The Path Integral over Relativistic Worldlines

B. Koch
with E. Muñoz and I. Reyes

based on:

Phys.Rev. D96 (2017) no.8, 085011
and arXiv:1706.05388.

Content

- PI of the RPP, Status
- Local Symmetry: Velocity Rotations
- Constructing the PI of the RPP
- Conclusion

The Path Integral

Propagator

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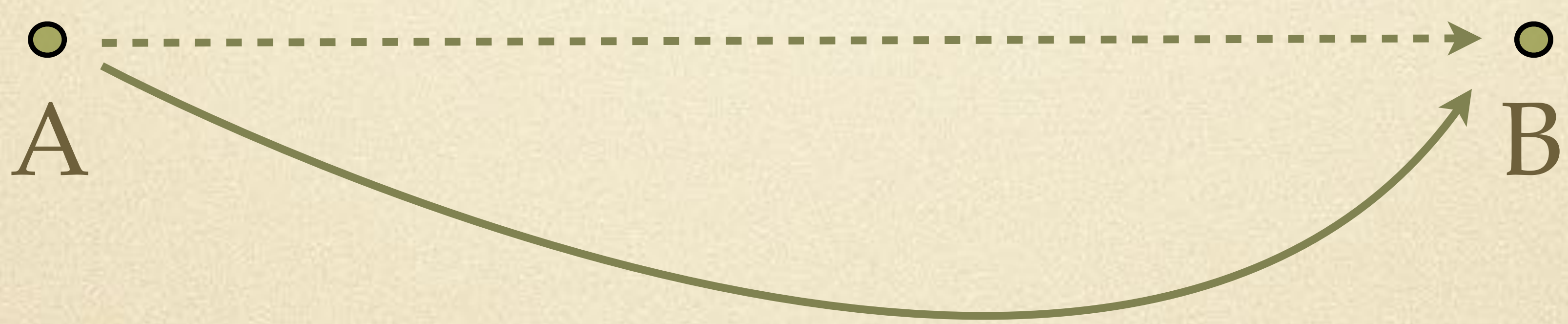

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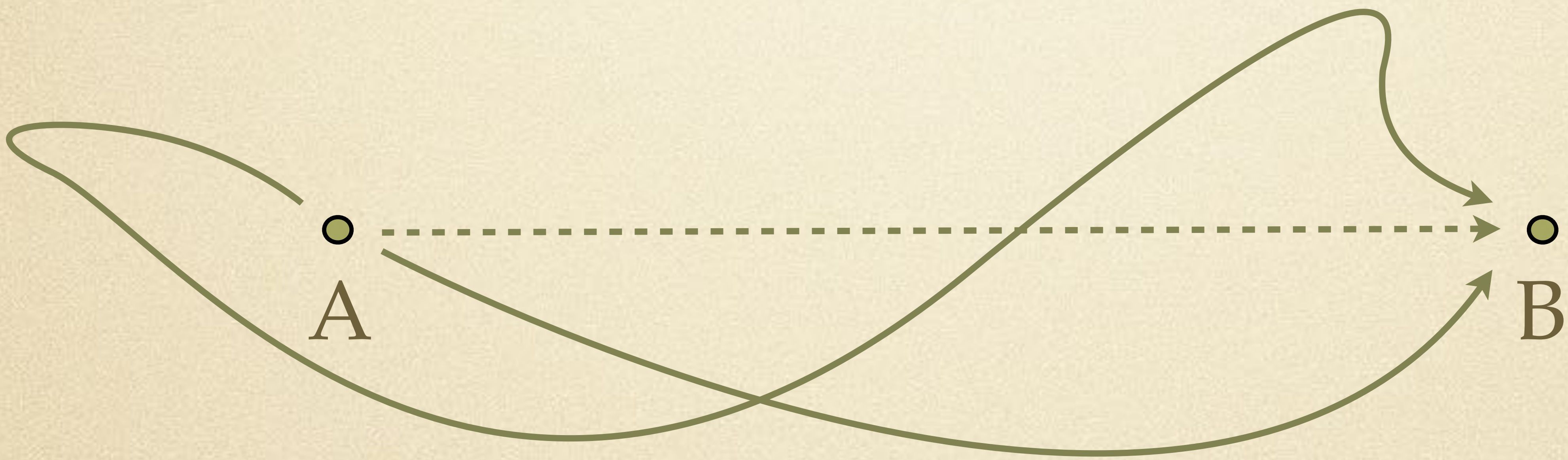

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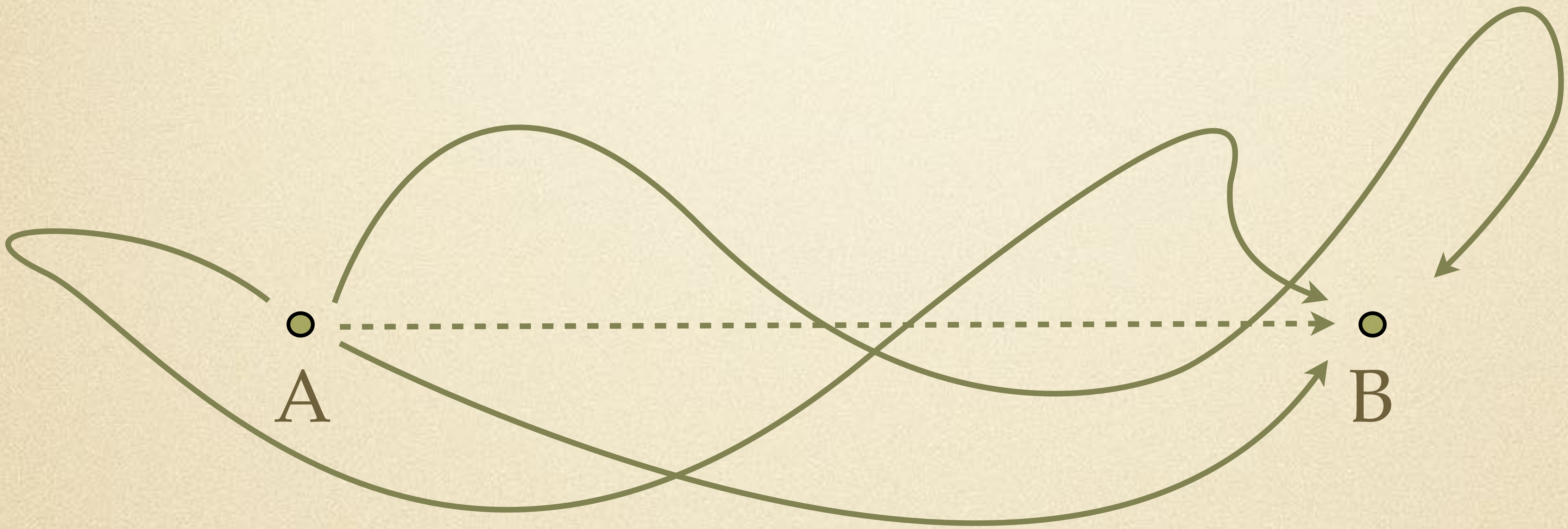

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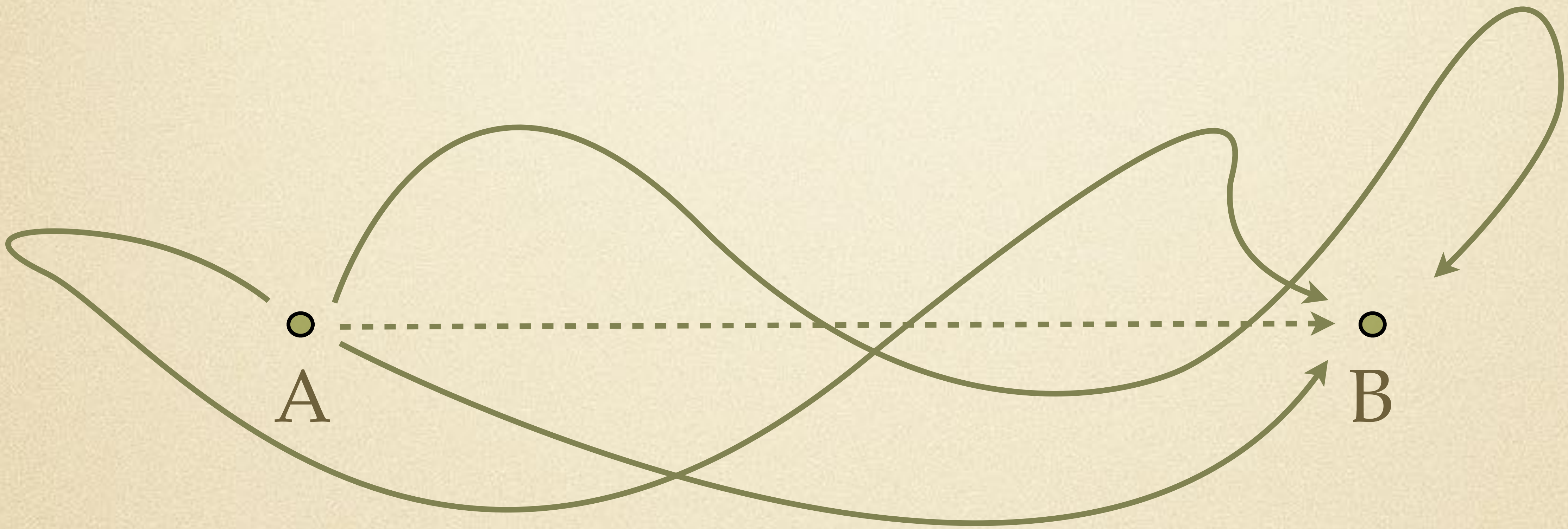

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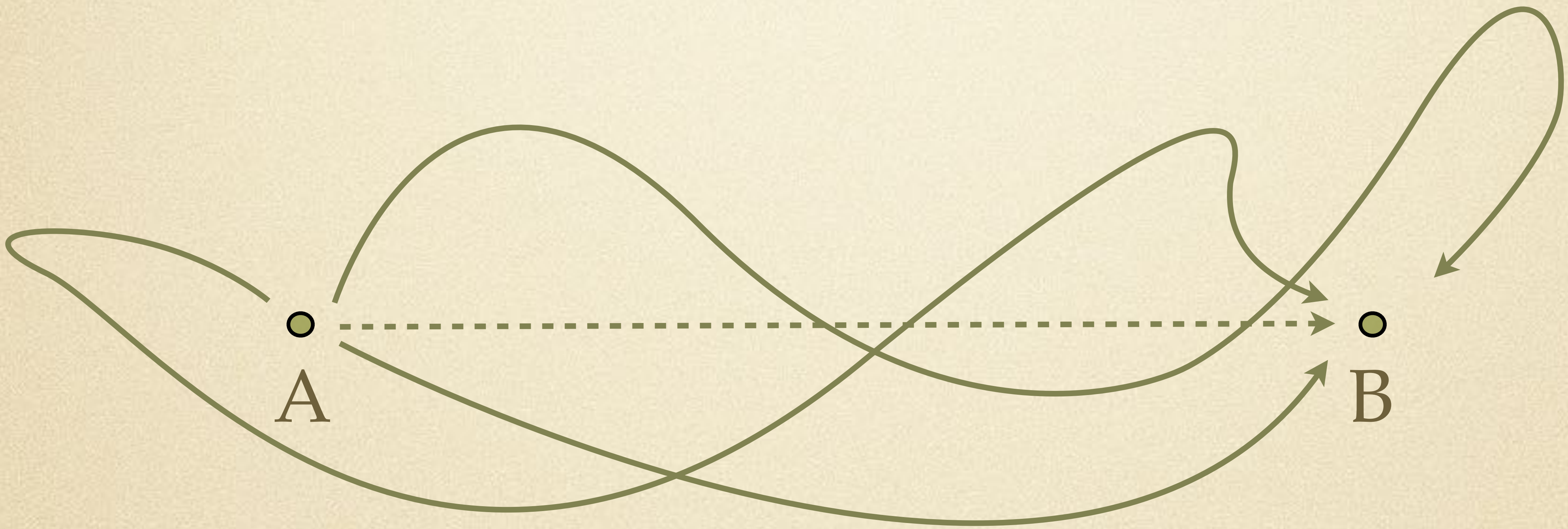

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(after Wick rotation)

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- Can be connected (Chapman, Kolmogorov)

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&

stepwise construction of PI

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No probability conservation

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Relativistic Propagator

„Solutions“ in the Literature

*1

*3

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- Hamiltonian formalism (classically equivalent) [*1]

