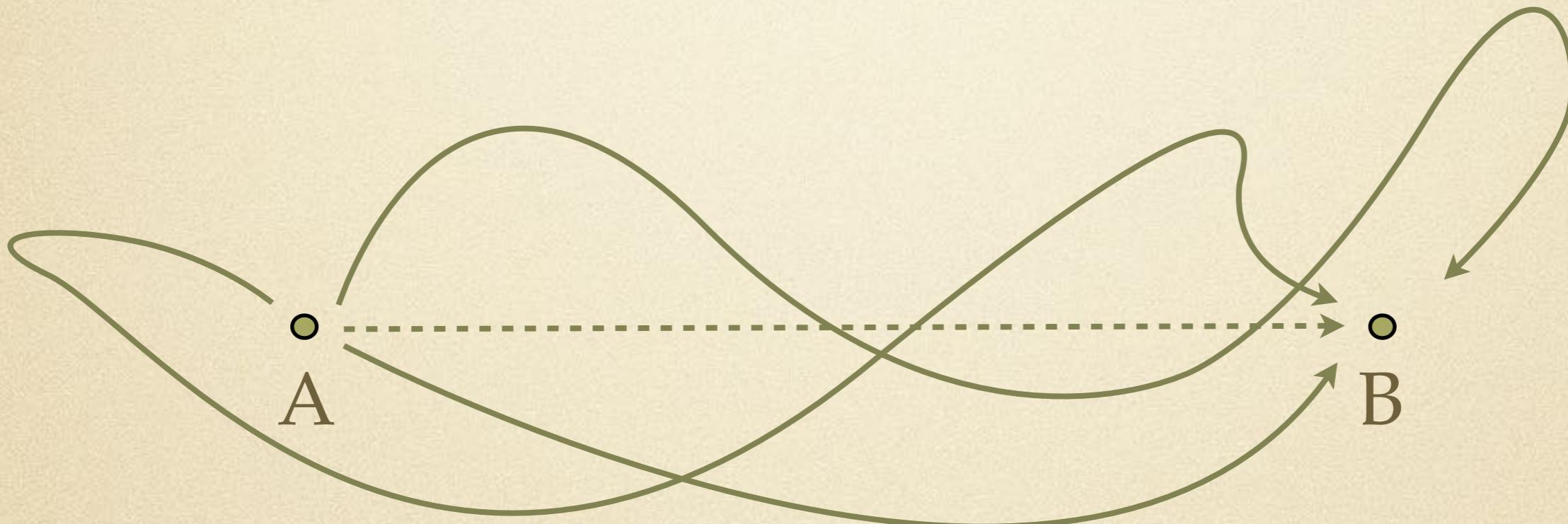
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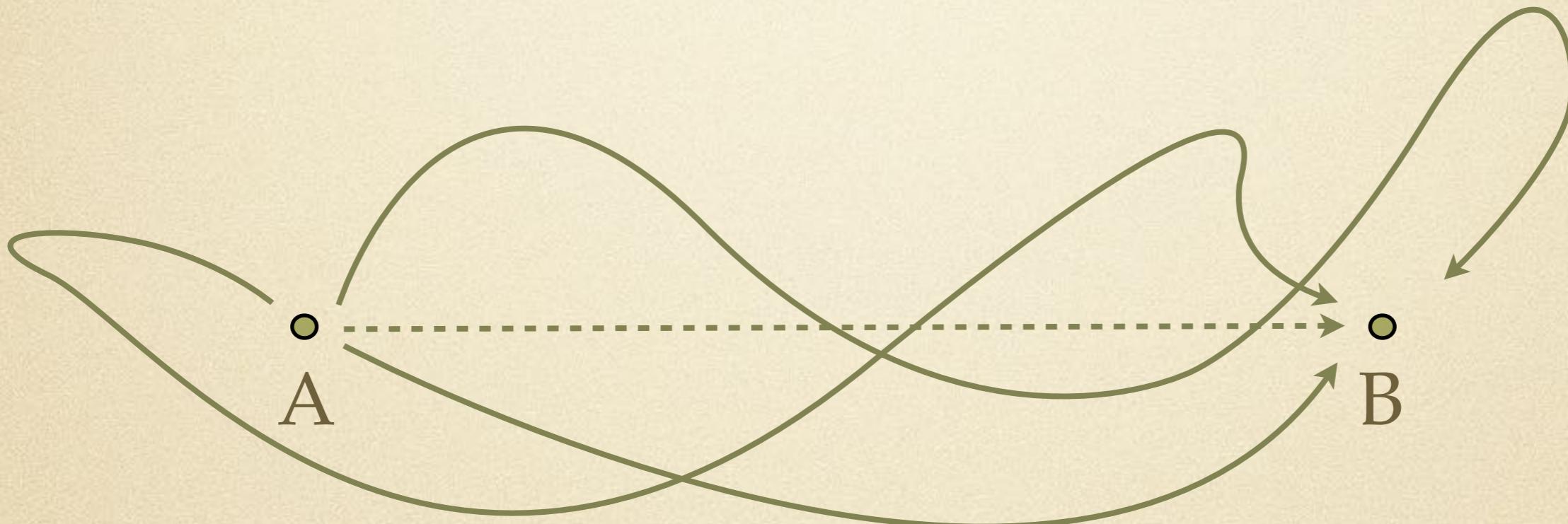
Propagator



$$\langle A | B \rangle \sim \int \mathcal{D}x \ e^{-S_{A,B}}$$

The Path Integral

Propagator



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³(after Wick rotation)

Non Relativistic Propagator

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Two Nice Features

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Two Nice Features

- Quadratic in field variable
- Can be connected (Chapman, Kolmogorov)

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Probability conservation
&

stepwise construction of PI

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No probability conservation
&

no stepwise construction of PI⁷

Relativistic Propagator

„Solutions“ in the Literature

*1

*3

Relativistic Propagator

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- Hamiltonian formalism (classically equivalent)^{*1}

^{*3}

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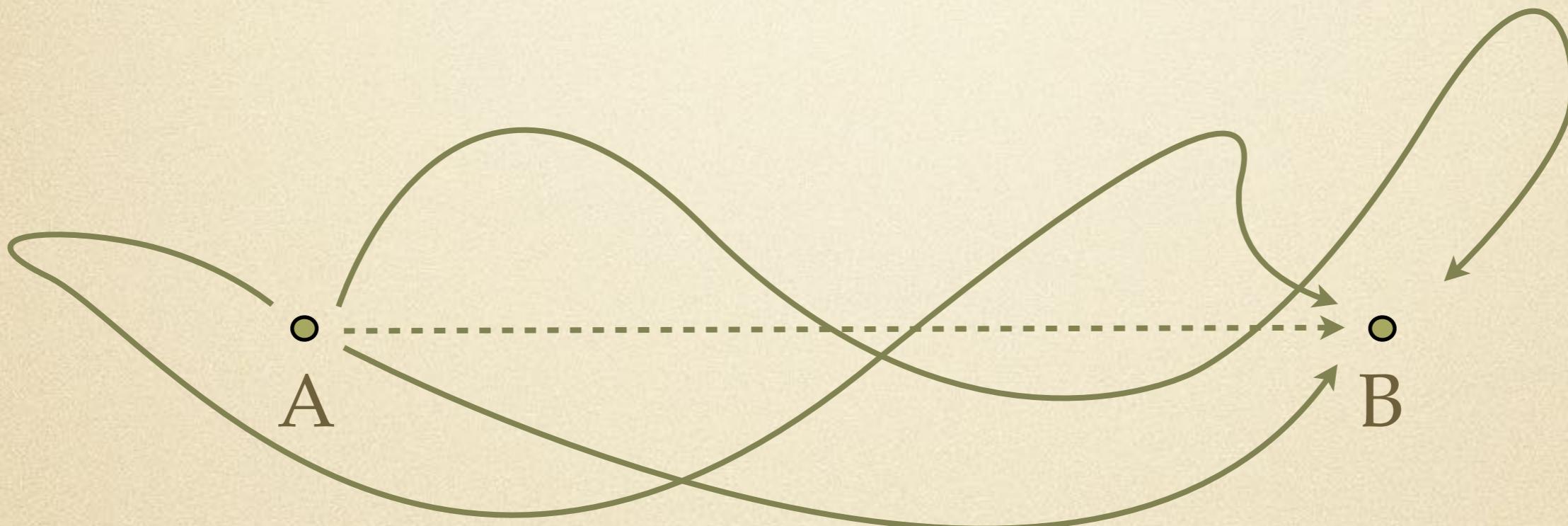
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Thats what we mostly do ...

Relativistic PI: our Proposal



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Three issues:

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Three issues:

- Issue 1: Local reparametrizations (known)
- Issue 2: Local velocity rotations (trivial?)
- Issue 3: Measure without anomalies

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using I1,I2,I3 works

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Functional *0
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Geometric *_00
stepwise proof

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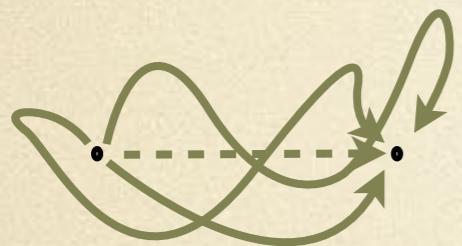
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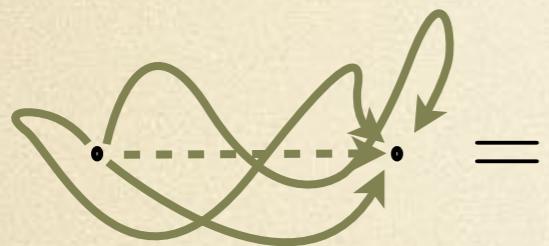
Stepwise proof

Stepwise proof



$$\langle A | B \rangle$$

Stepwise proof



$$\langle A | B \rangle =$$

Stepwise proof

The diagram illustrates a stepwise proof in a 2D space. It shows two points connected by a dashed green arrow. A solid green arrow starts from the left point and ends at the right point, representing a path or sequence of steps. The path is composed of several segments, some of which are curved, indicating a non-linear or stepwise progression.

$$\langle A | B \rangle = \langle A | B \rangle_1$$

Stepwise proof

The diagram illustrates a stepwise proof for a Feynman diagram. On the left, a complex loop diagram with two external lines is shown. A dashed line connects two vertices on the loop. An arrow points from this dashed line to the right, indicating the decomposition process. To the right of the equals sign, the diagram is decomposed into a sum of simpler diagrams. The first term is a single horizontal line with arrows pointing from left to right, representing a one-step reduction. The second term is a more complex diagram involving a triangle and a horizontal line, representing a two-step or higher-order reduction. This pattern continues with ellipses, showing the iterative nature of the proof.

$$\langle A | B \rangle = \langle A | B \rangle_1 + \langle A | B \rangle_2 + \dots$$

Stepwise proof

The diagram illustrates a stepwise proof for a Feynman diagram. On the left, a complex loop diagram is shown with a wavy line and two vertices connected by a dashed line. An arrow points from this diagram to an equals sign. To the right of the equals sign is a sum of terms. The first term is a single horizontal dashed line with a green arrow pointing right, followed by a plus sign. The second term is a more complex diagram consisting of a horizontal dashed line with a green arrow pointing right, a vertical solid line with a green arrow pointing down, and a diagonal solid line with a green arrow pointing up-right, all meeting at a central vertex. This is followed by another plus sign and an ellipsis.

$$\langle A|B \rangle = \langle A|B \rangle_1 + \langle A|B \rangle_2 + \dots$$

Don't count again!

Stepwise proof

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$$\langle A | B \rangle = \langle A | B \rangle_1 + \langle A | B \rangle_2 + \dots$$

Stepwise proof

The diagram illustrates a stepwise proof for a quantum state $\langle A | B \rangle$. On the left, a wavy green line with arrows at both ends represents the state $\langle A | B \rangle$. It is shown as a sum of two terms: a horizontal dashed line with arrows and a vertical dashed line with arrows. This is followed by a plus sign and a ellipsis. On the right, the same expression is repeated: a horizontal dashed line with arrows plus a vertical dashed line with arrows plus a ellipsis.

$$\langle A | B \rangle = \text{[diagram]} + \dots$$
$$\langle A | B \rangle = \langle A | B \rangle_1 + \langle A | B \rangle_2 + \dots$$

Strategy

Stepwise proof

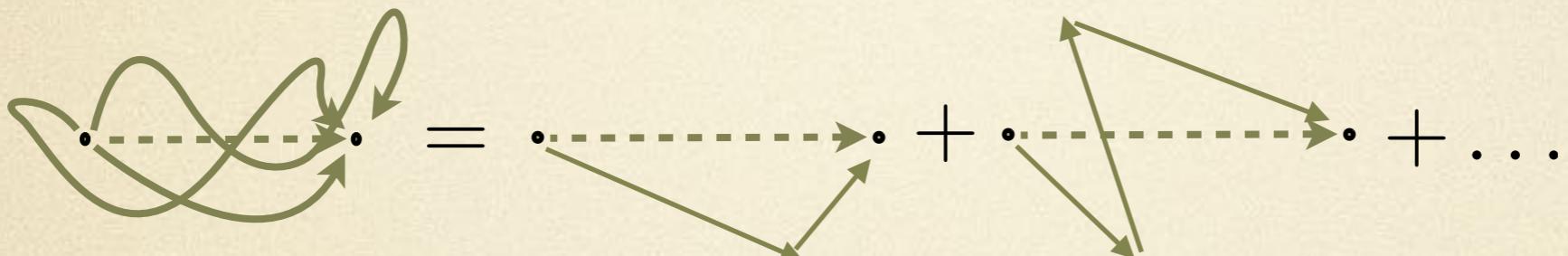
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$$\langle A|B \rangle = \langle A|B \rangle_1 + \langle A|B \rangle_2 + \dots$$

Strategy

- Clarify geometry meaning of I_{1,2,3}

Stepwise proof


$$\langle A|B \rangle = \langle A|B \rangle_1 + \langle A|B \rangle_2 + \dots$$

Strategy

- Clarify geometry meaning of I_{1,2,3}
- Calculate $\langle A|B \rangle_1$ using I_{2,3}

Stepwise proof

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Strategy

- Clarify geometry meaning of I_{1,2,3}
- Calculate $\langle A|B \rangle_1$ using I_{2,3}
- Show with I_{1,2} $\langle A|B \rangle_1$ contains $\langle A|B \rangle_2 \dots$

- Issue 1: Local reparametrizations (known)

$$S = \int d\lambda \sqrt{\left(\frac{dx^\mu}{d\lambda} \right)^2}$$

Invariant under $\lambda \rightarrow \lambda'(\lambda)$

We fix proper time such that

$$\tau = \tau(\lambda) \quad \text{with} \quad \left(\frac{dx^\mu}{d\tau} \right)^2 = 1$$

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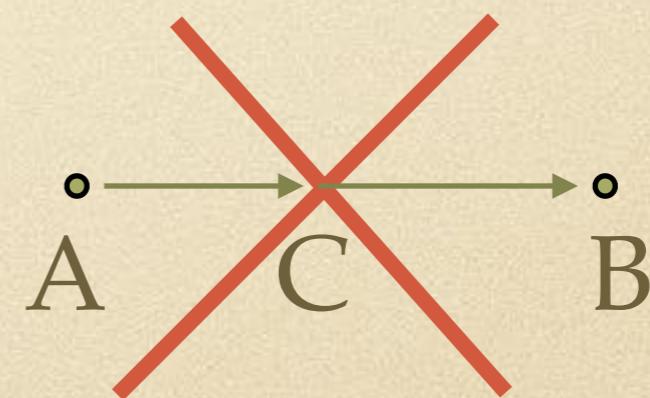
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- Issue 2: Local velocity rotations (trivial?)

$$S = \int d\lambda \sqrt{\left(\frac{dx^\mu}{d\lambda} \right)^2} = \int d\lambda \sqrt{v^\mu v_\mu}$$

Invariant under $v^\mu \rightarrow v'^\mu = \Lambda_\nu^\mu(\lambda) v^\nu$

with $v'^\mu v'_\mu = v^\mu v_\mu$

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Factor out of PI

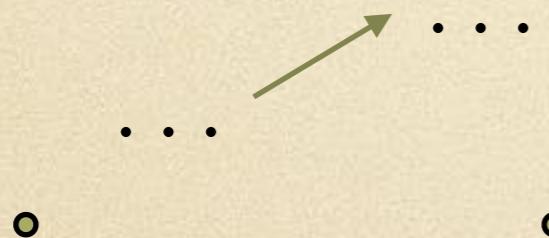
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Factor out of PI



if $S = S'$ and $\mathcal{L} = \mathcal{L}'$!

- Issue 3: Measure without anomalies

When performing transformation

$$v^\mu \rightarrow v'^\mu = \Lambda_\nu^\mu(\lambda) v^\nu$$

define right measure invariant under this symmetry:

$$\mathcal{D}x \rightarrow \mathcal{D}x' = \mathcal{D}x$$

Geometric example for two step propagator

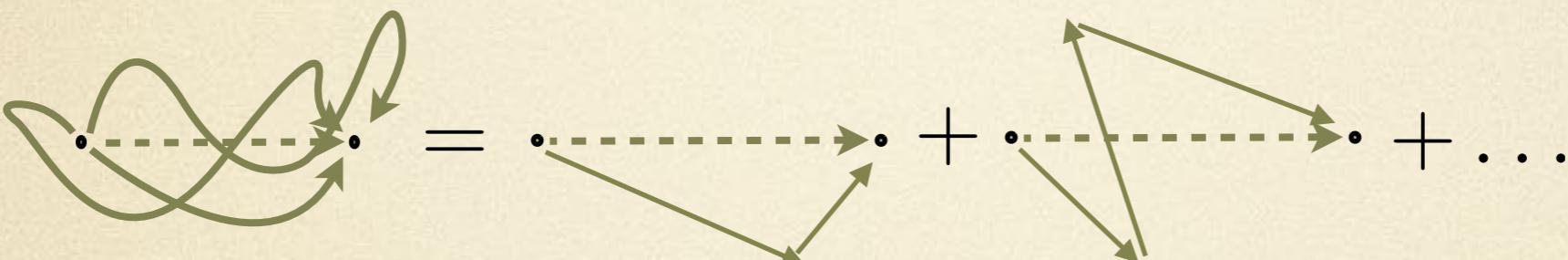
Stepwise proof

$$\langle A|B \rangle = \langle A|B \rangle_1 + \langle A|B \rangle_2 + \dots$$

Strategy

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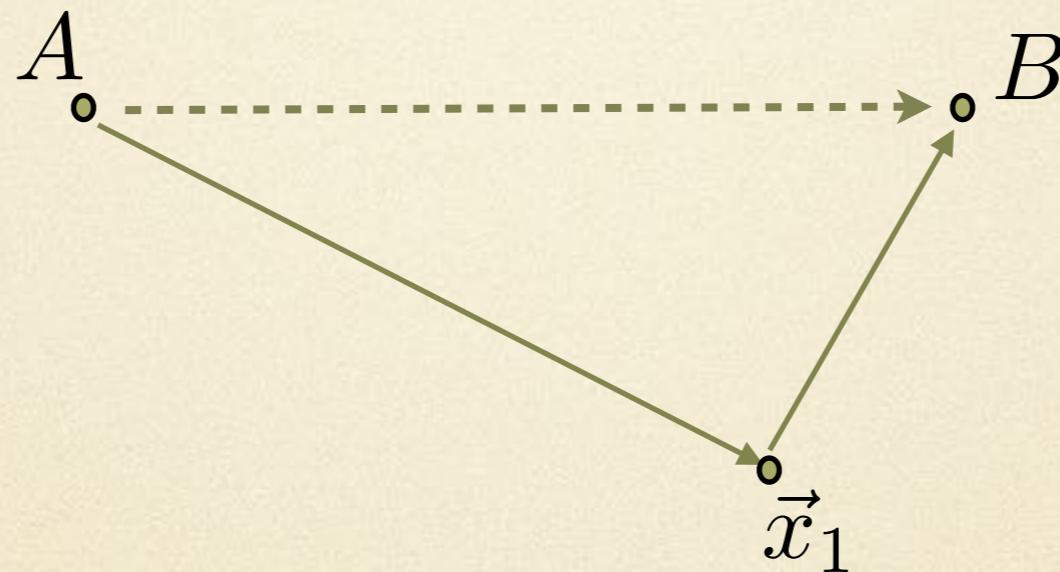
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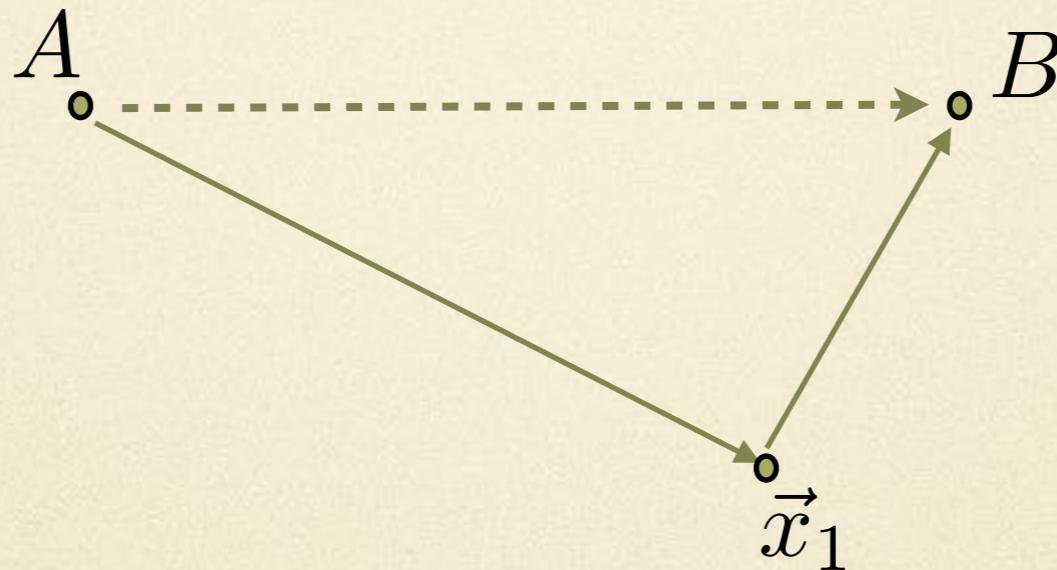
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$$\langle A|B\rangle_1 = N \int d^2x_1 \Delta_1 e^{-(S_{A,\vec{x}_1} + S_{\vec{x}_1,B})}$$

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Change of integration coordinates

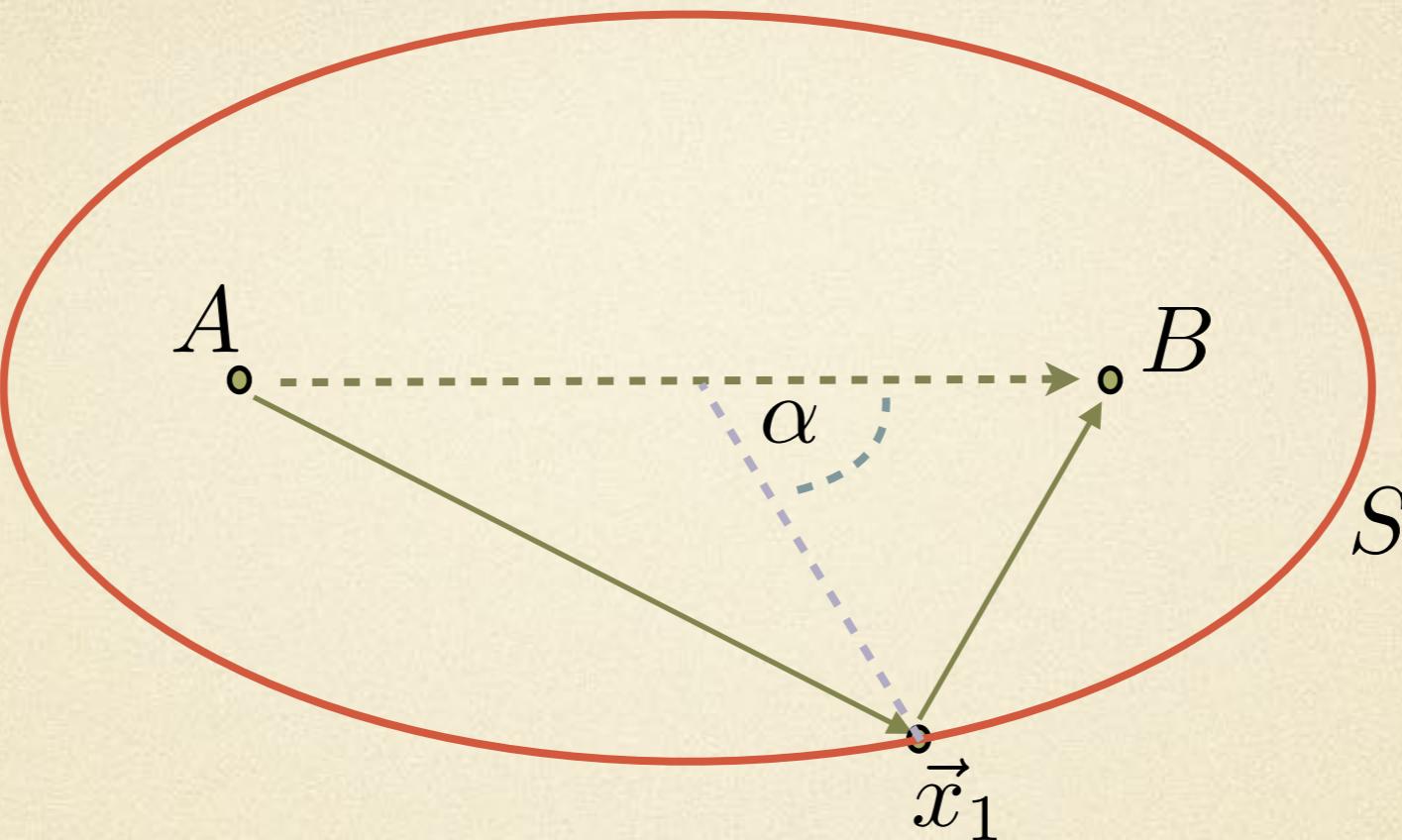
$$(x_1, y_1) \rightarrow (S, \alpha)$$

$$x_1 = \frac{S}{2M} \cos(\alpha)$$

$$y_1 = \frac{x_f}{2} \sqrt{\left(\frac{S}{x_f M}\right)^2 - 1} \cdot \sin(\alpha)$$

$$x_f = |\vec{B} - \vec{A}|$$

- Calculate $\langle A|B \rangle_1$ using I2,3



$$\langle 0, \vec{x}_f \rangle_1 = \mathcal{N}_{2,1}(t_{i,f}) \int_{x_f M}^{\infty} dS \int_0^{2\pi} d\alpha \cdot \Delta_1 \frac{2 (S/M)^2 - x_f^2 (1 + \cos(2\alpha))}{8 \sqrt{(S)^2 - (x_f M)^2}} \exp [-S]$$