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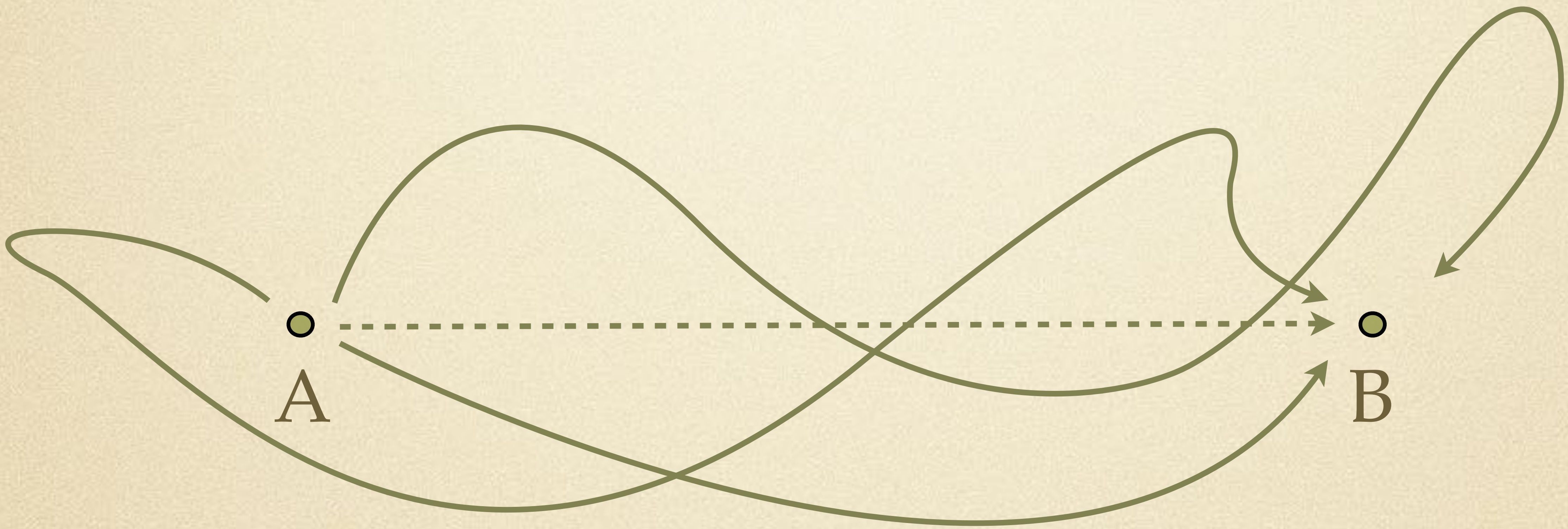
Thats what we mostly do …

Relativistic PI: our Proposal



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- Issue 1: Local reparametrizations (known)
- Issue 2: Local velocity rotations (trivial?)
- Issue 3: Measure without anomalies

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stepwise proof

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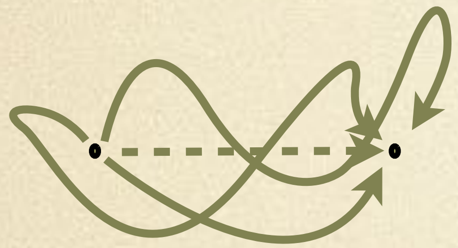
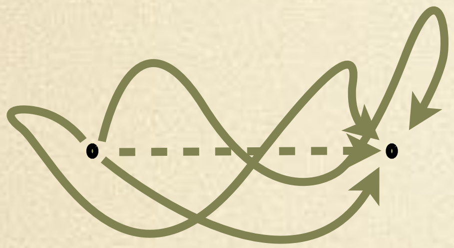


$\langle A|B \rangle$

Stepwise proof

 $=$

$$\langle A|B\rangle \quad =$$

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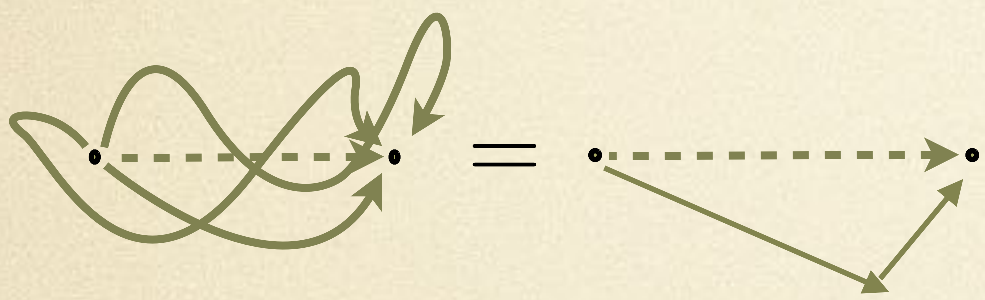
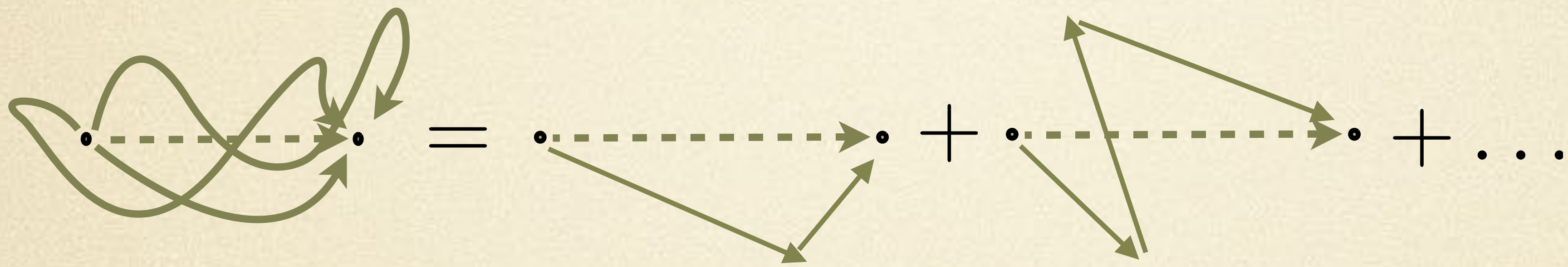


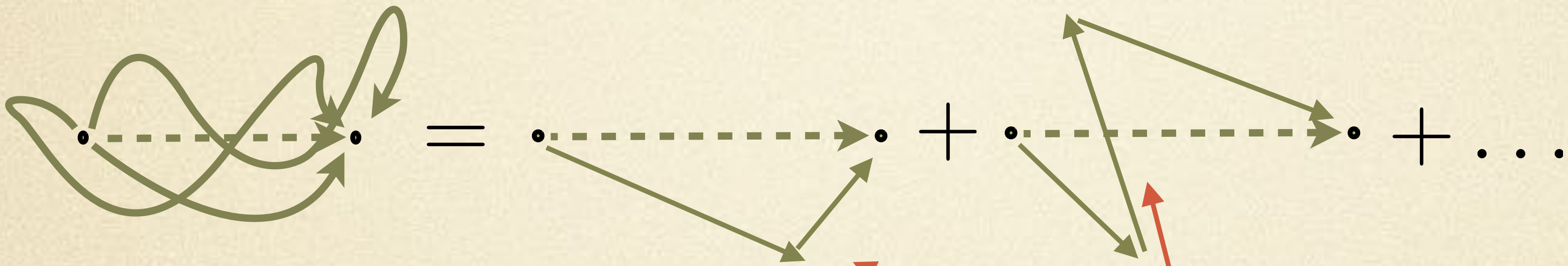
$$\langle A|B\rangle \quad = \quad \langle A|B\rangle_1$$

Stepwise proof



$$\langle A|B\rangle = \langle A|B\rangle_1 + \langle A|B\rangle_2 + \ldots$$

Stepwise proof



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Don't count again!

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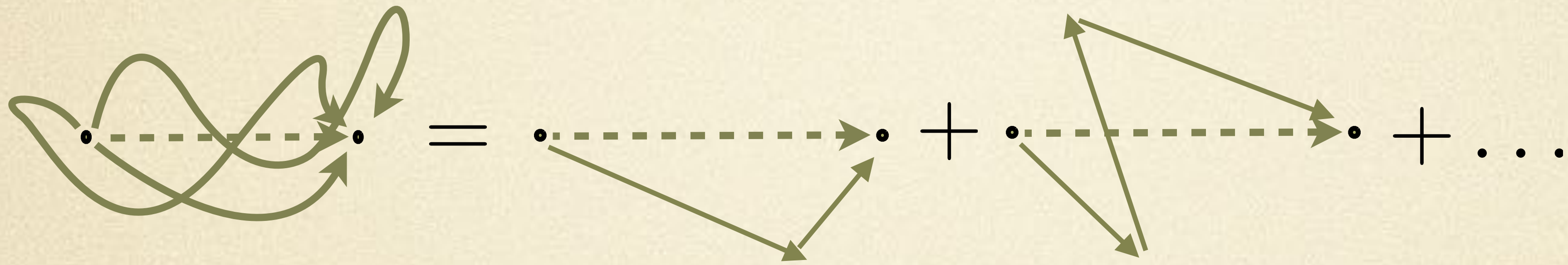

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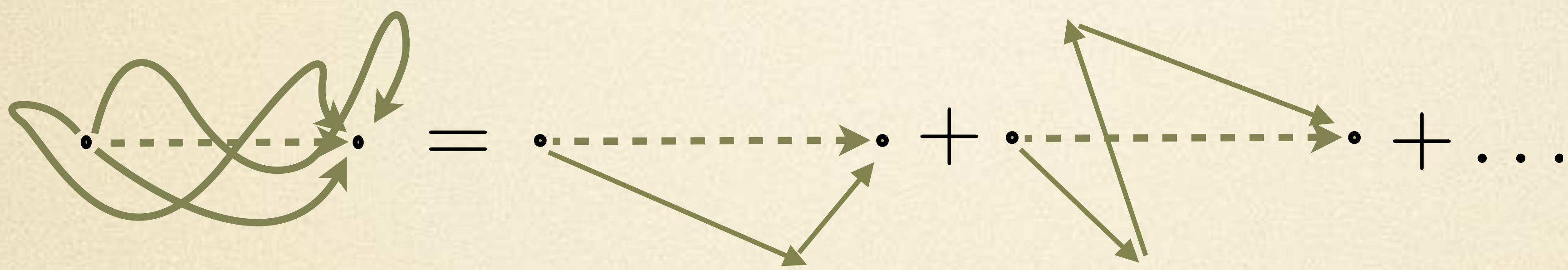

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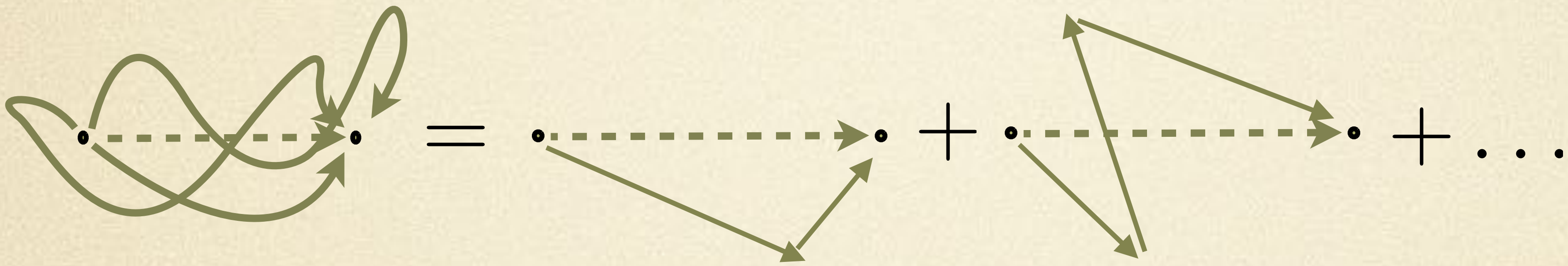

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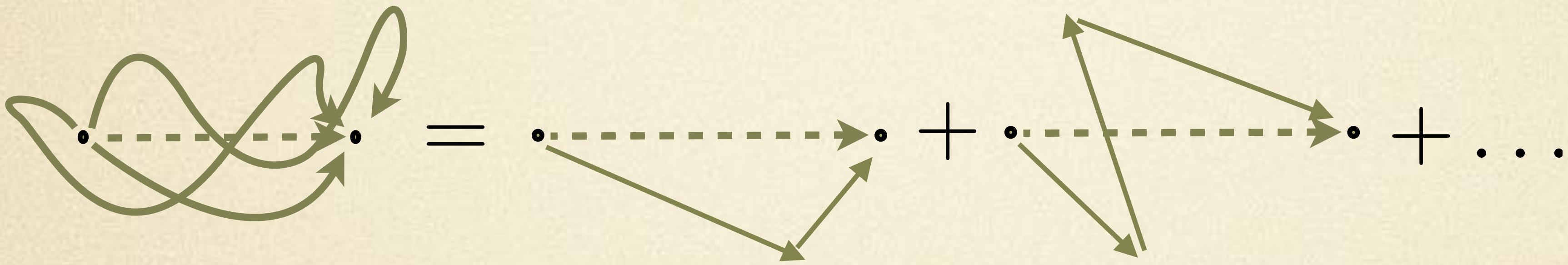