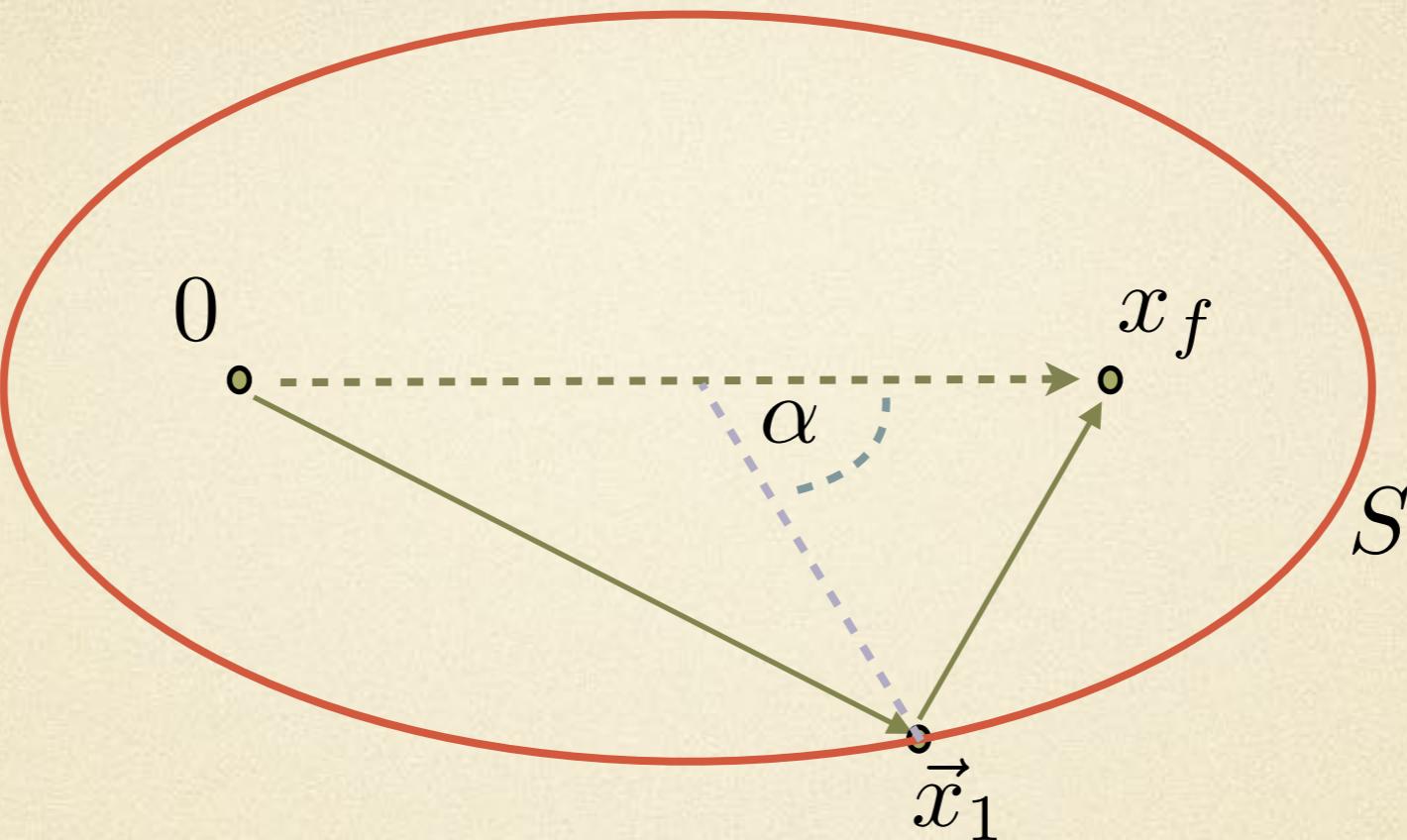
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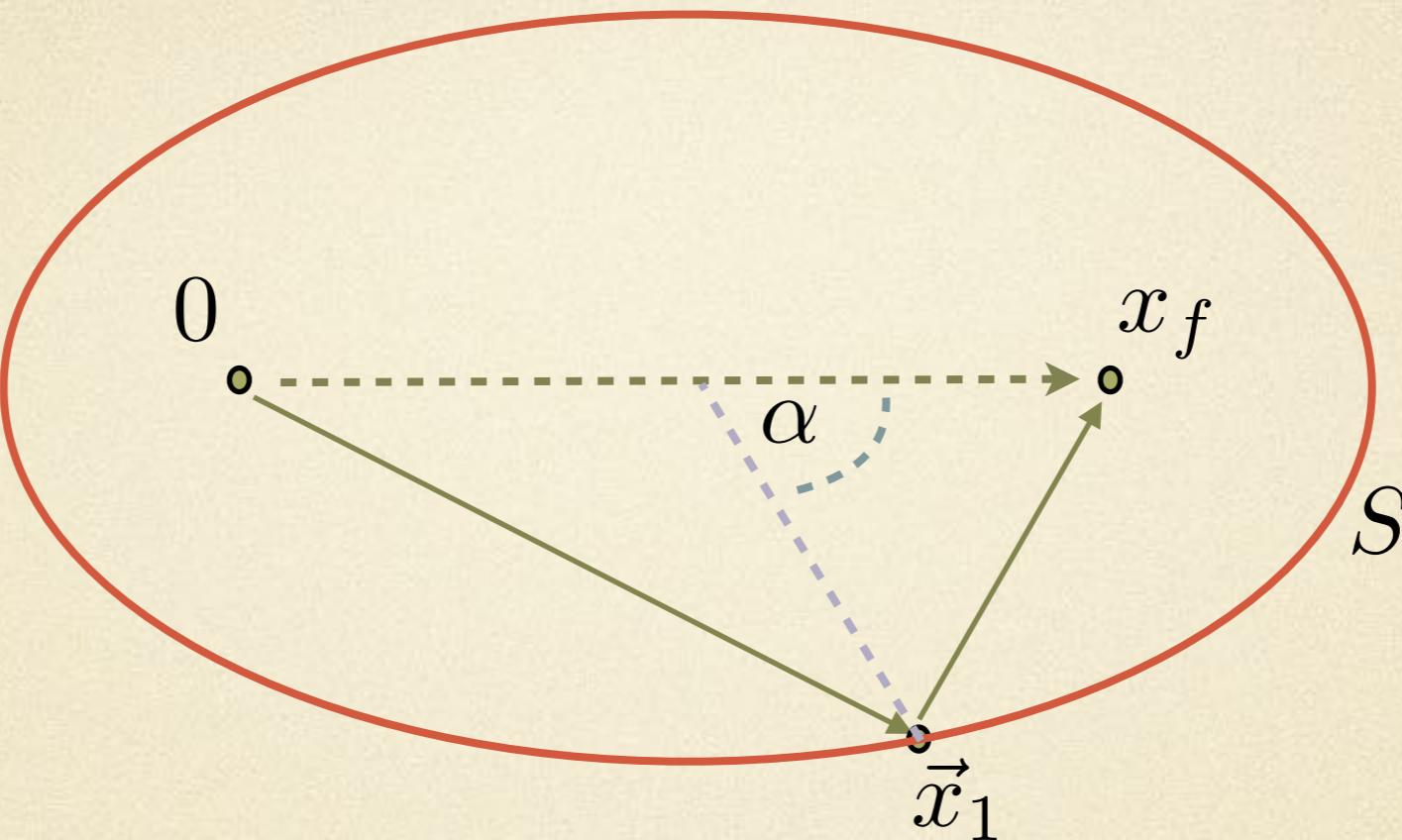
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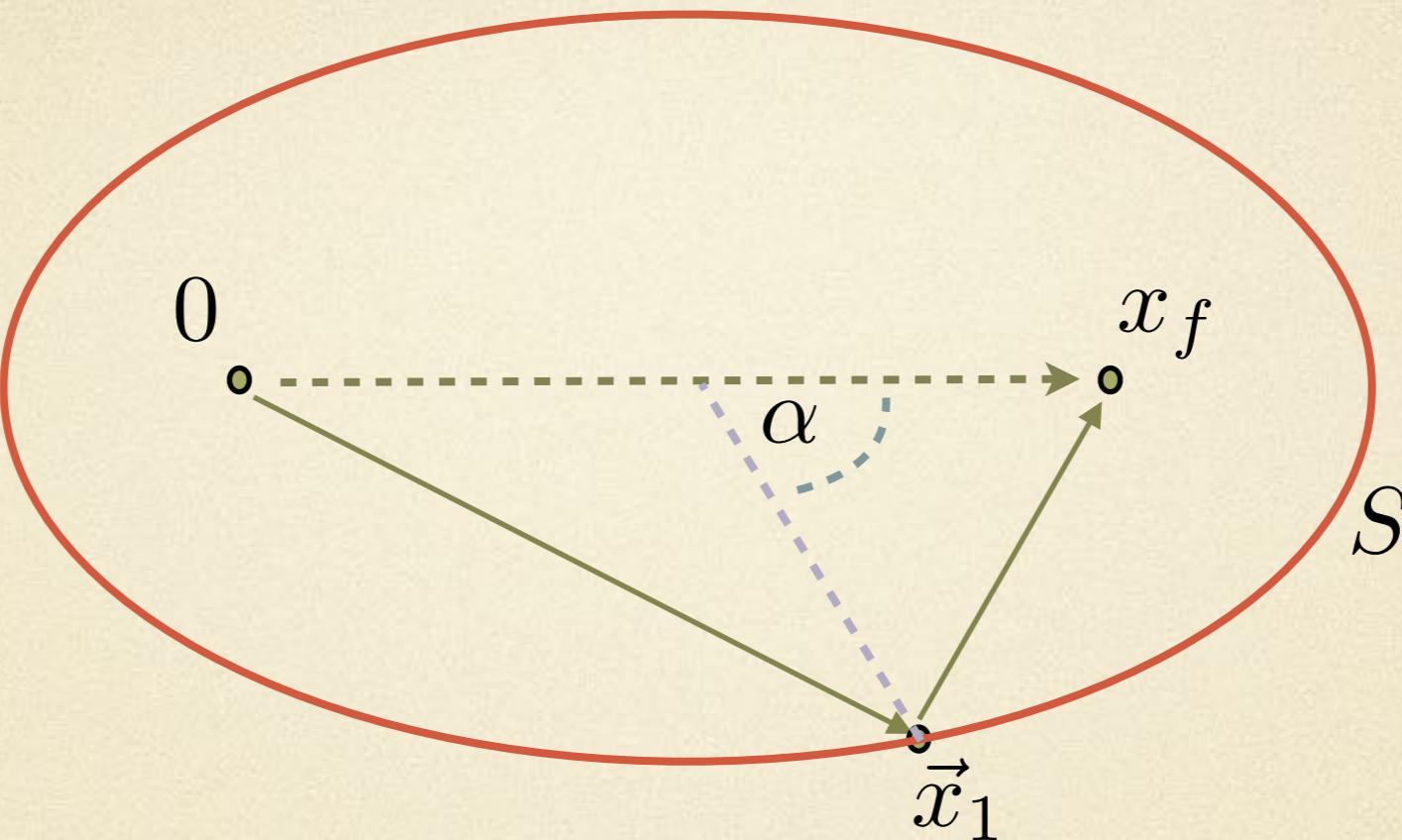
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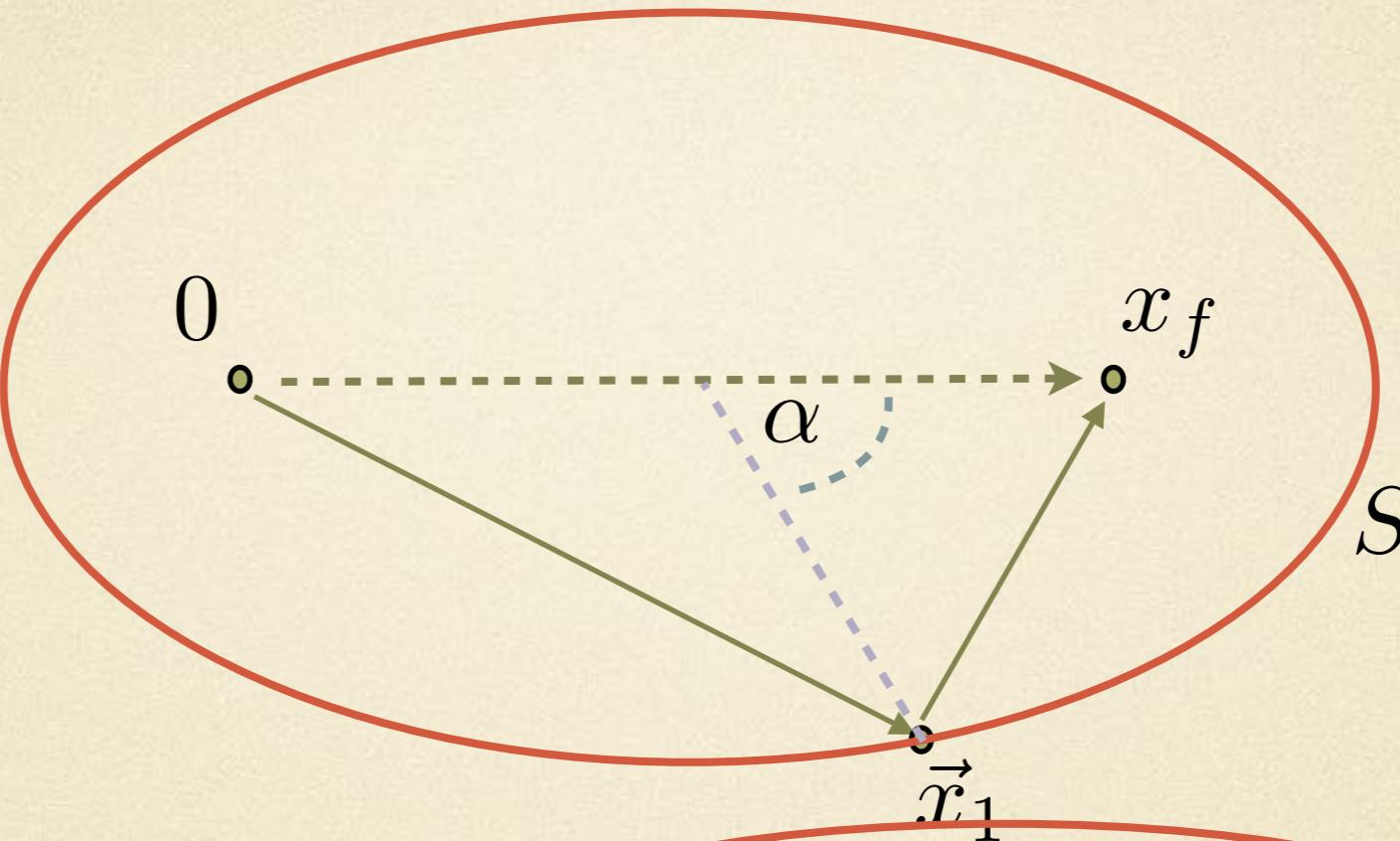


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I2 velocity rotation!