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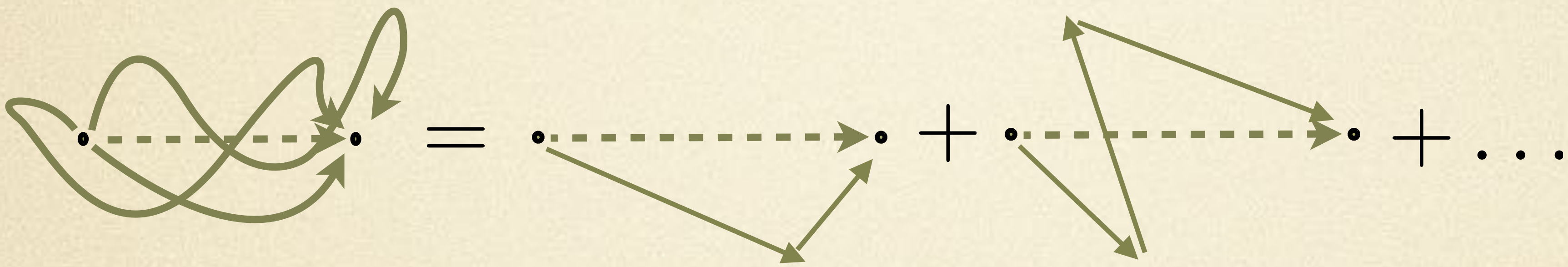

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- Calculate $\langle A|B\rangle_1$ using I2,3
- Show with I1,2 $\langle A|B\rangle_1$ contains $\langle A|B\rangle_2 \ldots$

- Issue 1: Local reparametrizations (known)

$$S = \int d\lambda \sqrt{\left(\frac{dx^\mu}{d\lambda}\right)^2}$$

Invariant under $\lambda \to \lambda'(\lambda)$

We fix proper time such that

$\tau = \tau(\lambda)$ with $\left(\frac{dx^\mu}{d\tau}\right)^2 = 1$

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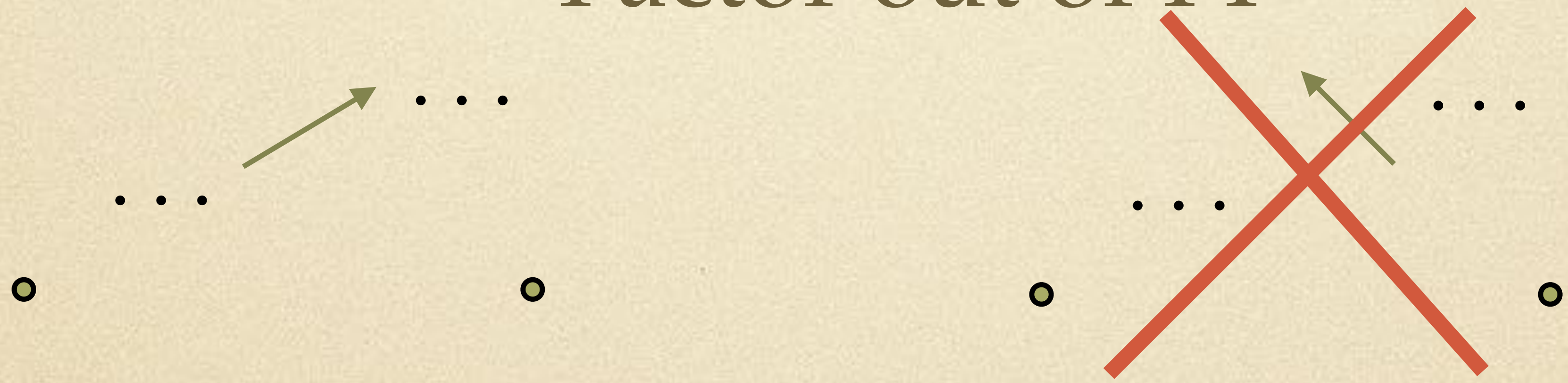

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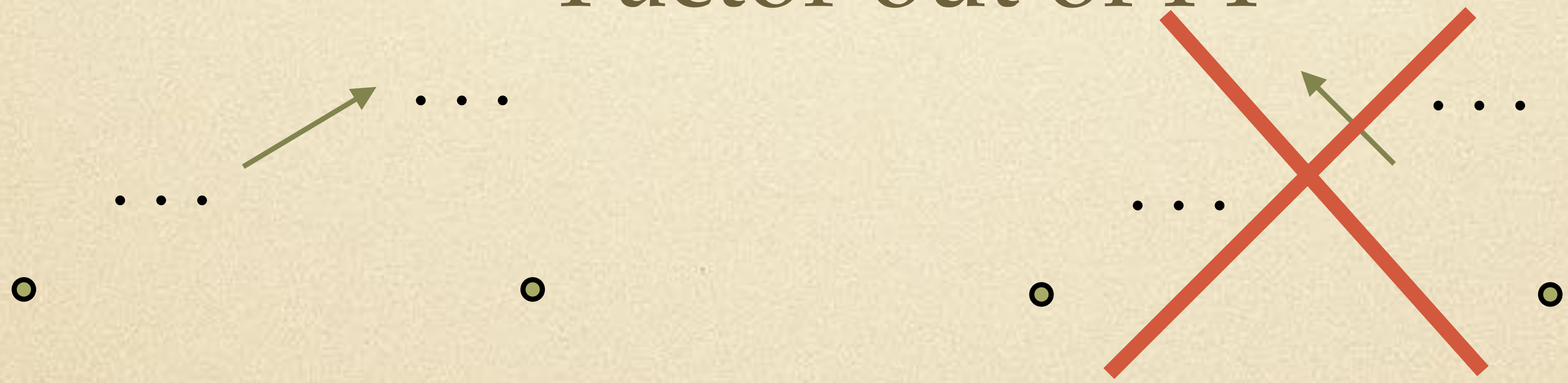


if $S = S'$ and $\mathcal{L} = \mathcal{L}'$!

- Issue 3: Measure without anomalies

When performing transformation

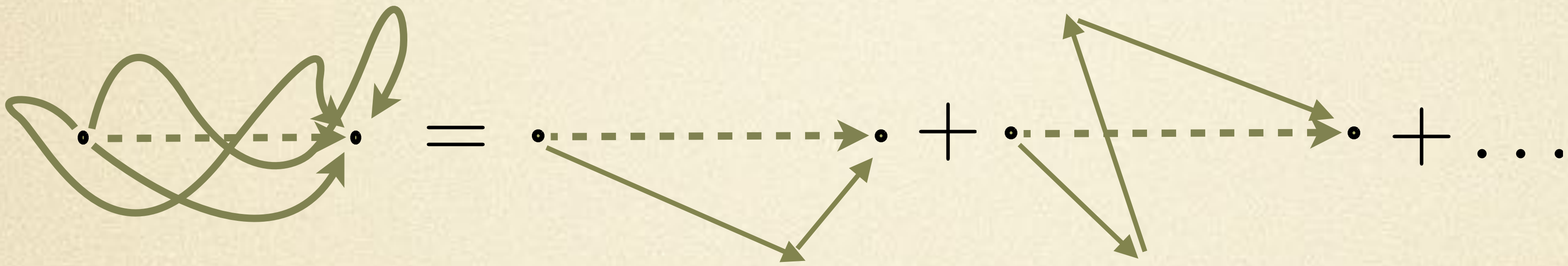
$$v^\mu \to v'^\mu = \Lambda^\mu{}_\nu(\lambda) v^\nu$$

define right measure invariant under this symmetry:

$$\mathcal{D}x \to \mathcal{D}x' = \mathcal{D}x$$

Geometric example for two step propagator

Stepwise proof



$$\langle A|B\rangle = \langle A|B\rangle_1 + \langle A|B\rangle_2 + \dots$$

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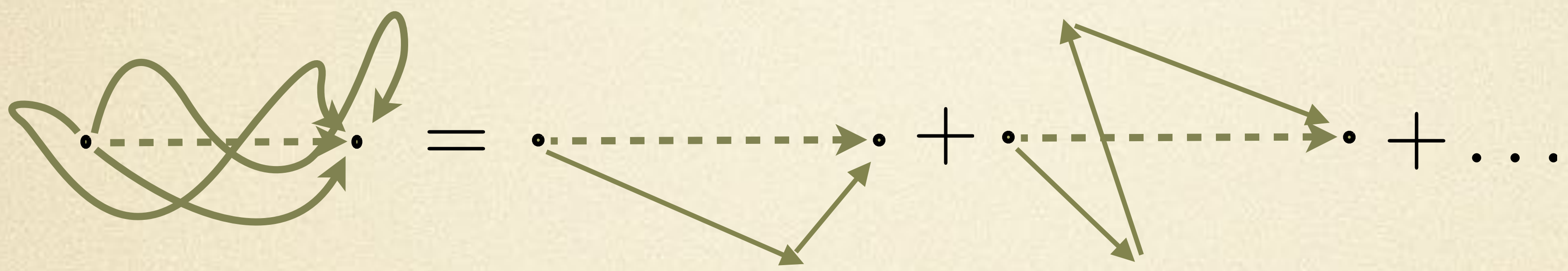

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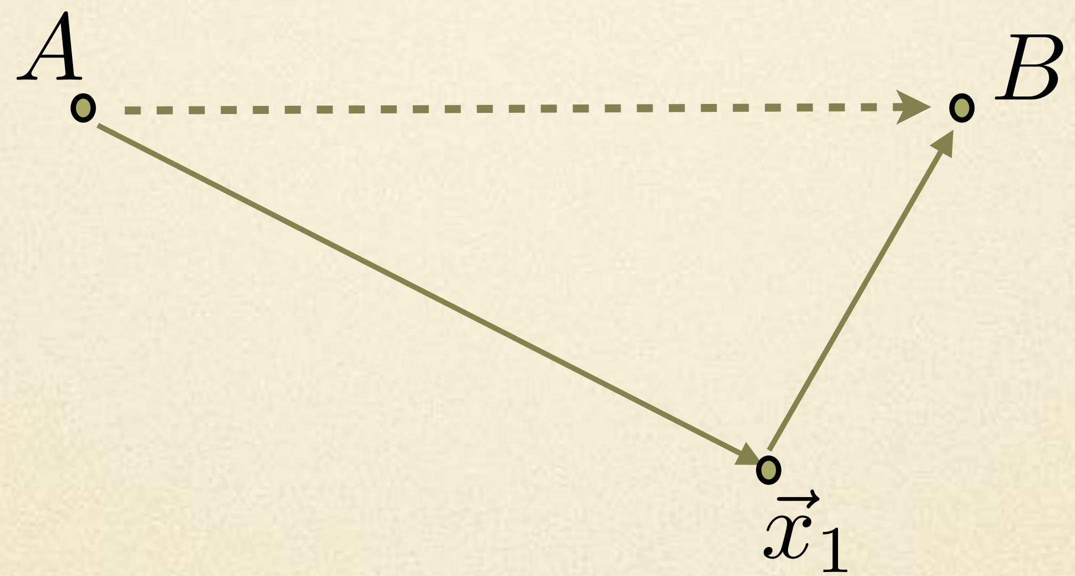

$$\langle A|B\rangle_1 = N \int d^2x_1 \, \Delta_1 \, e^{-(S_{A,\vec{x}_1} + S_{\vec{x}_1,B})}$$

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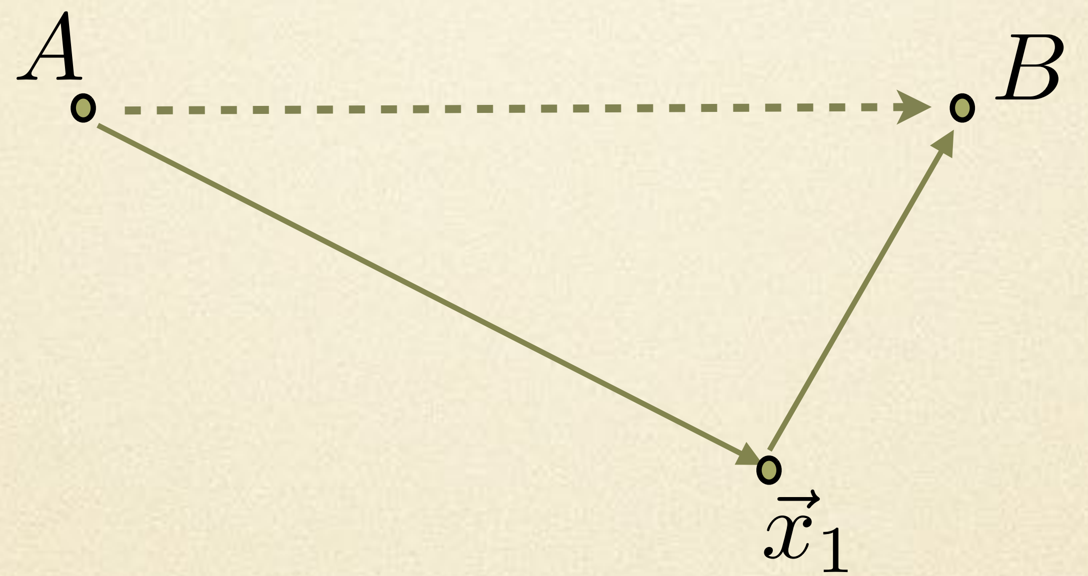

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Change of integration coordinates

$$(x_1, y_1) \to (S, \alpha)$$

$$x_1 = \frac{S}{2M}\cos(\alpha) \qquad y_1 = \frac{x_f}{2}\sqrt{\left(\frac{S}{x_f M}\right)^2 - 1} \cdot \sin(\alpha)$$

$$x_f = |\vec{B} - \vec{A}|$$

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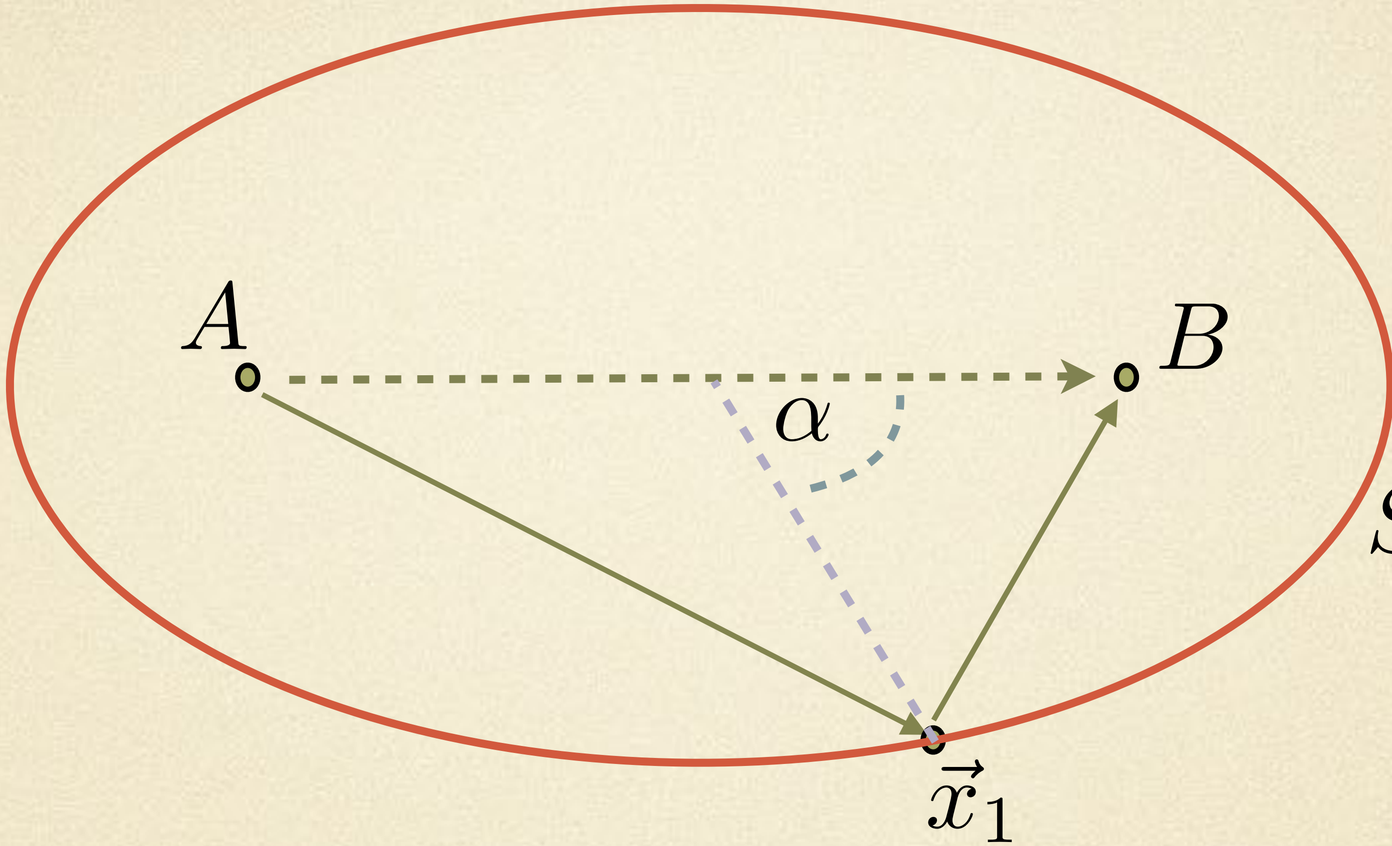

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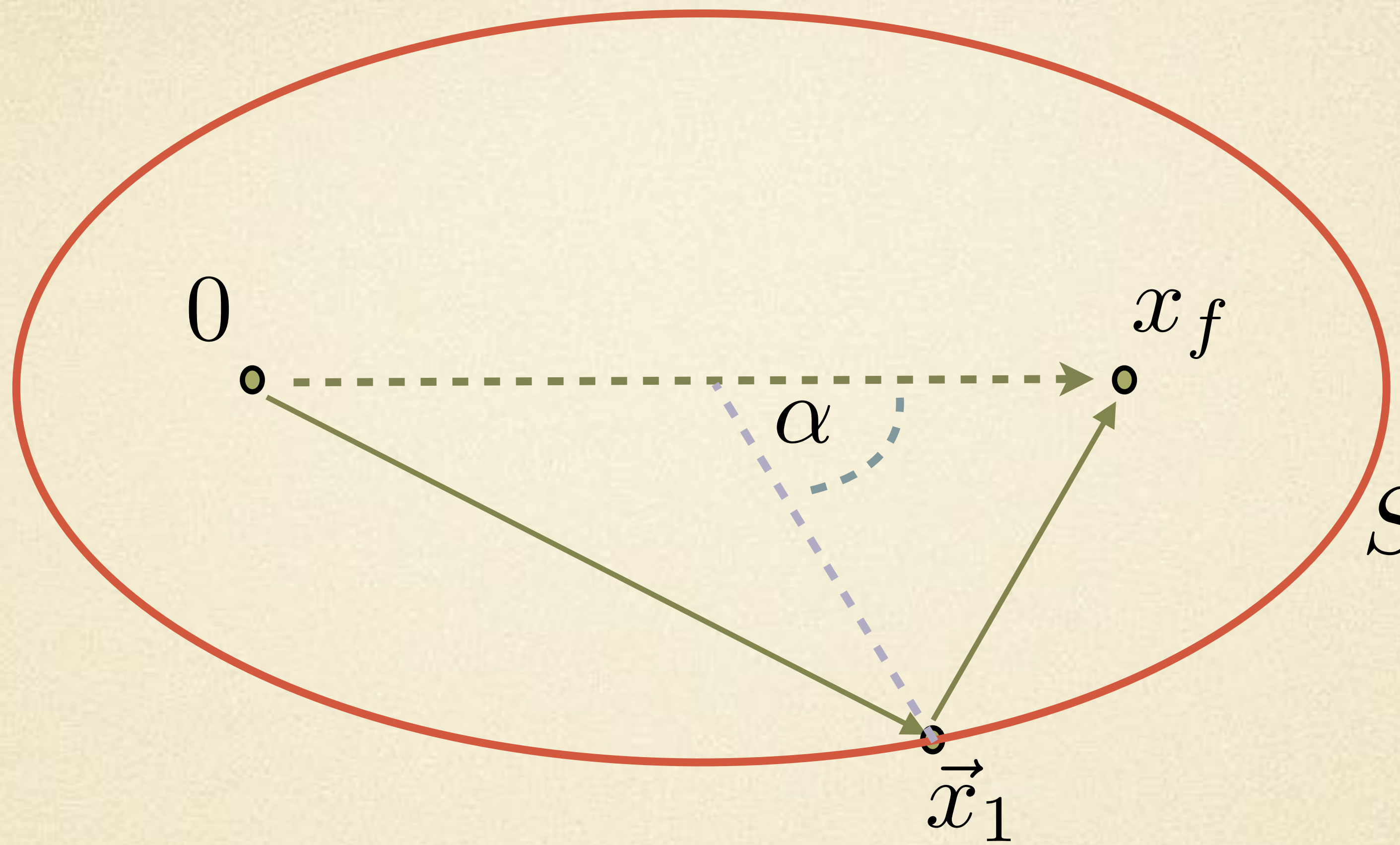

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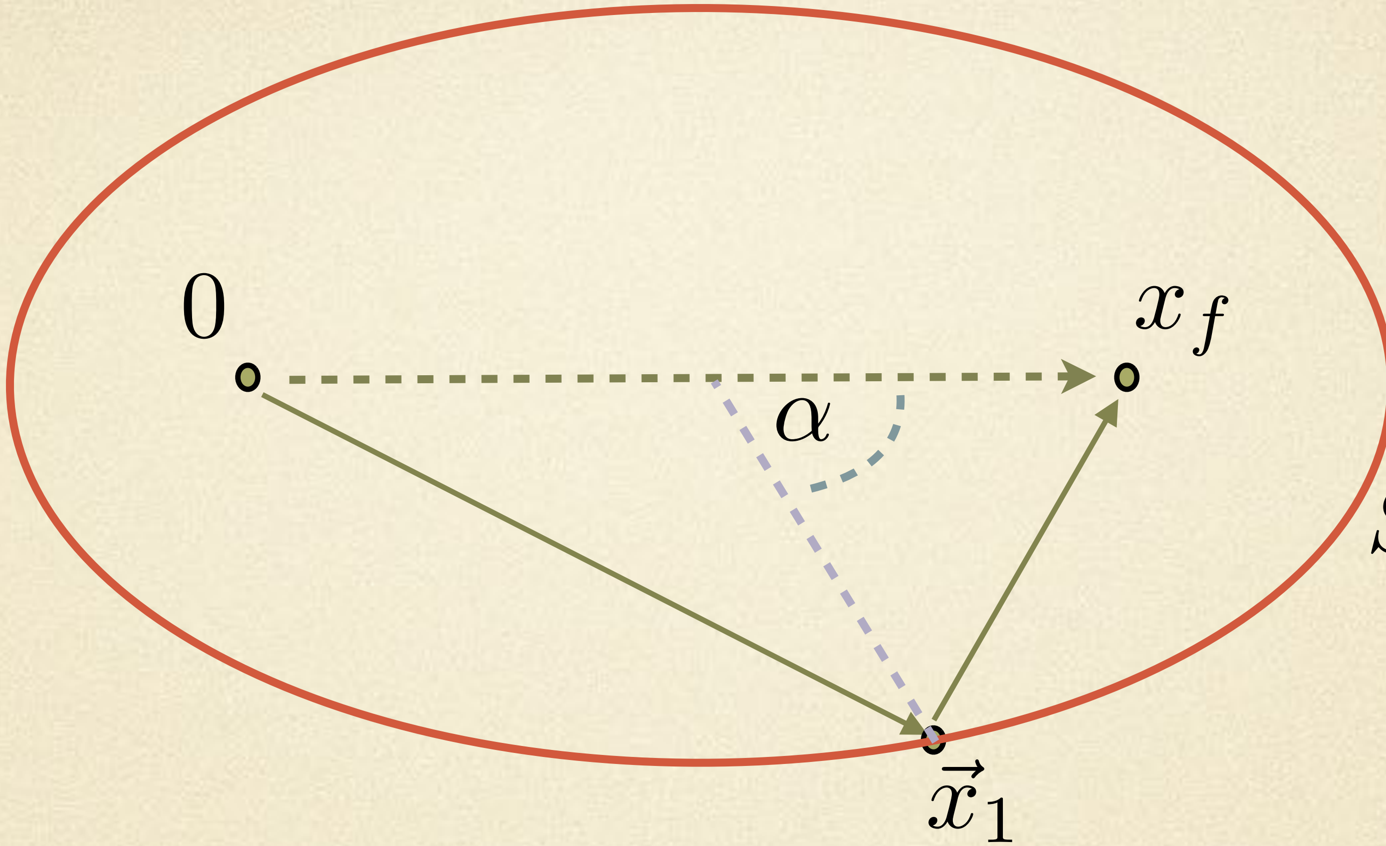