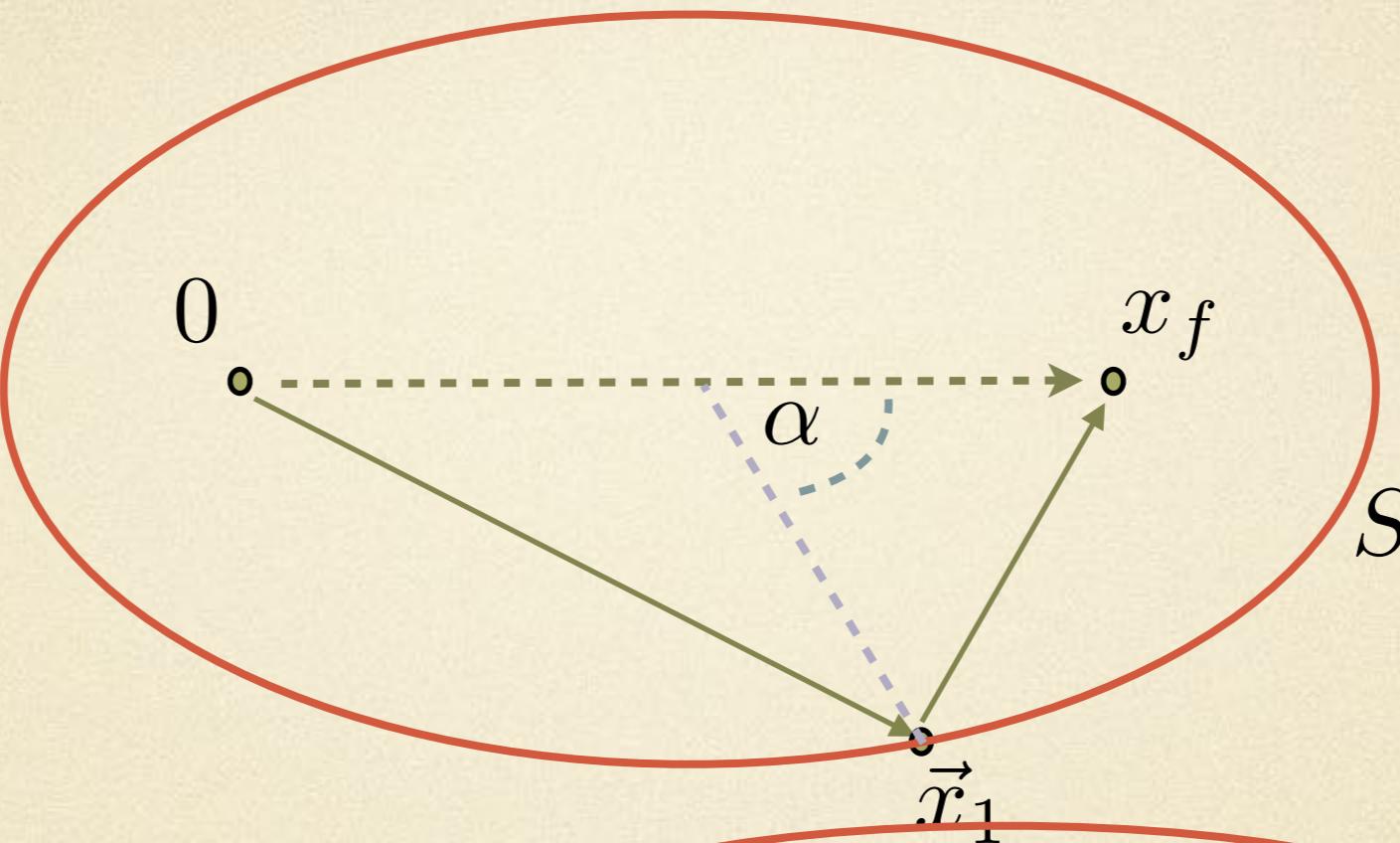
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- I2 velocity rotation!
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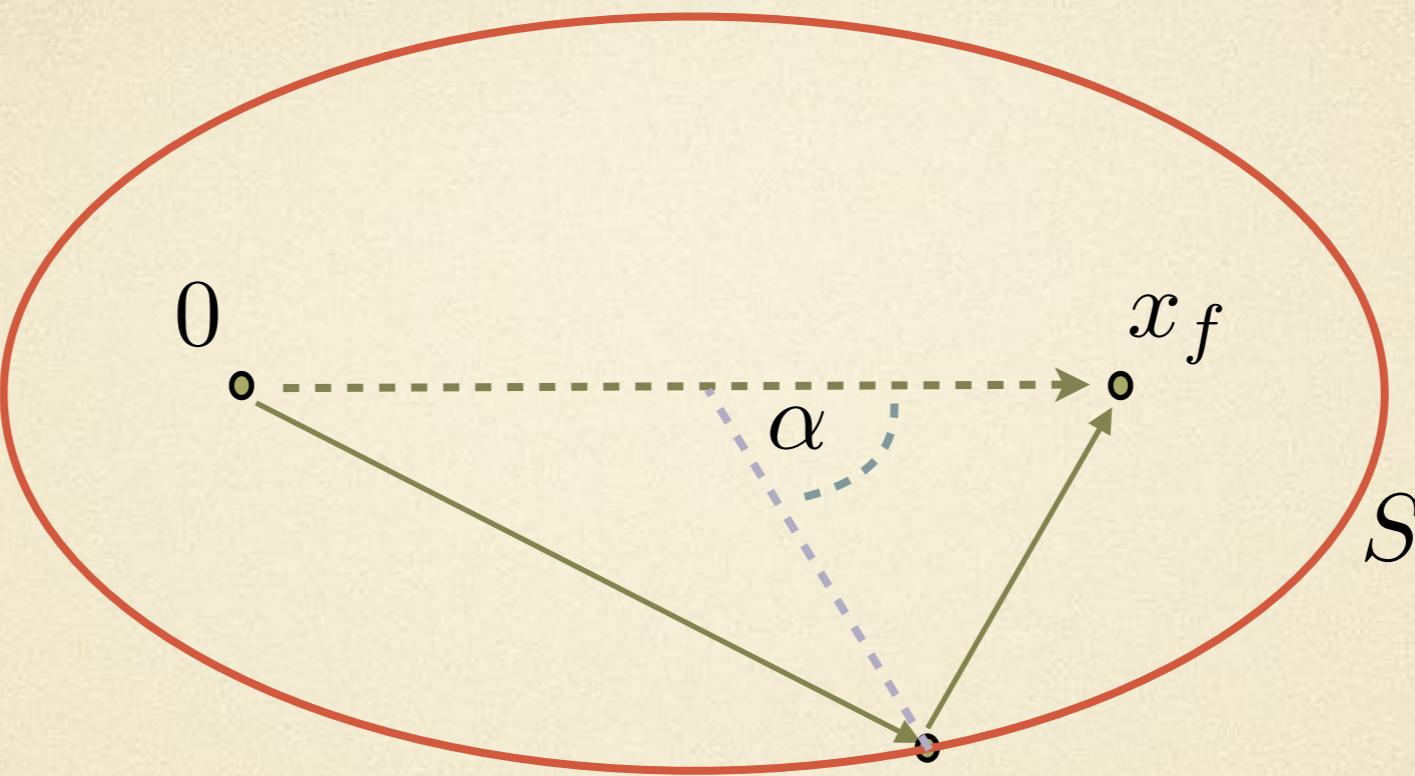
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$$\Delta_1^{-1} \equiv |\vec{x}_1 - 0| \cdot |\vec{x}_f - \vec{x}_1| \text{ cancels anomaly}$$

- Calculate $\langle A|B \rangle_1$ using I2,3

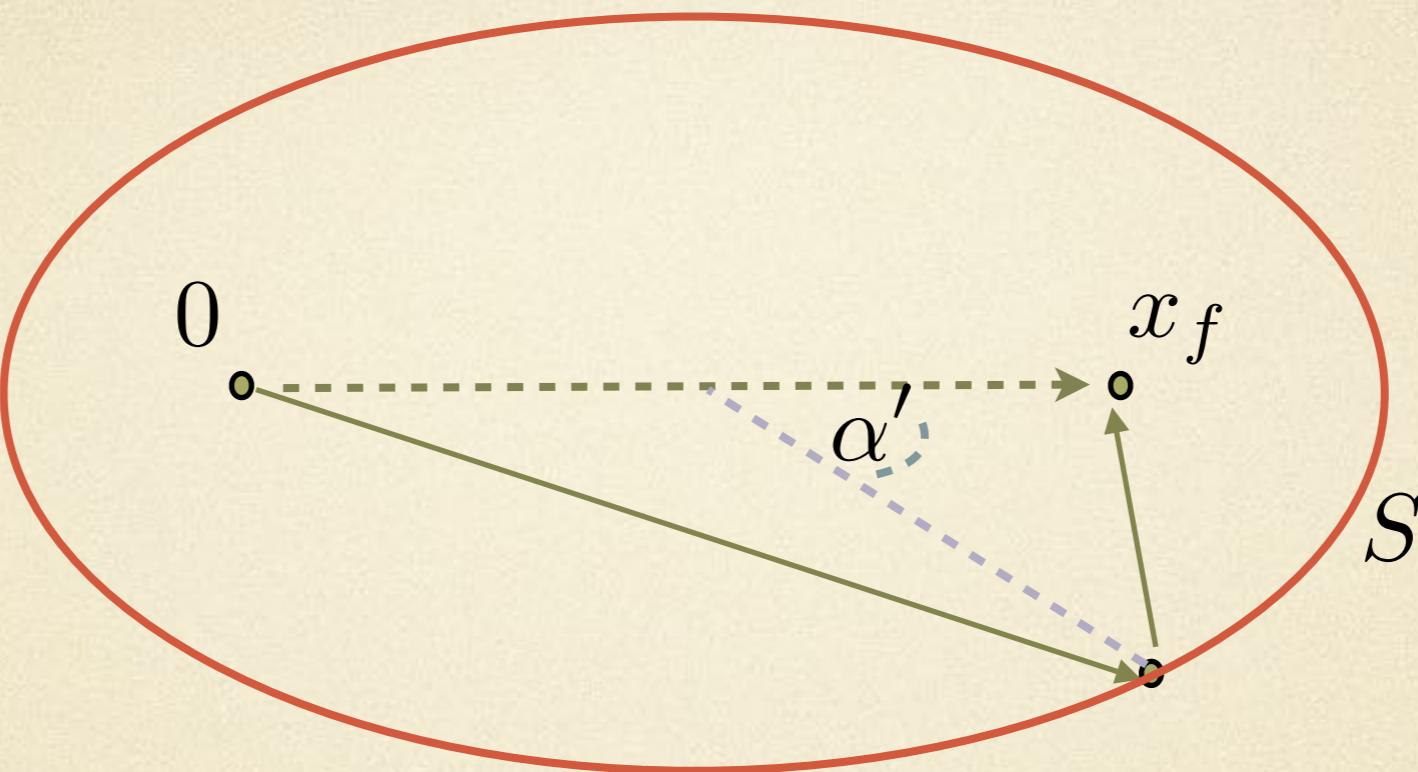


$$\langle 0, \vec{x}_f \rangle_1 = \mathcal{N}_{2,1} \left(\int_0^{2\pi} d\alpha \right) \cdot \int_{x_f M}^{\infty} dS \frac{1}{\sqrt{(S)^2 - (x_f M)^2}} \exp [-S]$$

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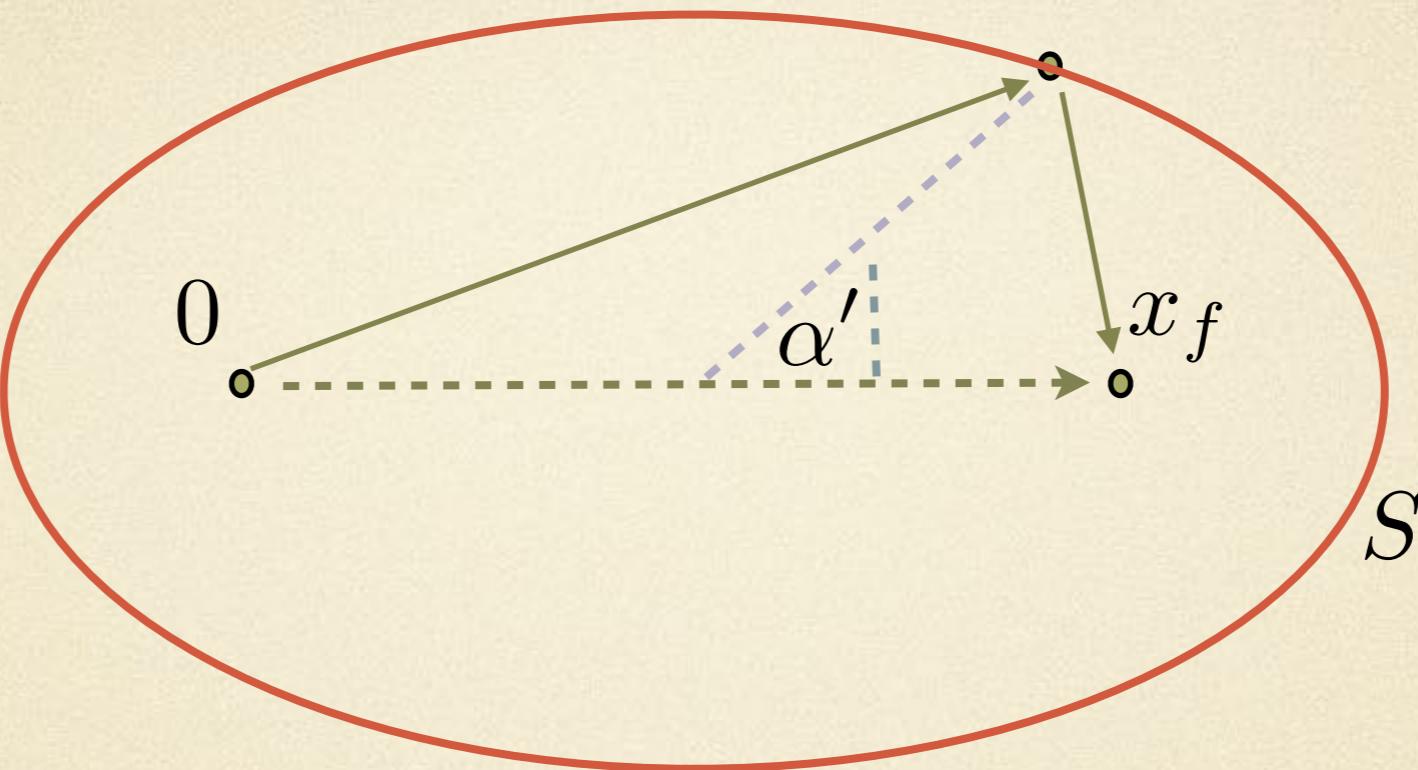