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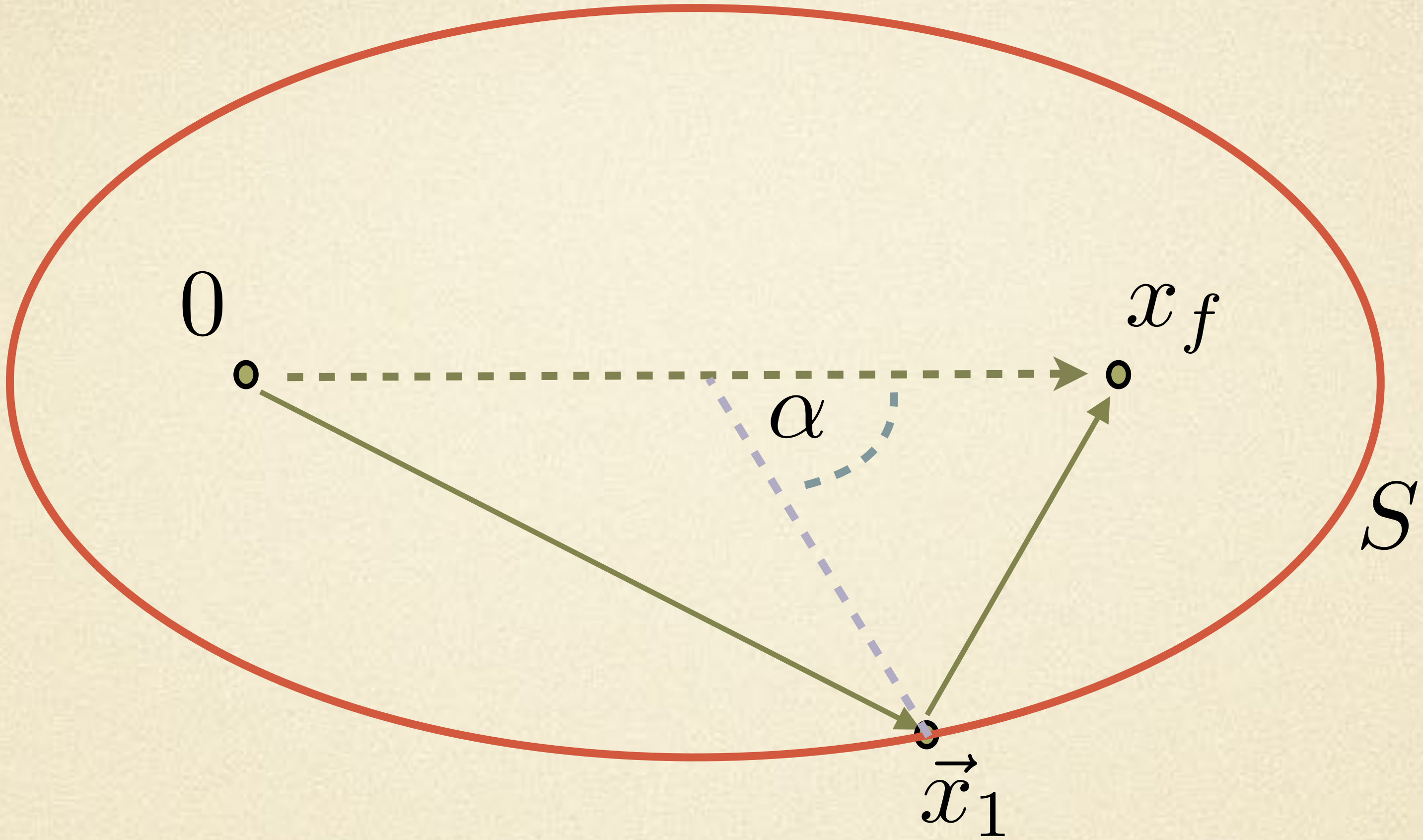

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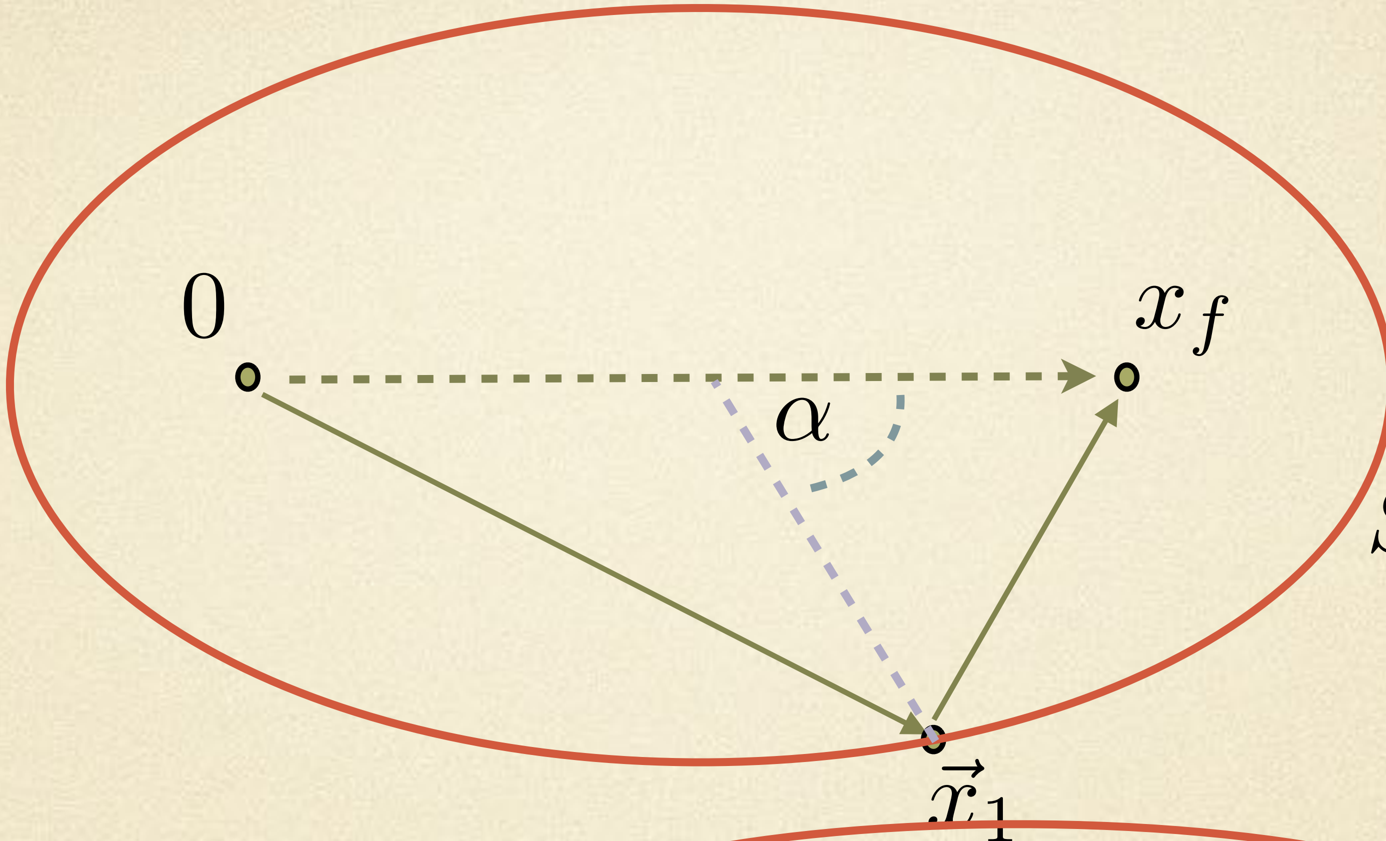

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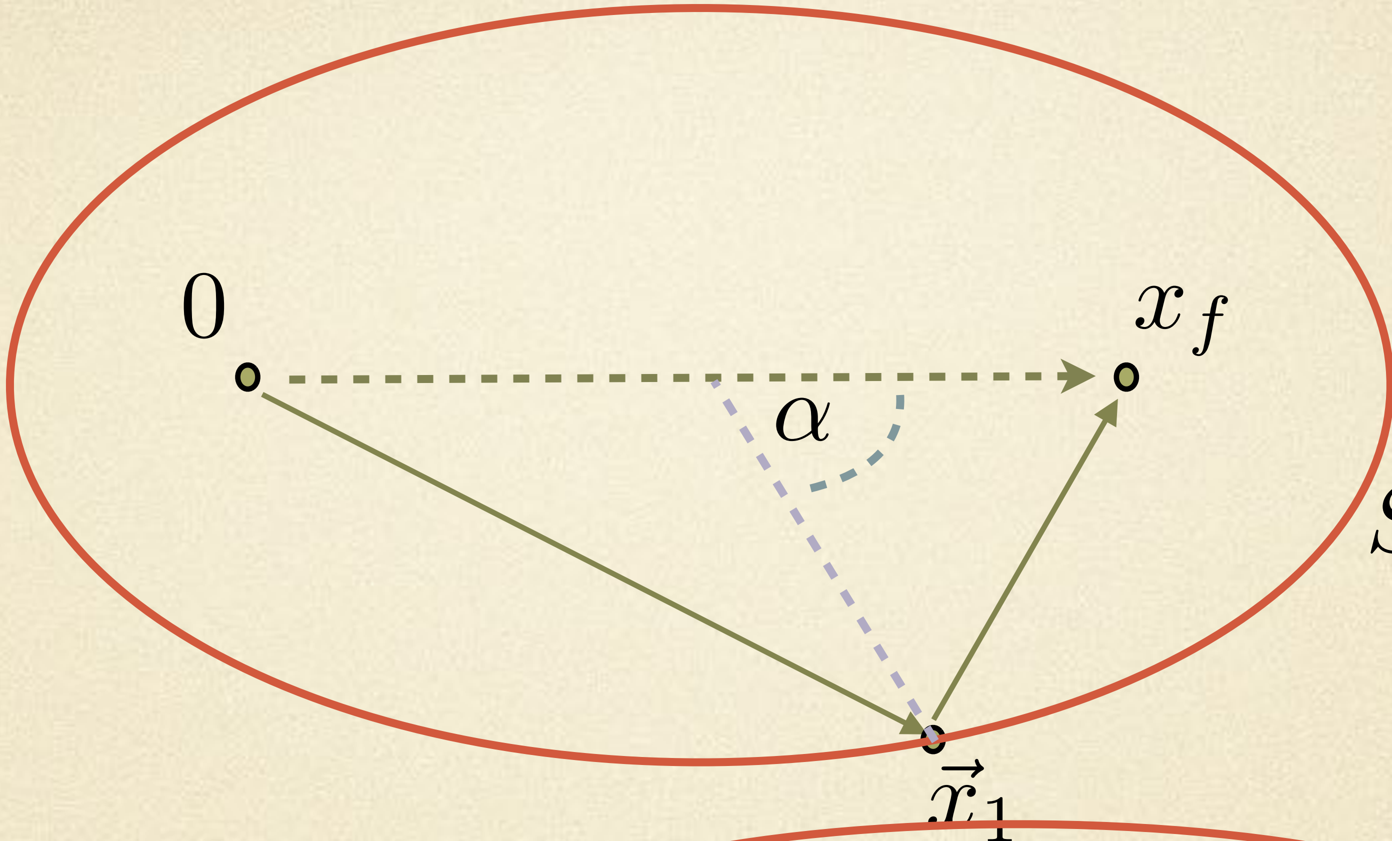

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$\Delta_1^{-1} \equiv |\vec{x}_1 - 0| \cdot |\vec{x}_f - \vec{x}_1|$ cancels anomaly

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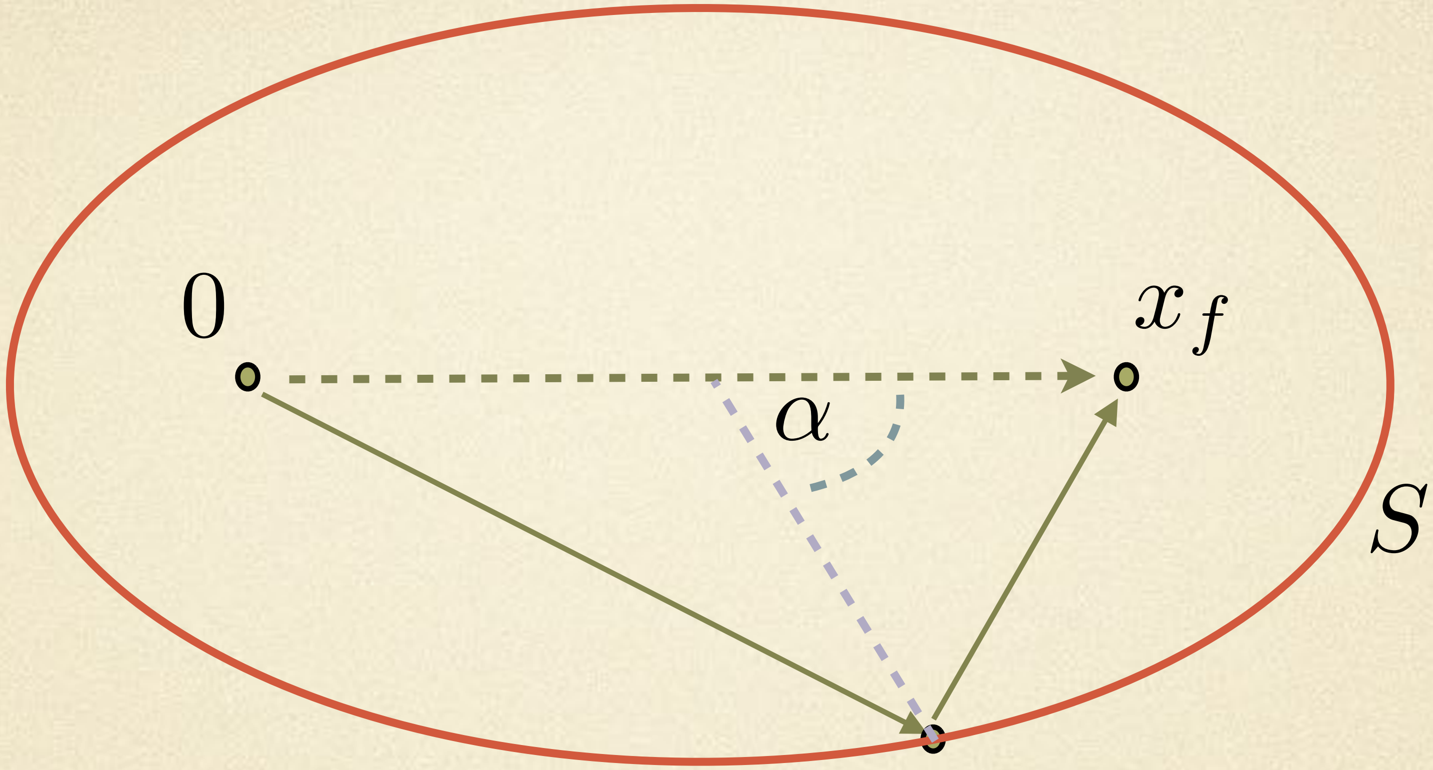