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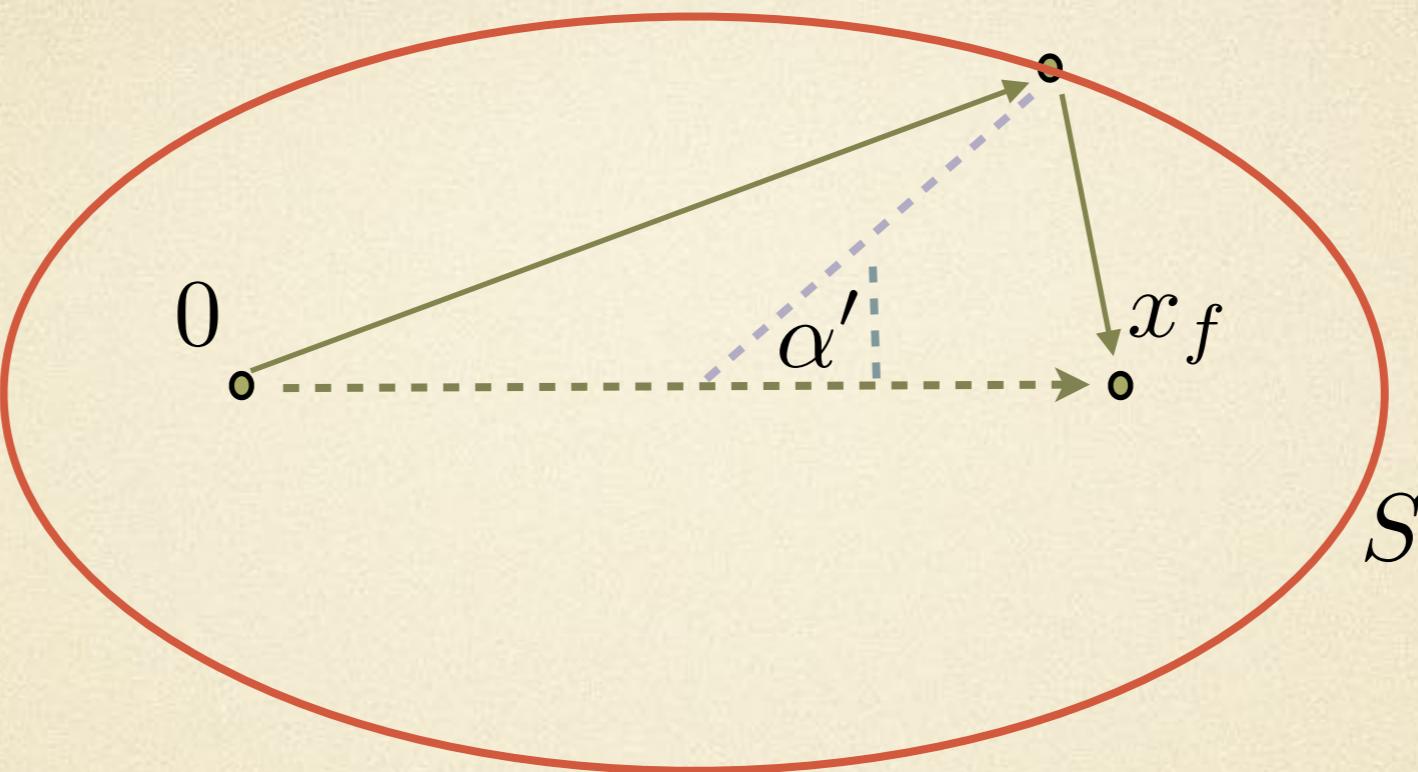


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The diagram shows a complex interaction represented by two nodes connected by multiple curved arrows forming loops. This is equated to a sum of simpler interactions, each represented by a pair of nodes connected by a single arrow. The first term in the sum is a horizontal dashed arrow between two nodes. Subsequent terms show more complex configurations, such as a node connected to another via a triangle or a diamond shape. The entire expression is followed by a plus sign and three dots, indicating a series expansion.

$$\langle A|B\rangle = \langle A|B\rangle_1 + \langle A|B\rangle_2 + \dots$$

- Show with I1,2 $\langle A|B \rangle_1$ contains $\langle A|B \rangle_2 \dots$

The diagram illustrates the decomposition of a complex interaction into simpler components. On the left, a green wavy line with arrows at both ends connects two vertices. A dashed horizontal line with arrows at both ends connects the same two vertices. This is followed by an equals sign. To the right of the equals sign is a large orange thumbs-up icon. Below the icon is a sum symbol ($+$). The first term in the sum is a green line with arrows connecting the two vertices. The second term is a green triangle with arrows pointing from the bottom-left vertex to the top vertex and from the top vertex to the bottom-right vertex. This is followed by another sum symbol ($+$) and three dots (\dots). Below this, the equation $\langle A|B \rangle = \langle A|B \rangle_1 + \langle A|B \rangle_2 + \dots$ is written.

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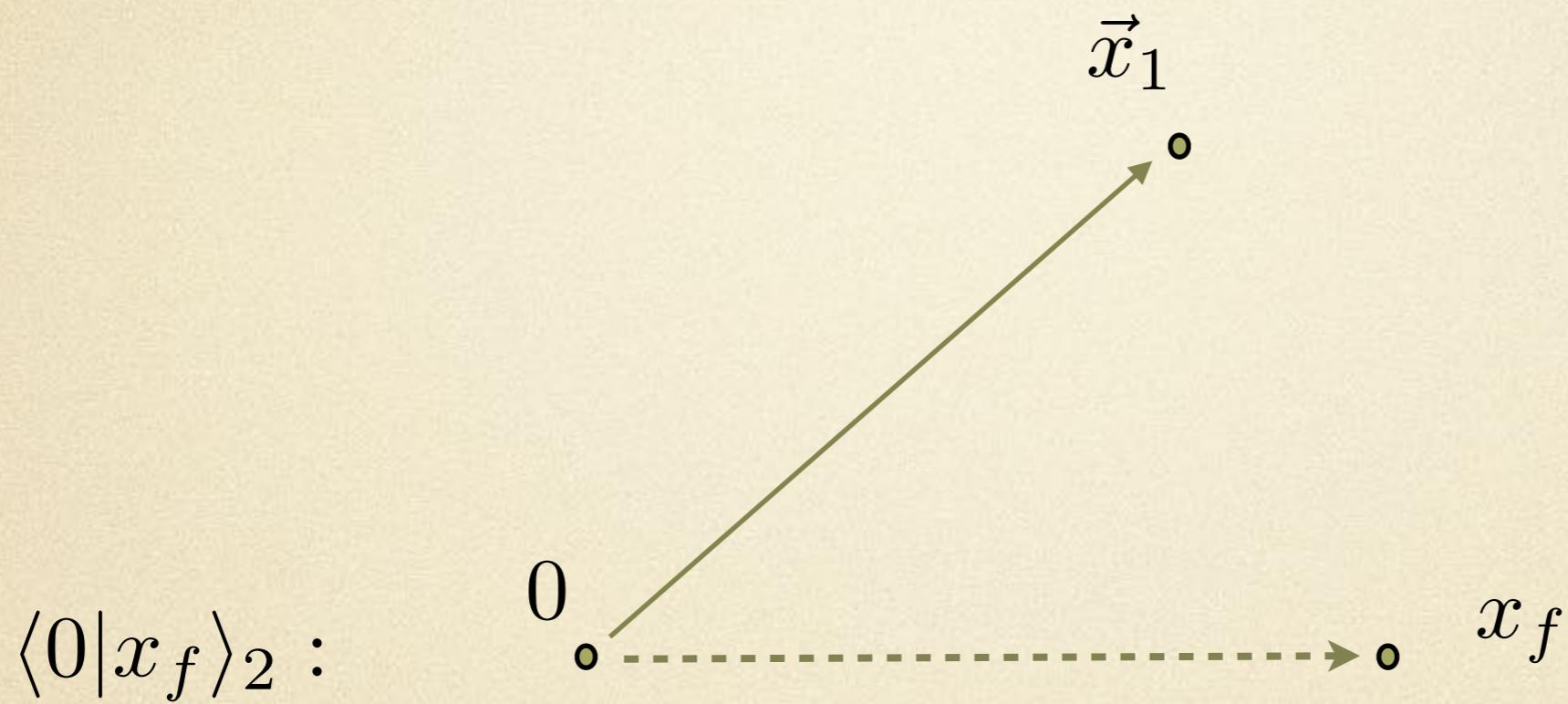
$$\langle A|B \rangle = \text{ thumbs-up icon } + \dots$$

$$\langle A|B \rangle = \langle A|B \rangle_1 + \langle A|B \rangle_2 + \dots$$

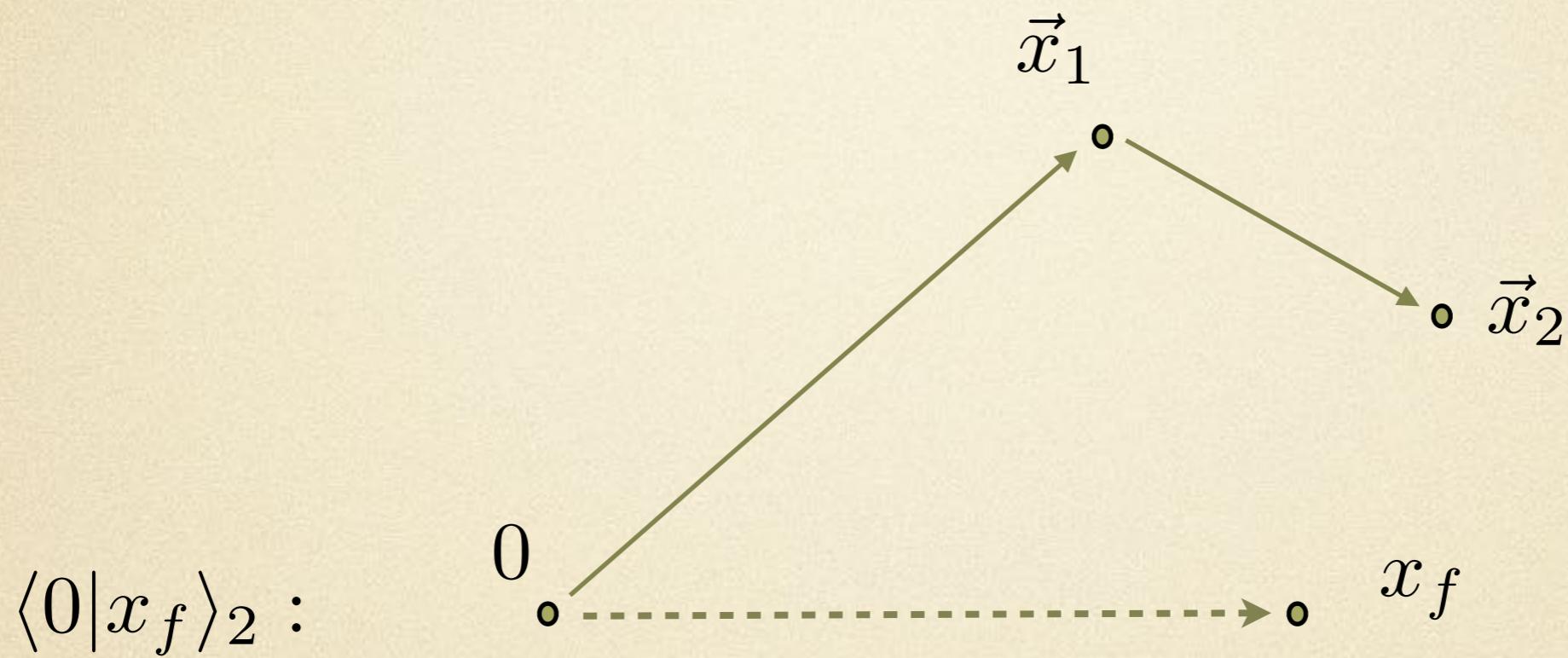
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$$\langle 0|x_f\rangle_2 : \quad 0 \circ \text{-----} \rightarrow \circ \quad x_f$$

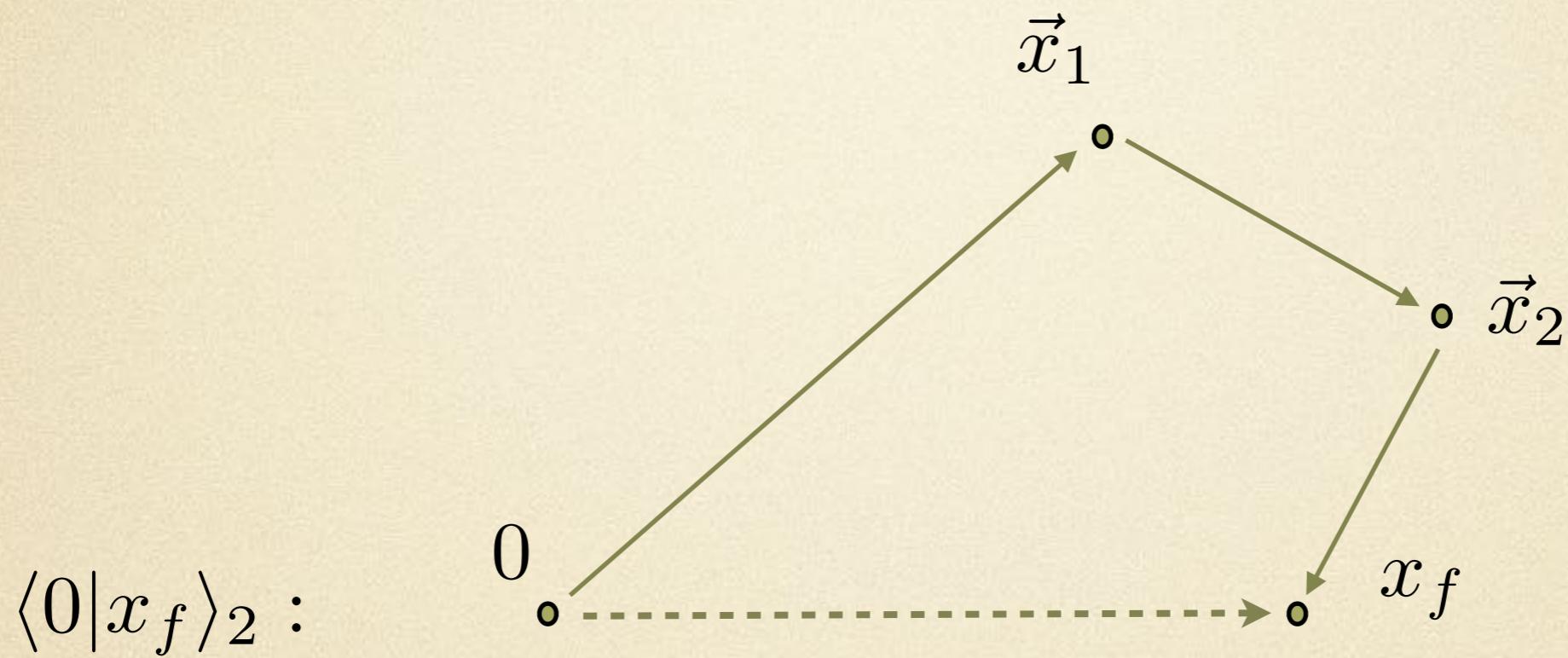
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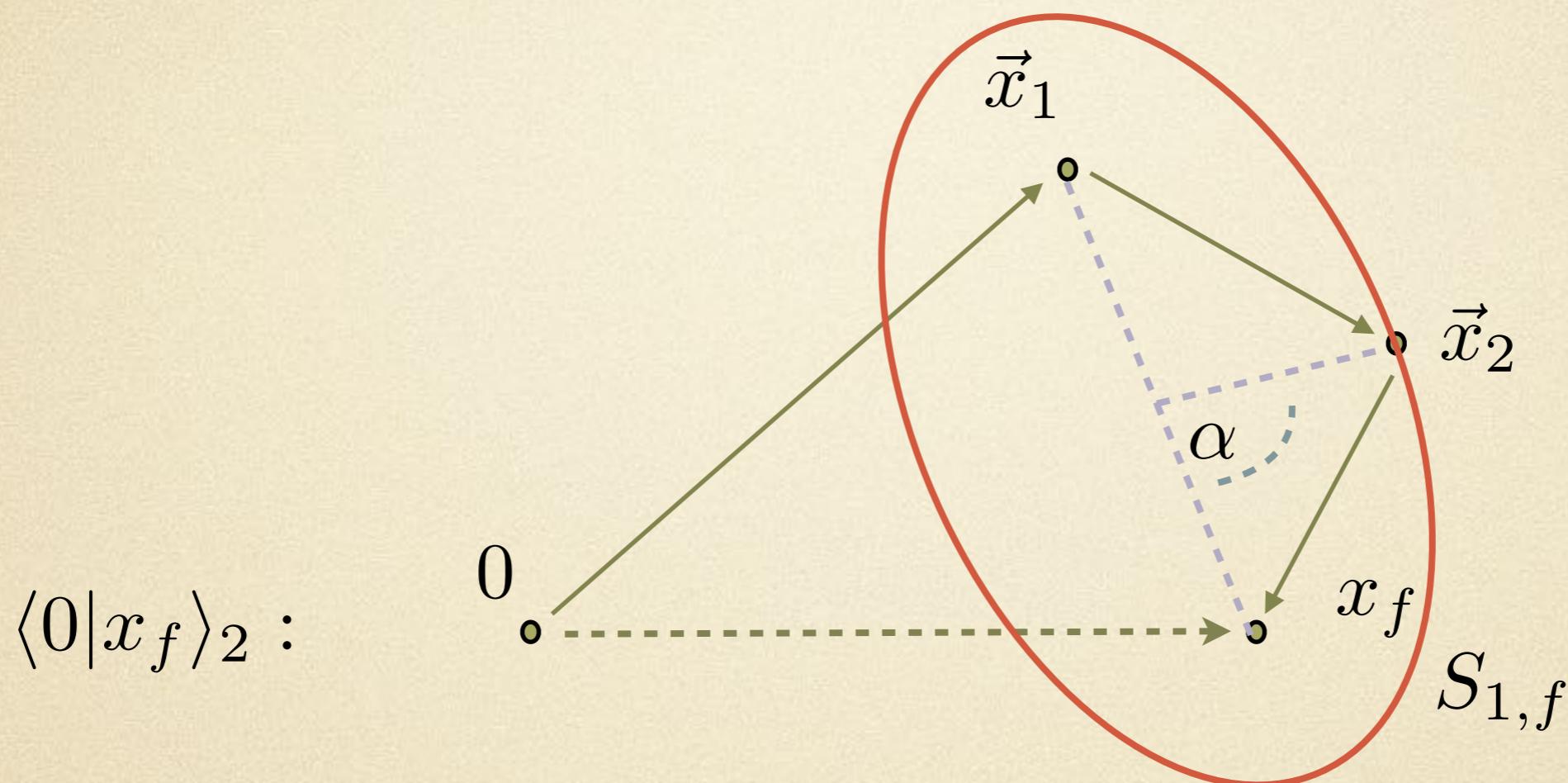
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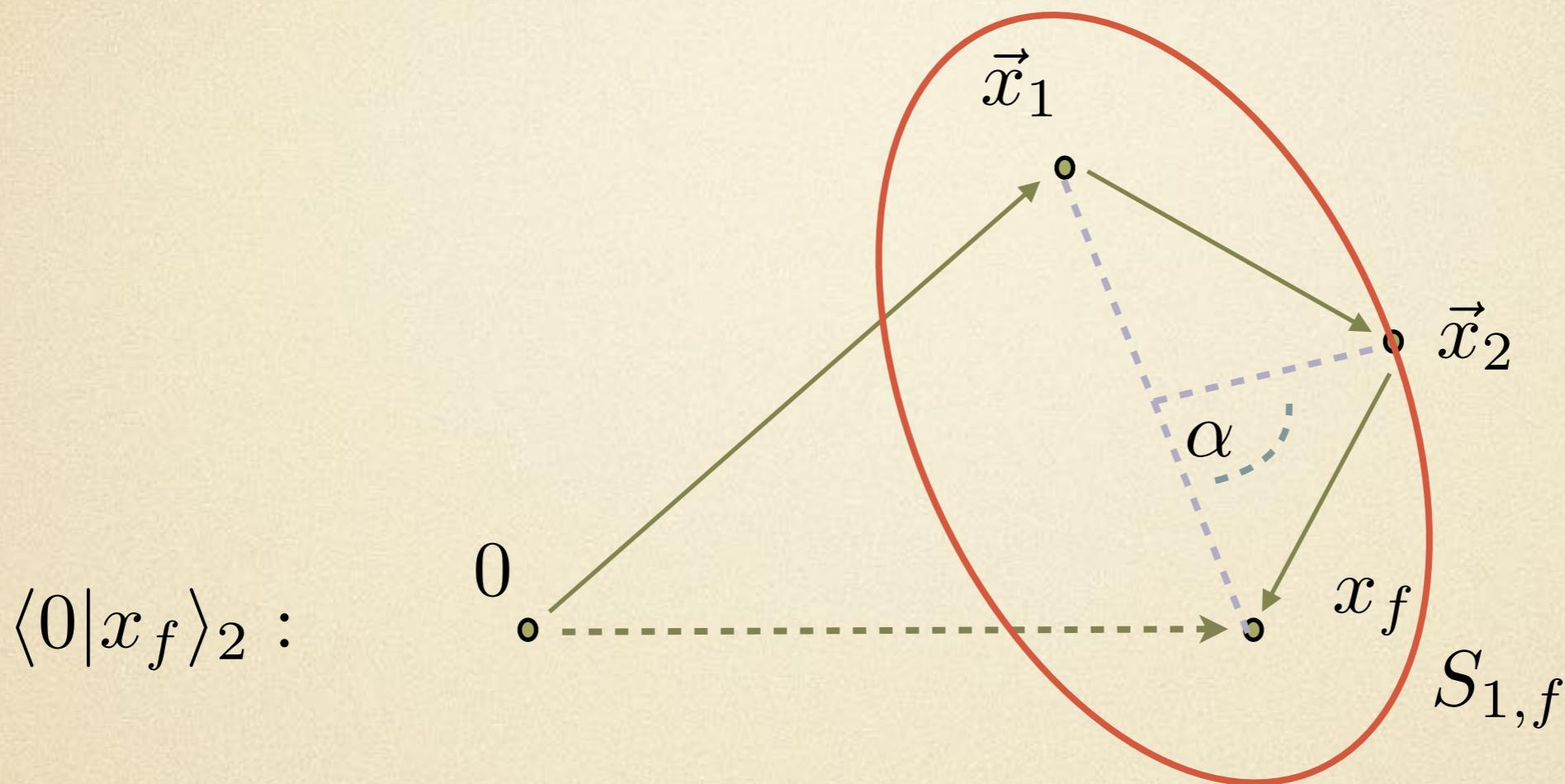
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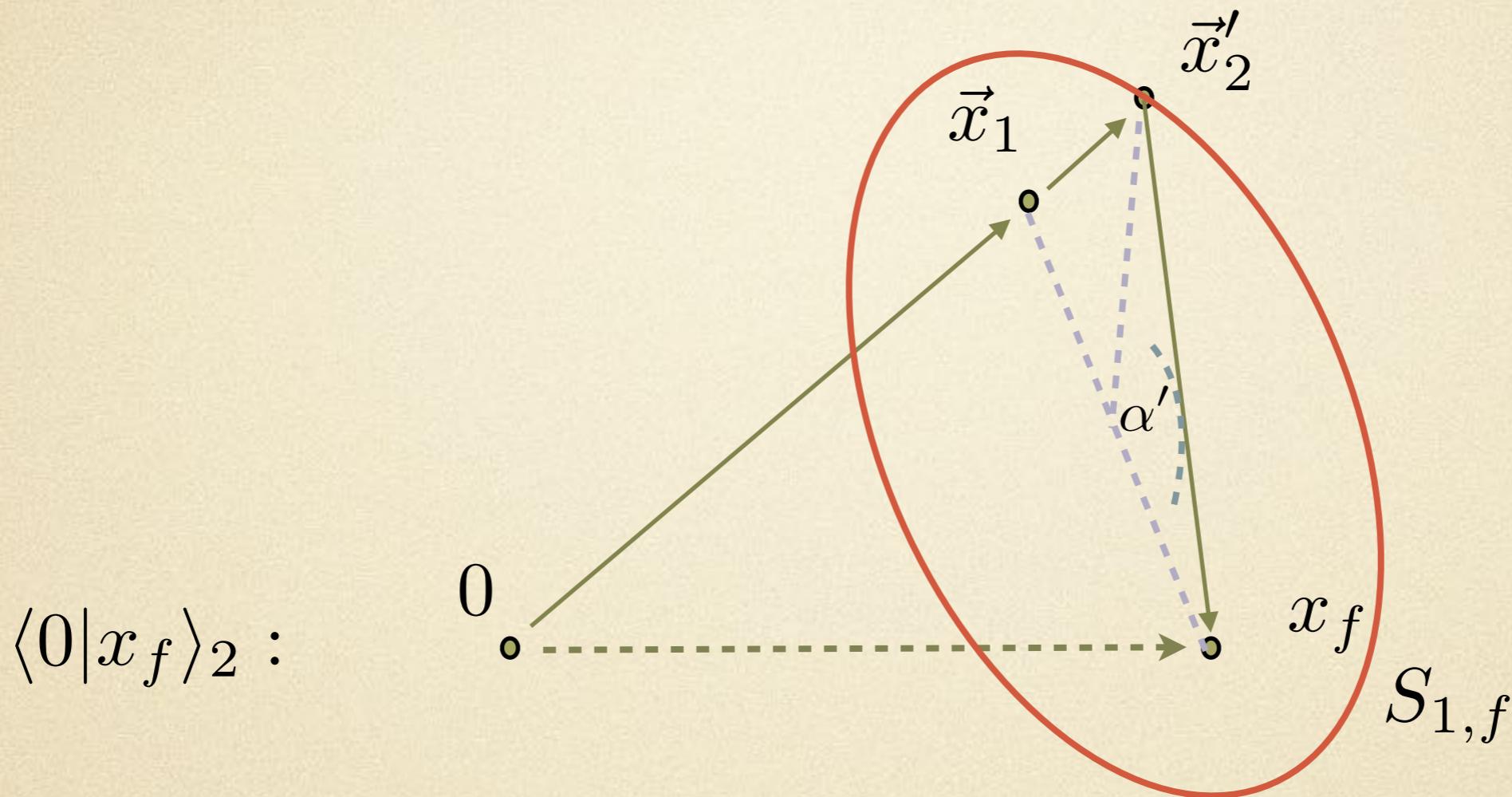