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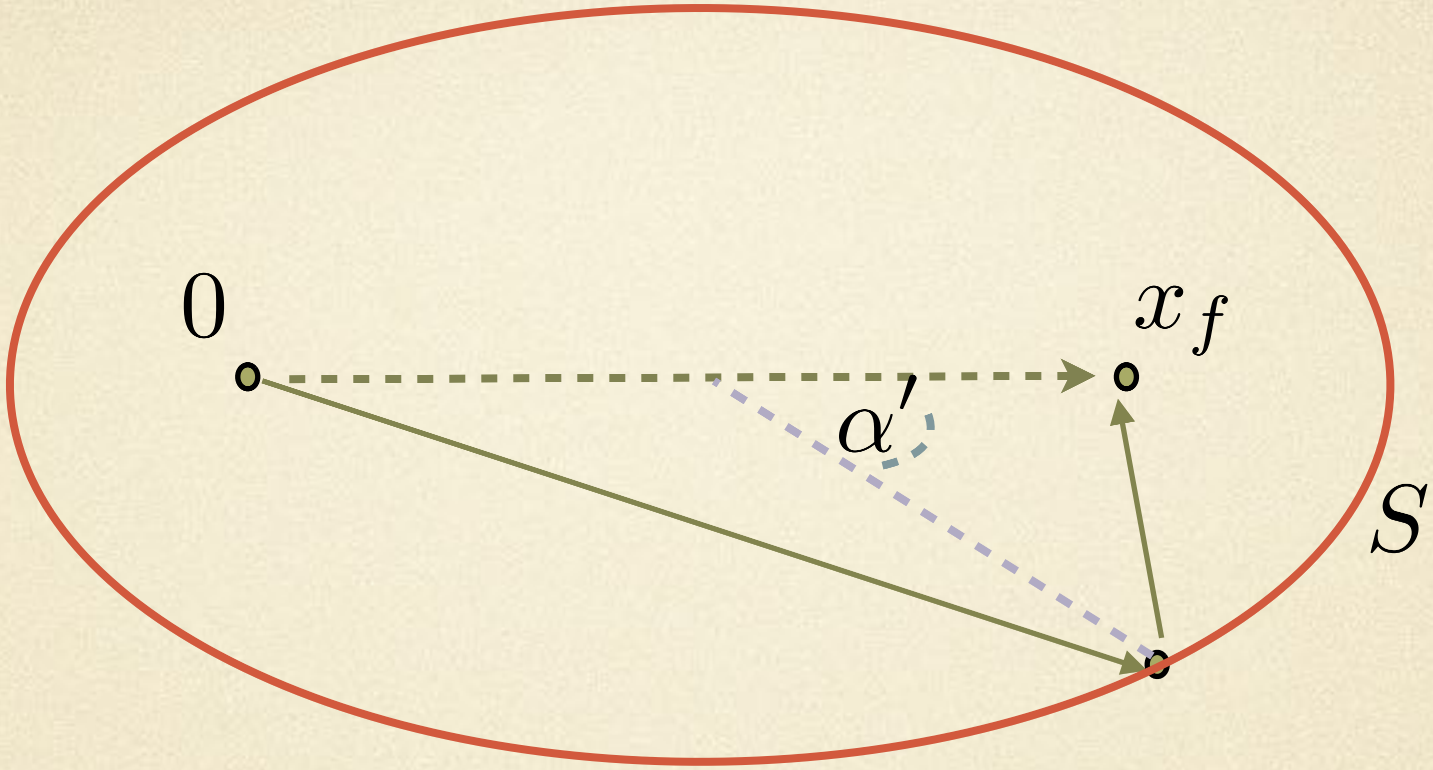

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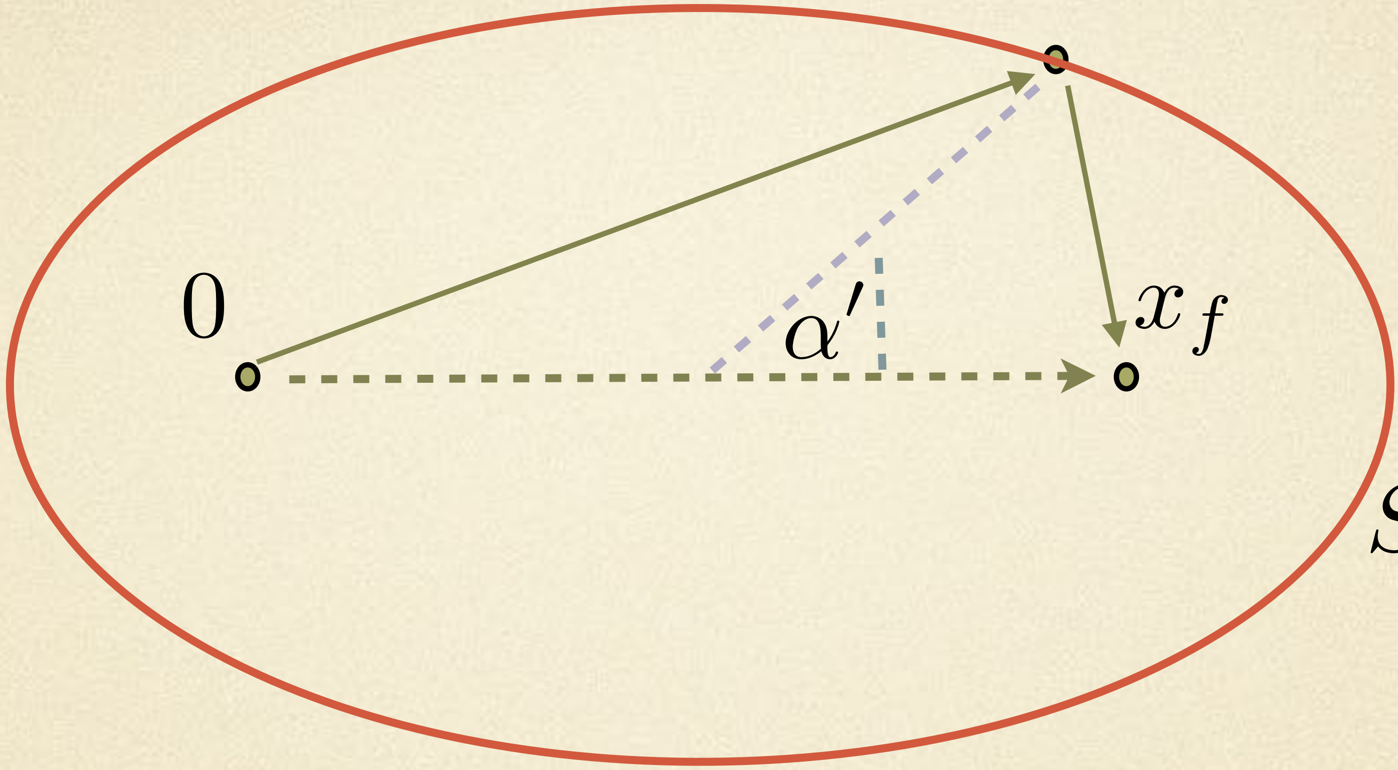

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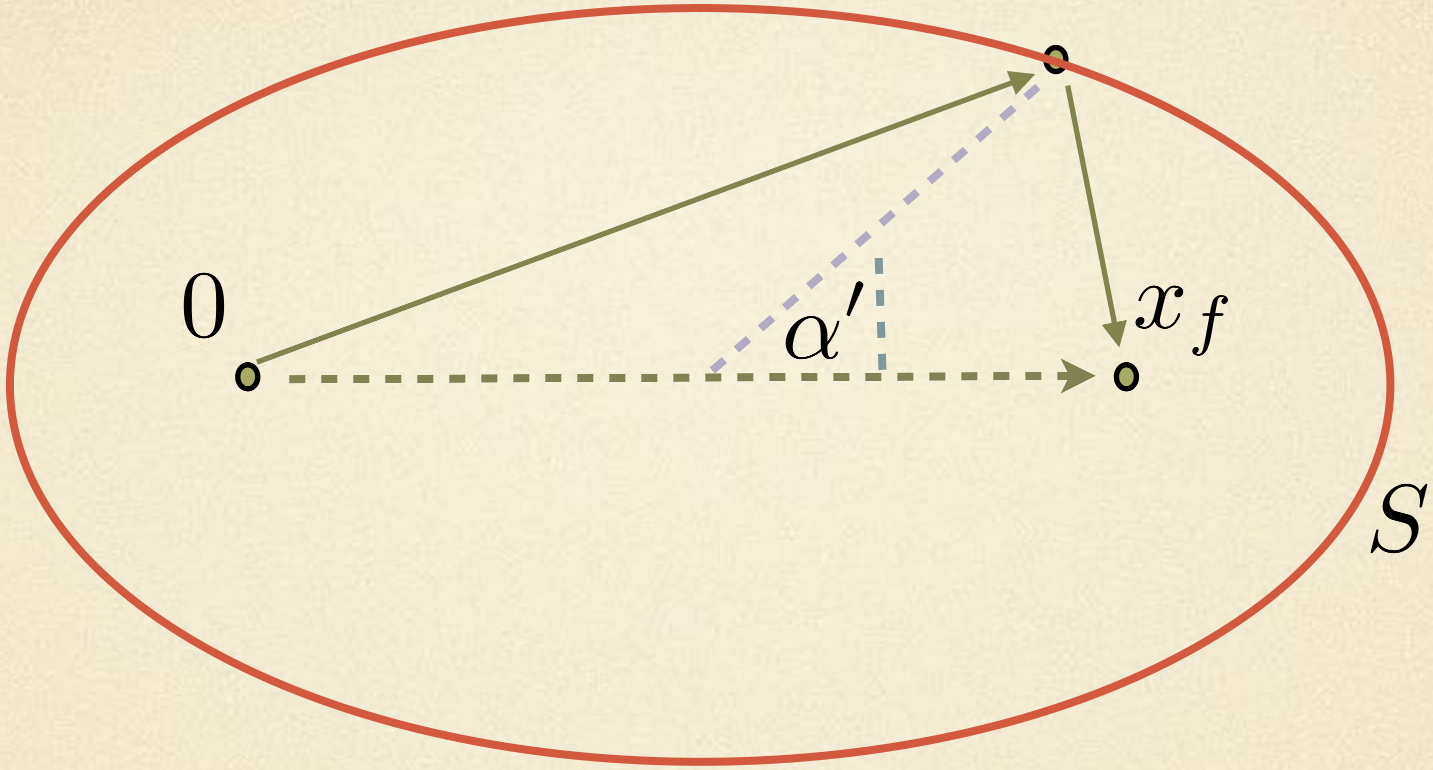

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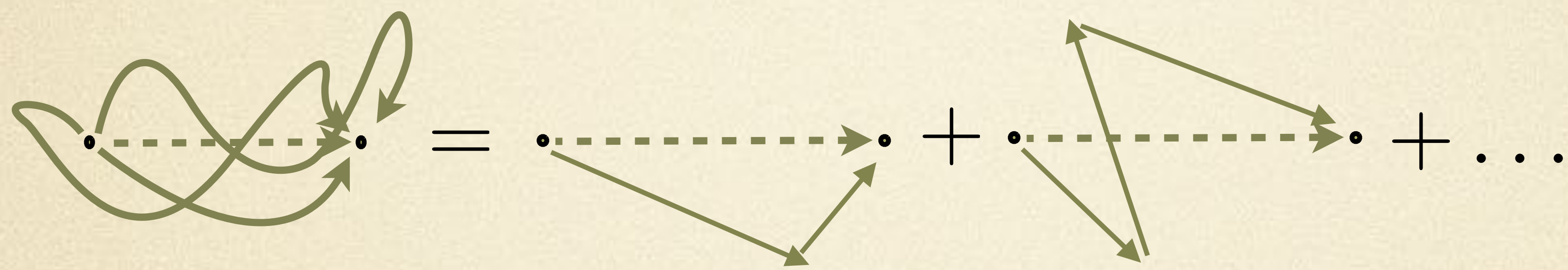

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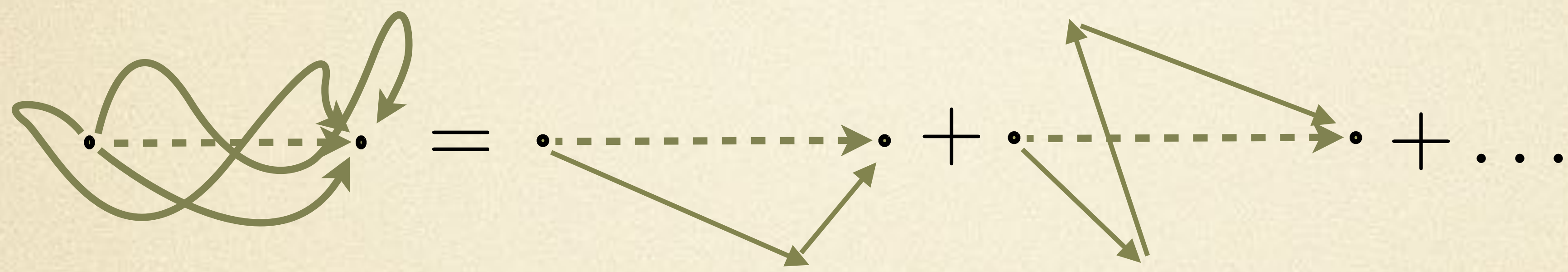