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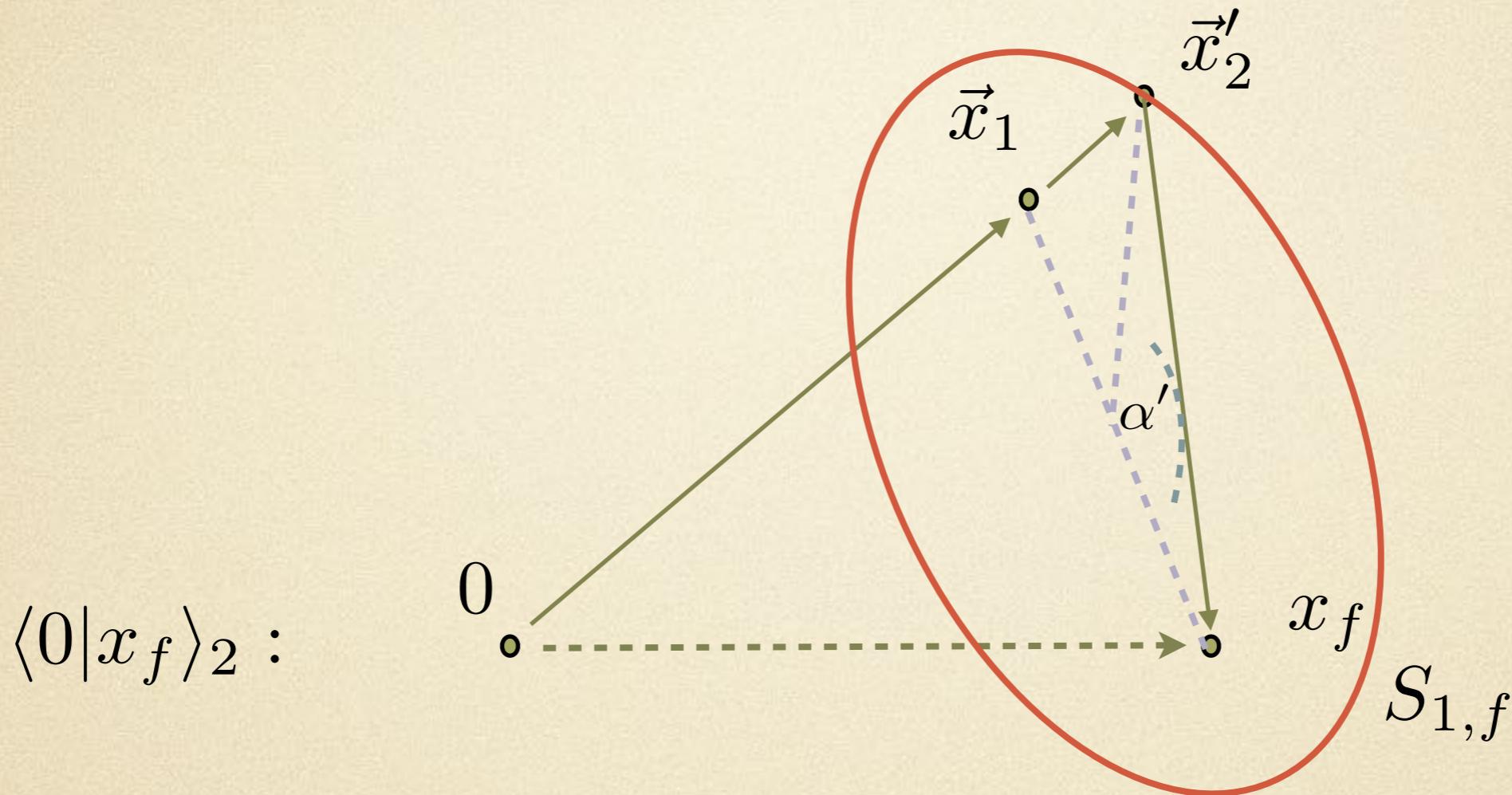
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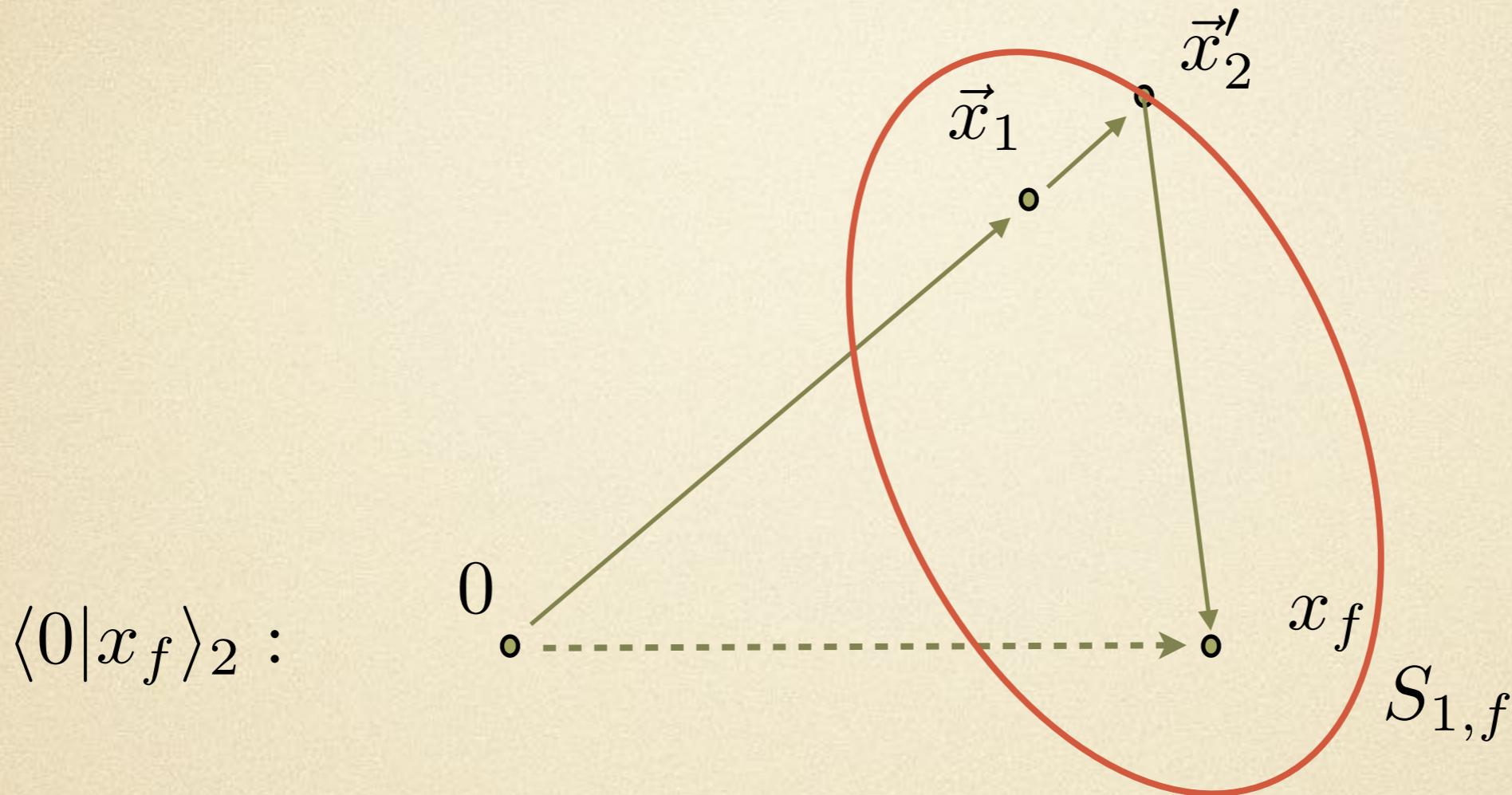
- For $\alpha \rightarrow \alpha'$ nothing changes (I2) $\vec{x}_2 \rightarrow \vec{x}'_2$

- Show with I1,2 $\langle A|B \rangle_1$ contains $\langle A|B \rangle_2 \dots$



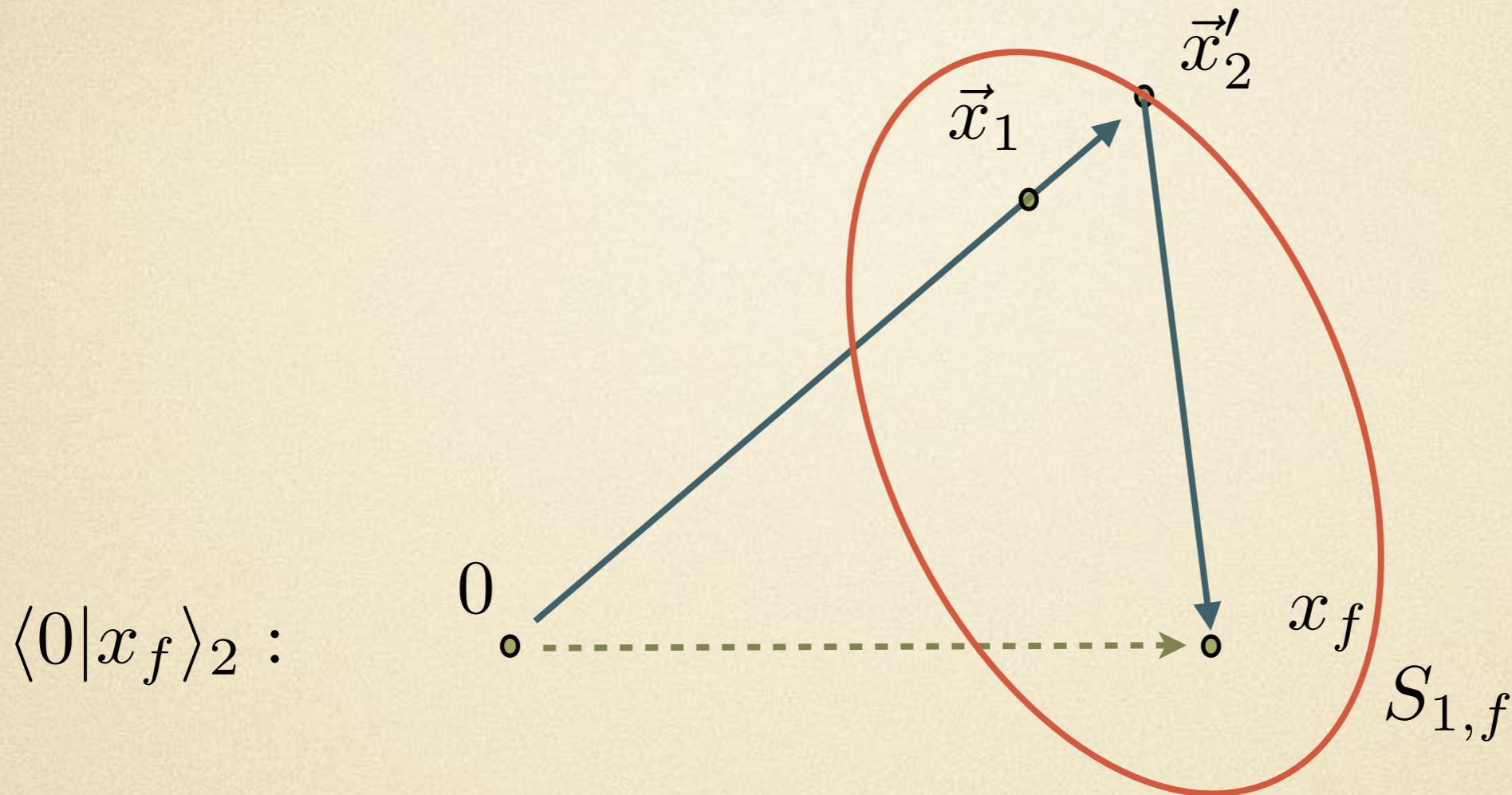
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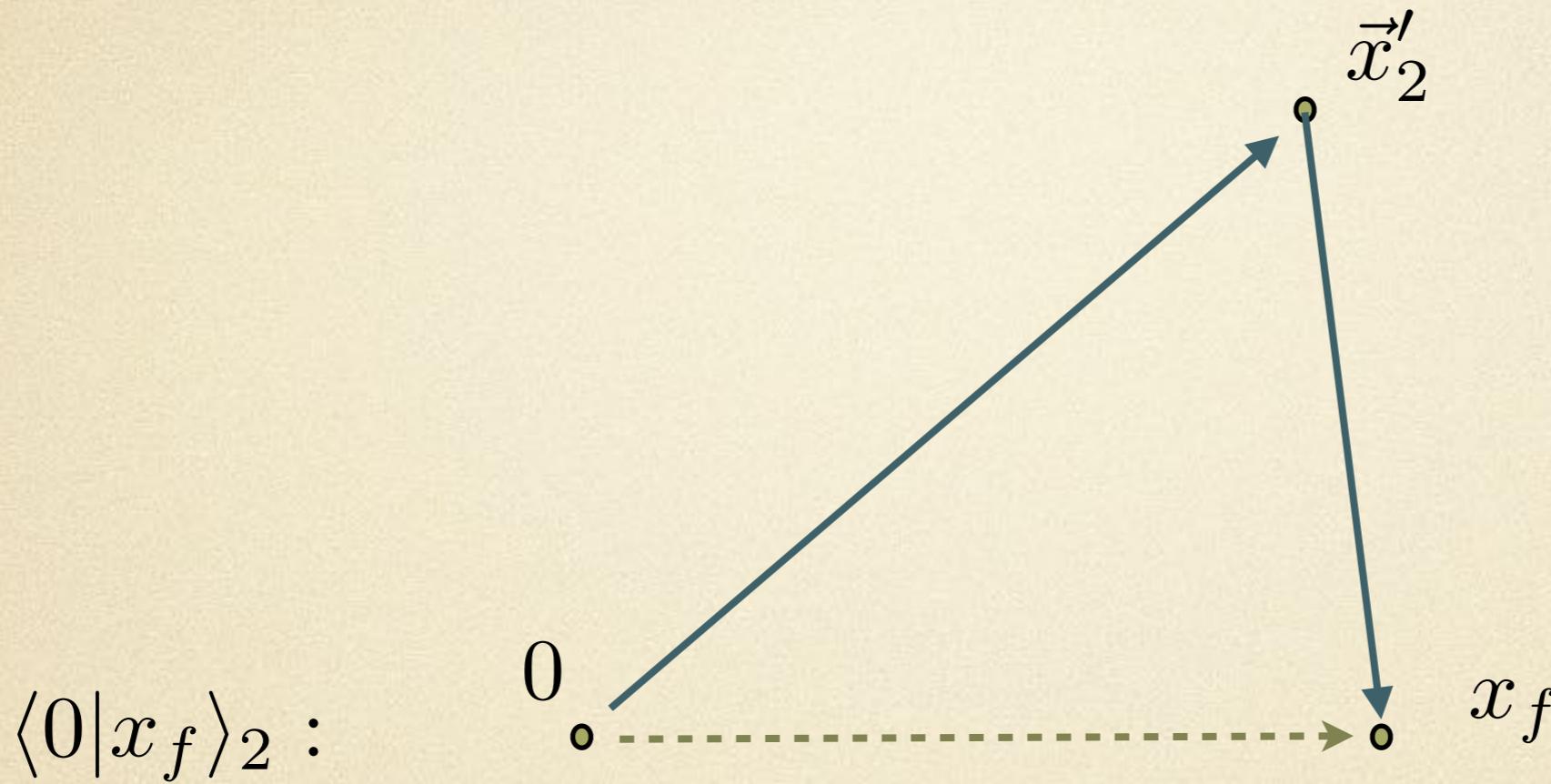
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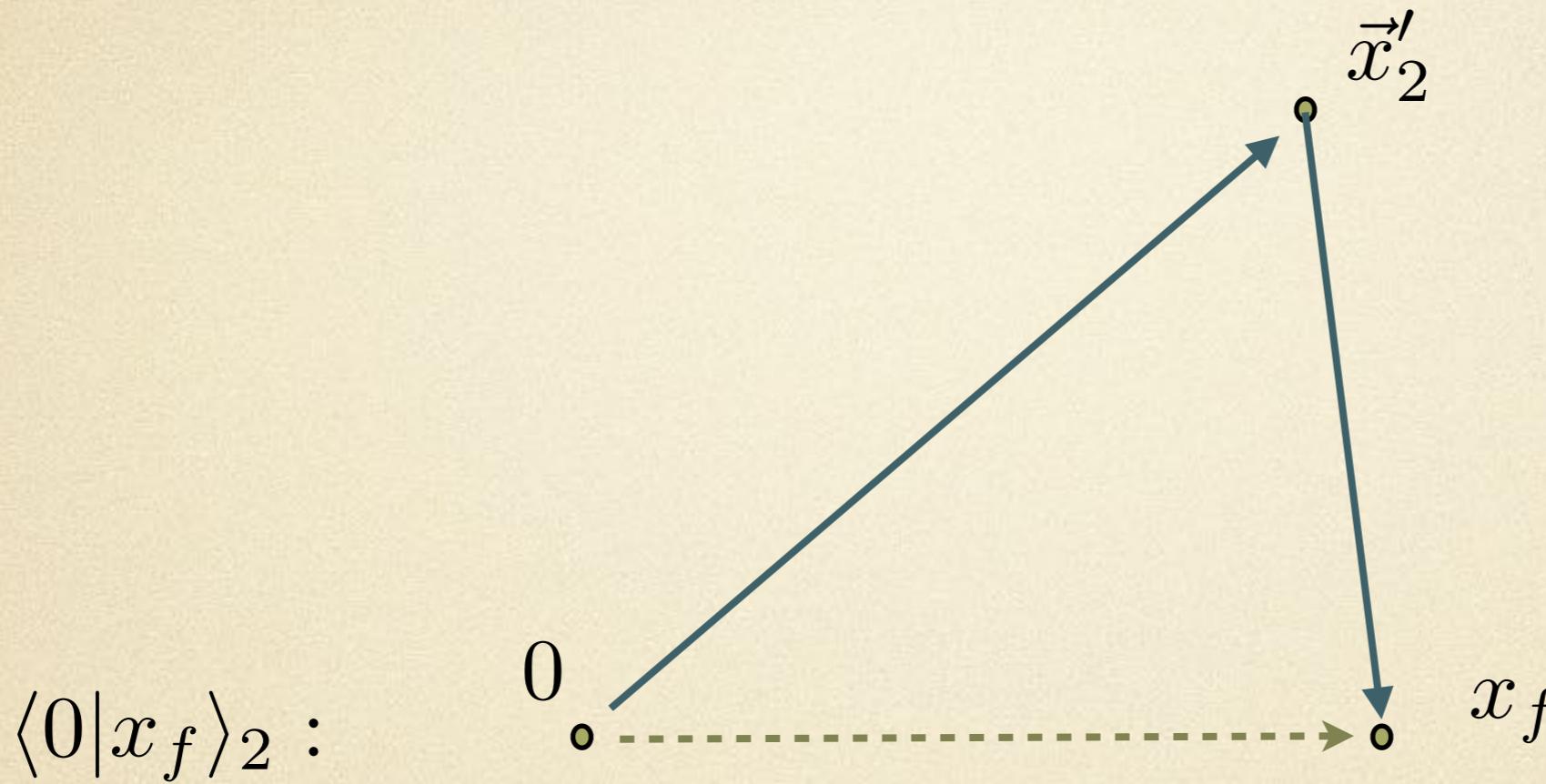
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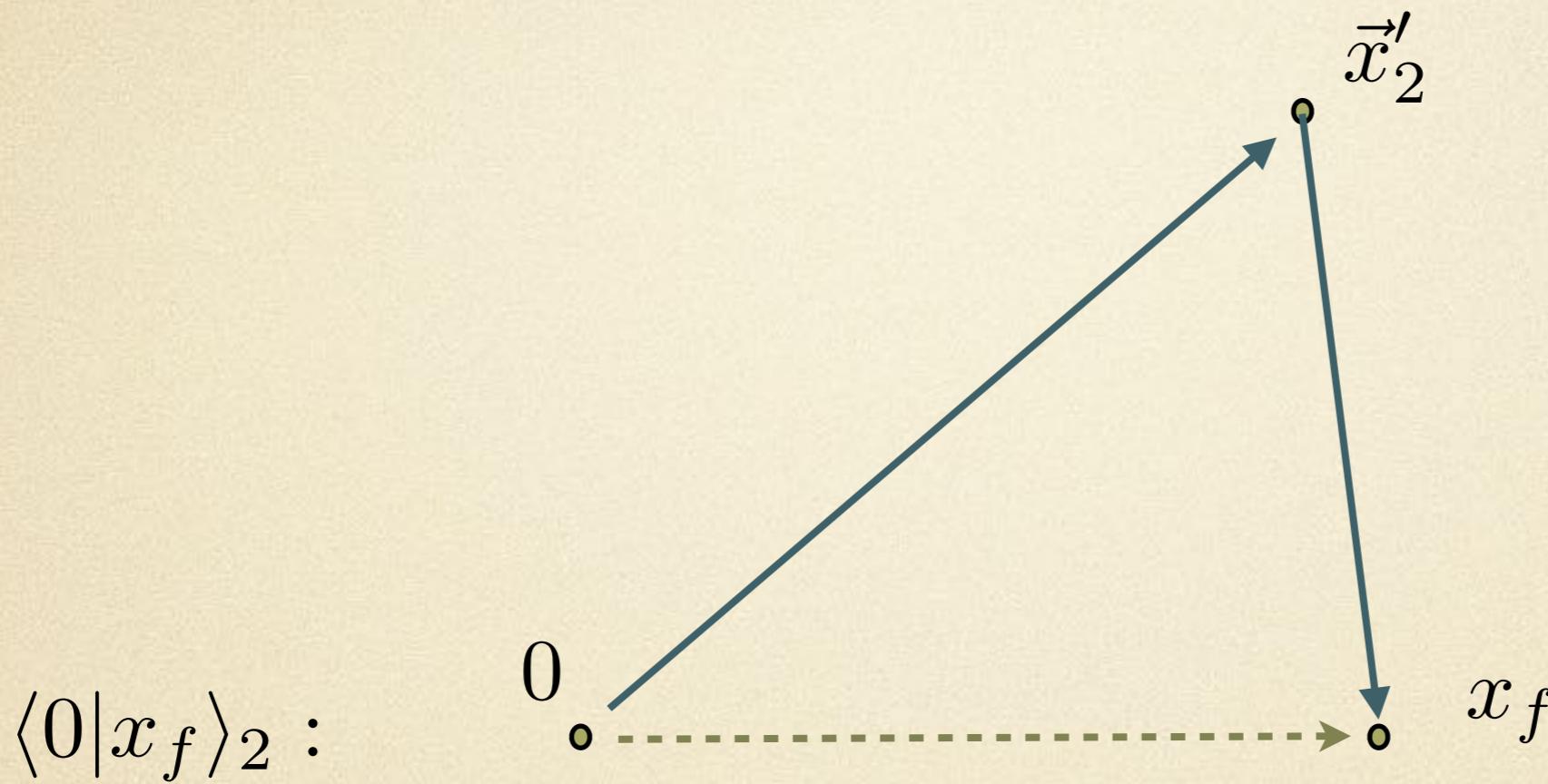
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Stepwise proof

The diagram shows a green wavy line segment starting from a point on the left and ending at a point on the right. This segment is decomposed into several straight line segments with arrows, representing a stepwise approximation. Below the diagram, the mathematical expression $\langle A|B \rangle = \langle A|B \rangle_1 + \langle A|B \rangle_2 + \dots$ is shown, where each term corresponds to one of the straight segments in the diagram.

$$\langle A|B \rangle = \langle A|B \rangle_1 + \langle A|B \rangle_2 + \dots$$

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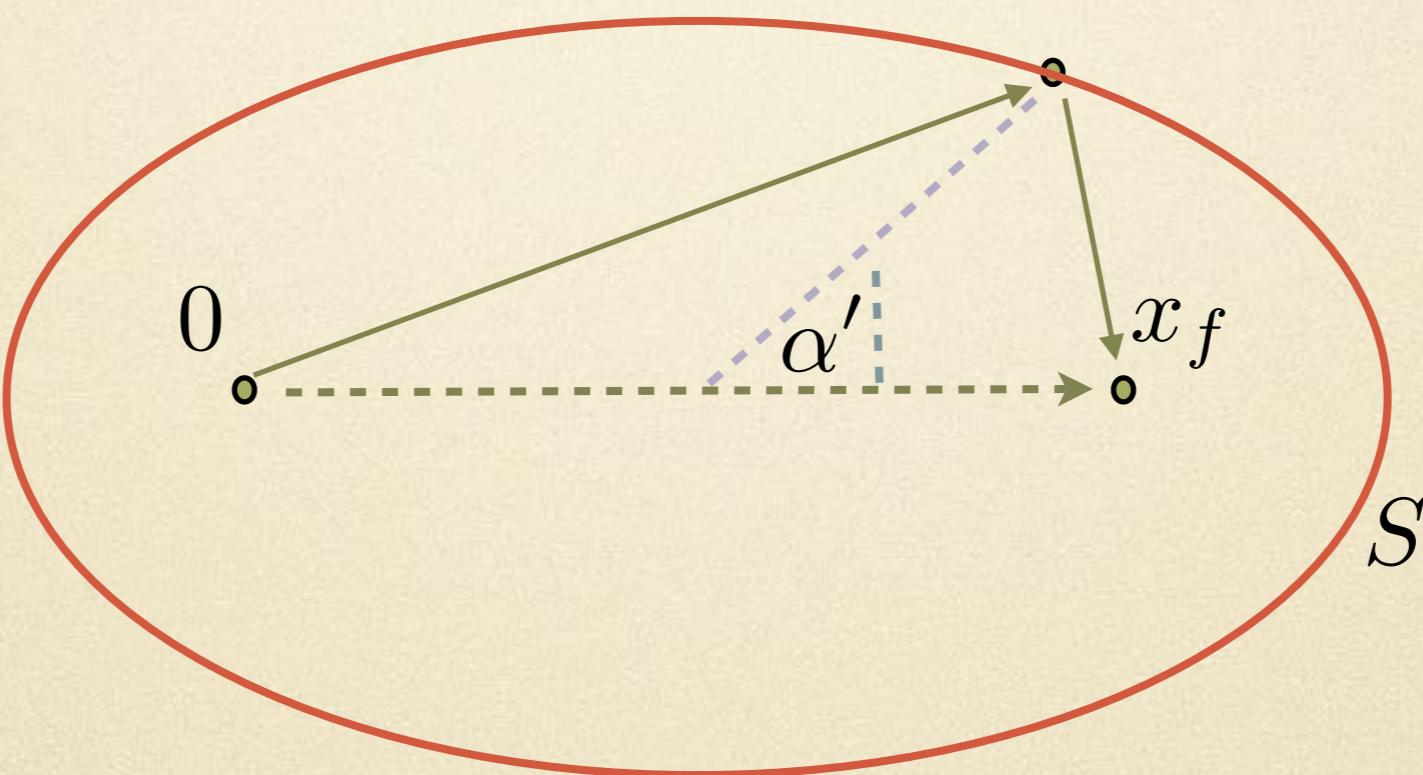
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Concluding Comments

- Generalization to D dimensions
- PI of RPP action can be done, considering I₁,I₂,I₃
- Chapman Kolmogorov becomes „trivial“
- Future work ...

Thank You



Literature

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