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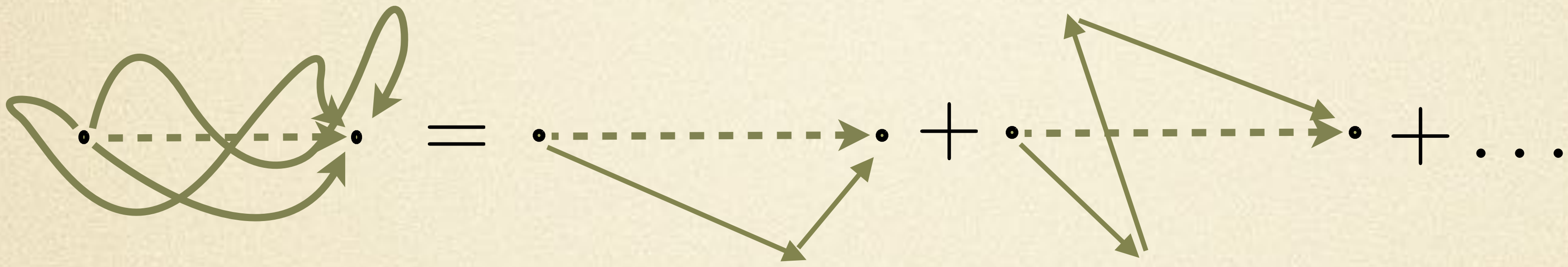

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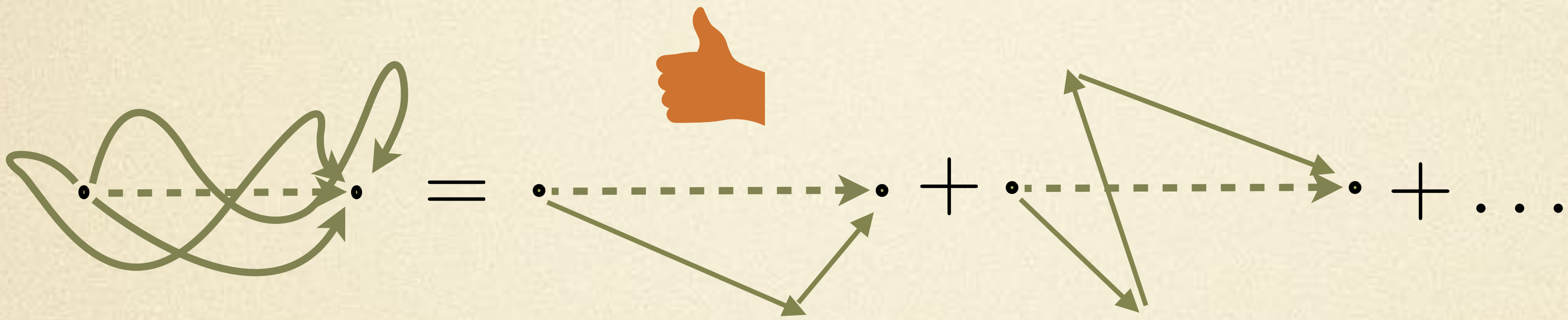

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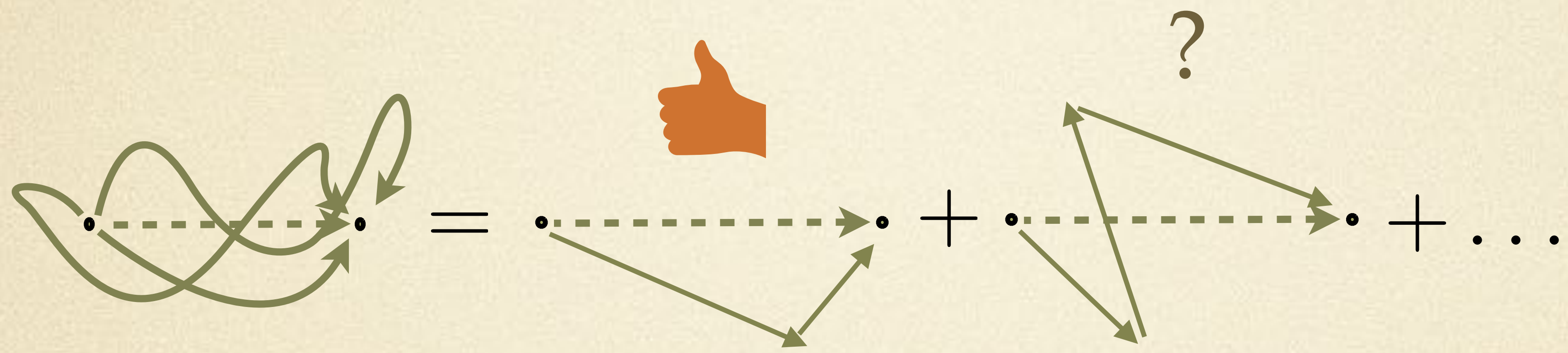

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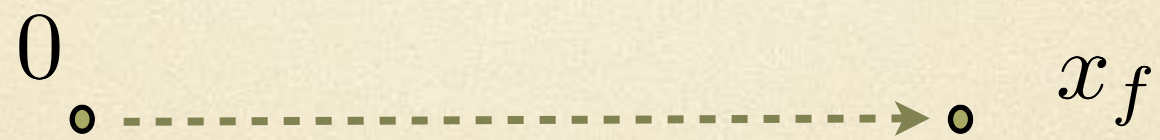

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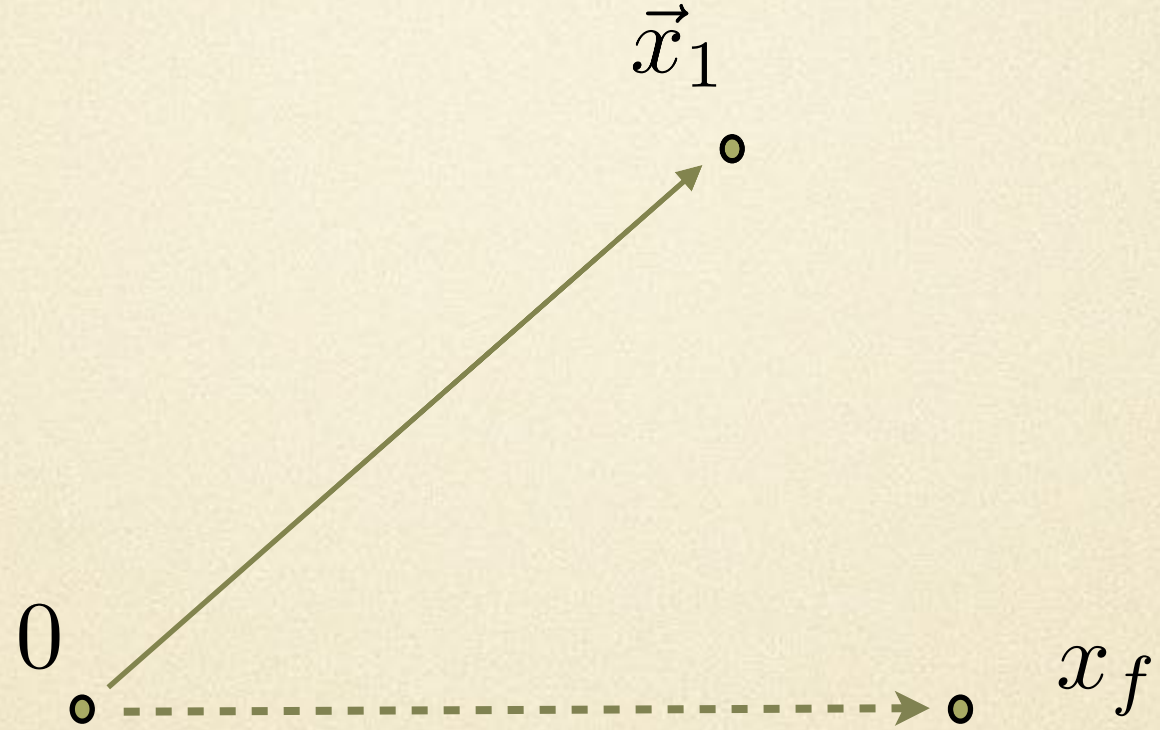

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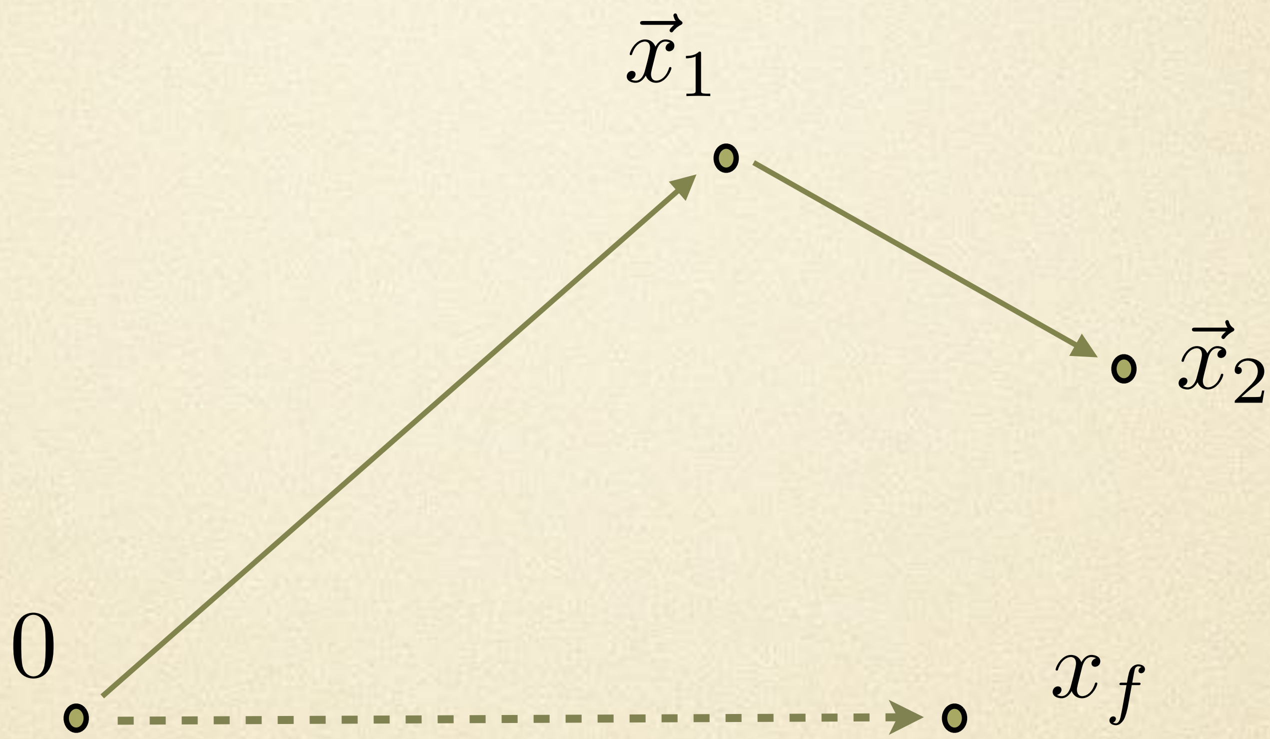

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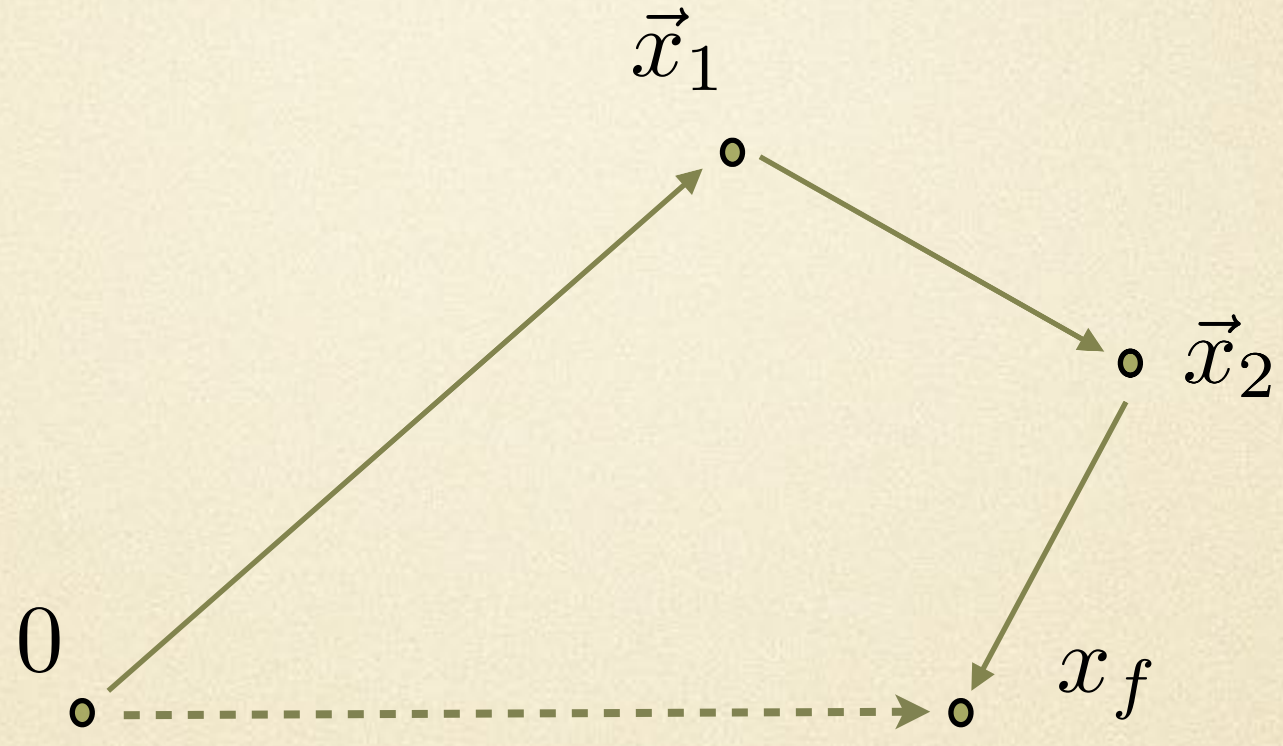

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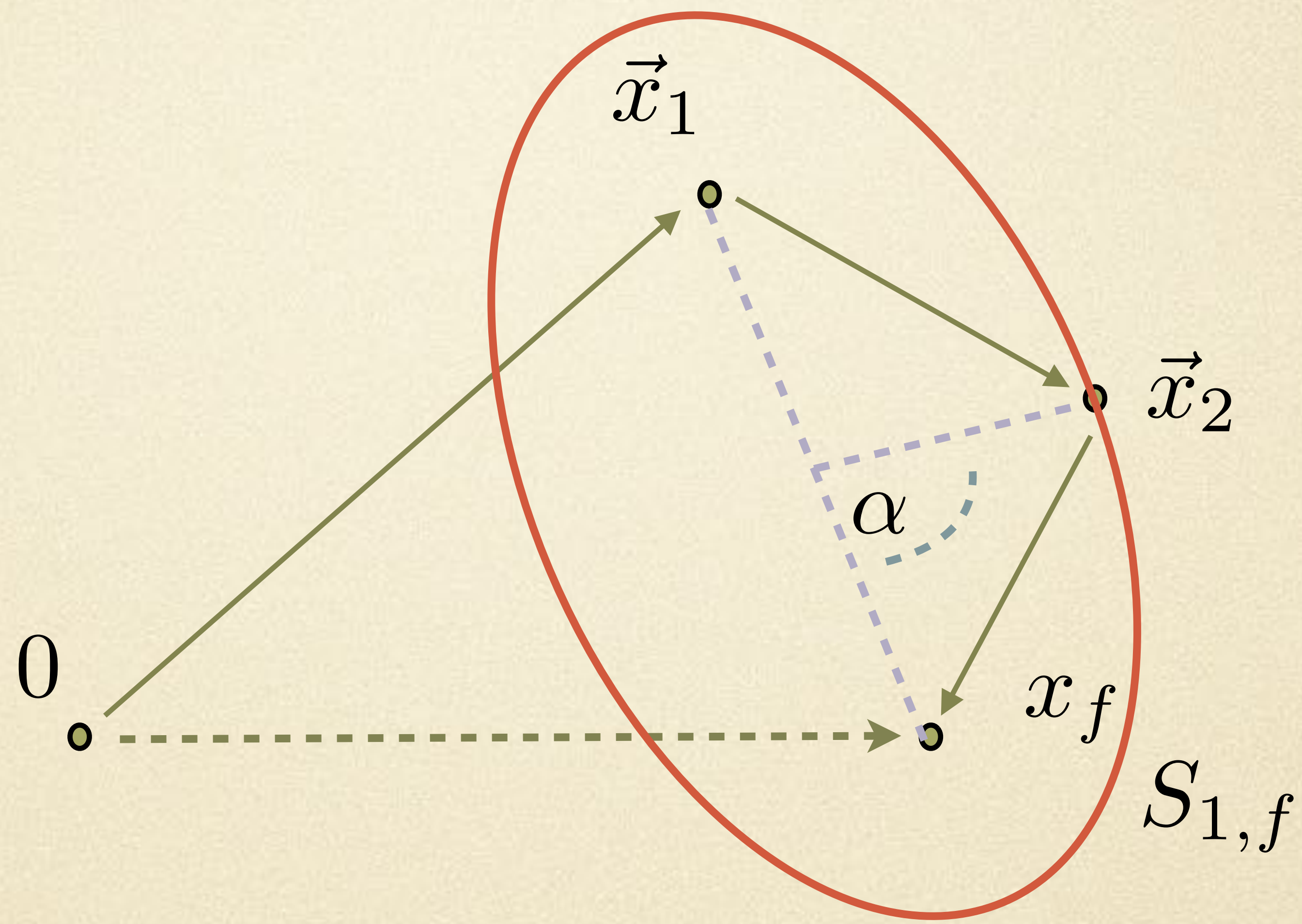

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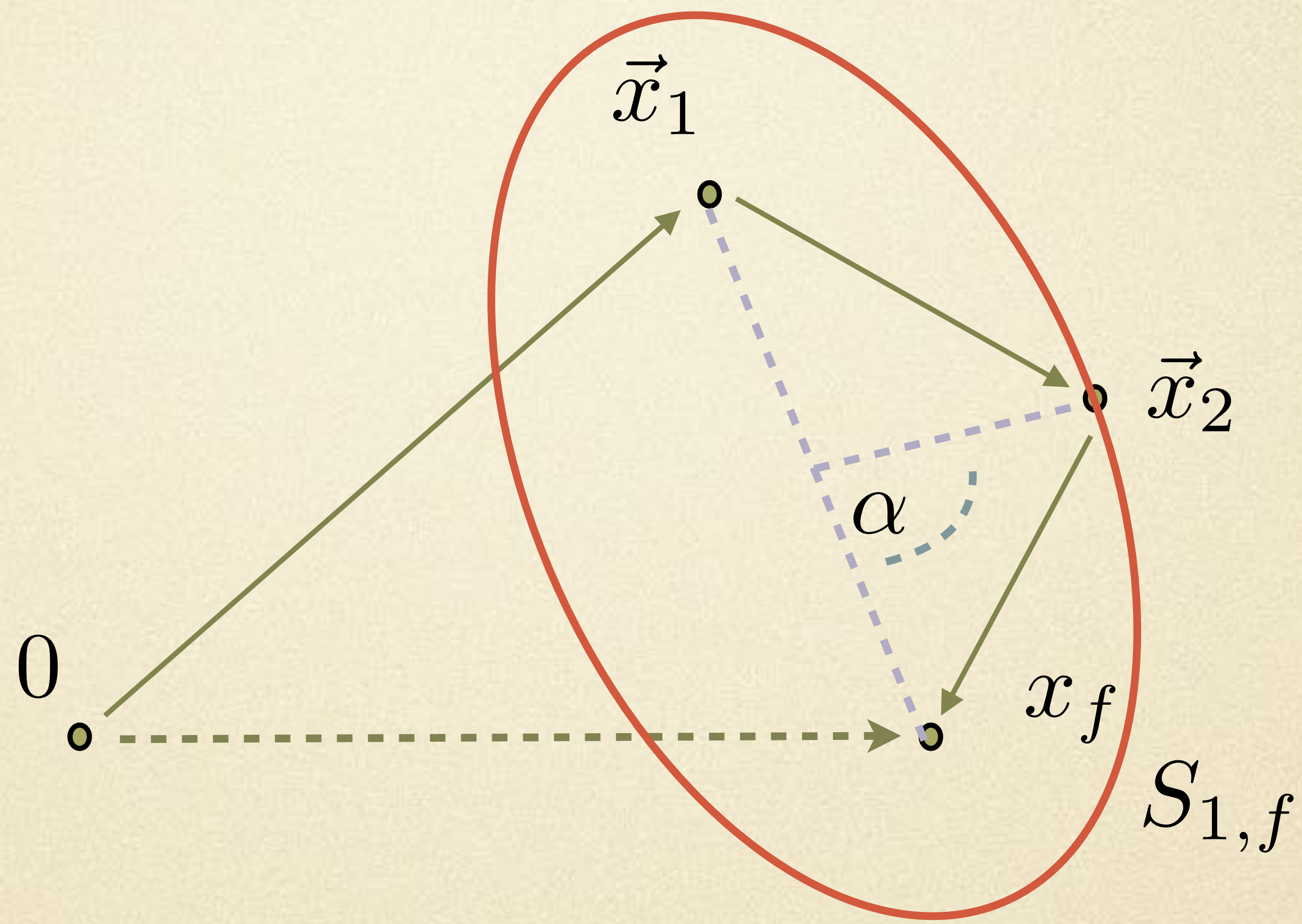

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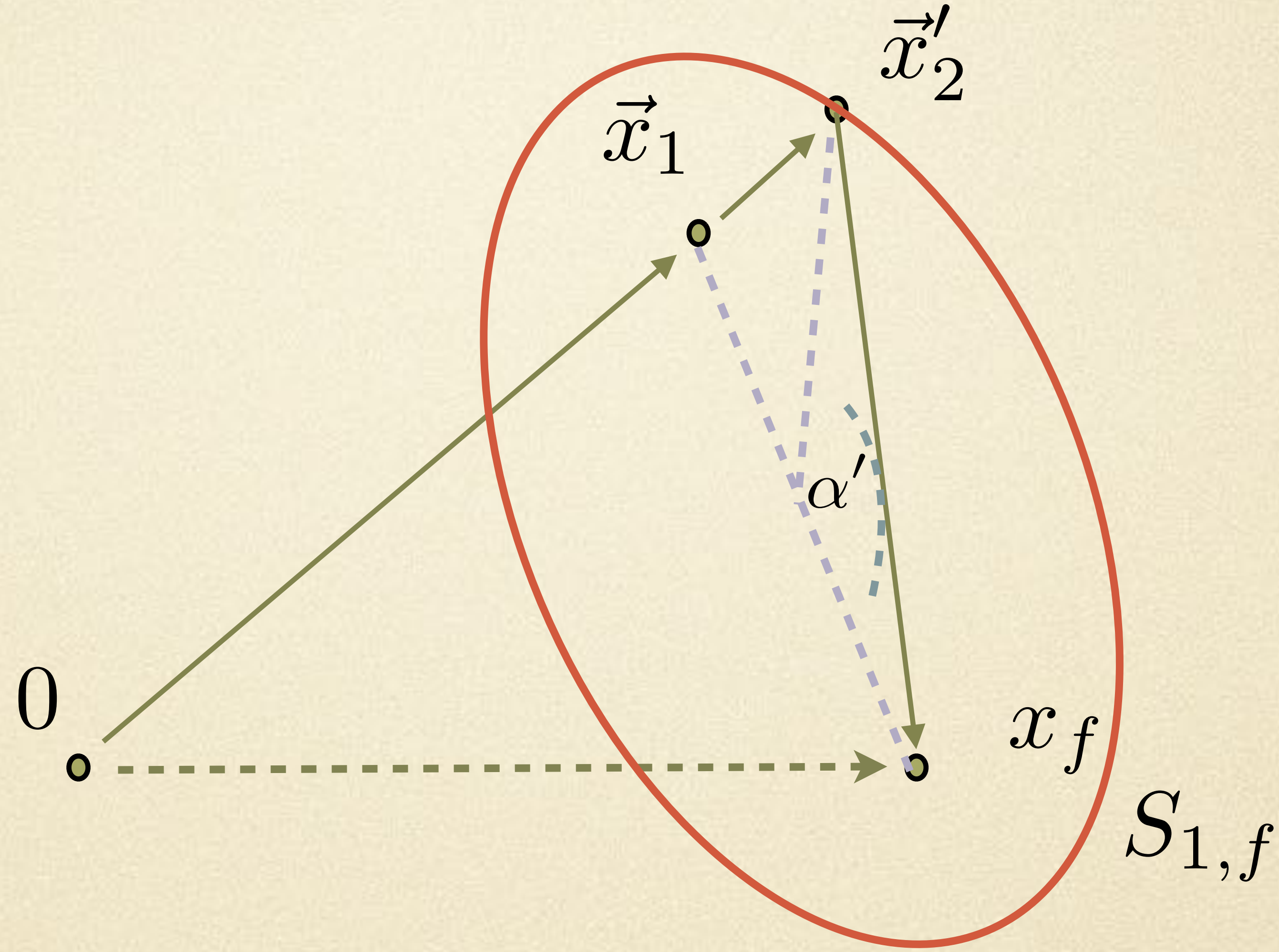

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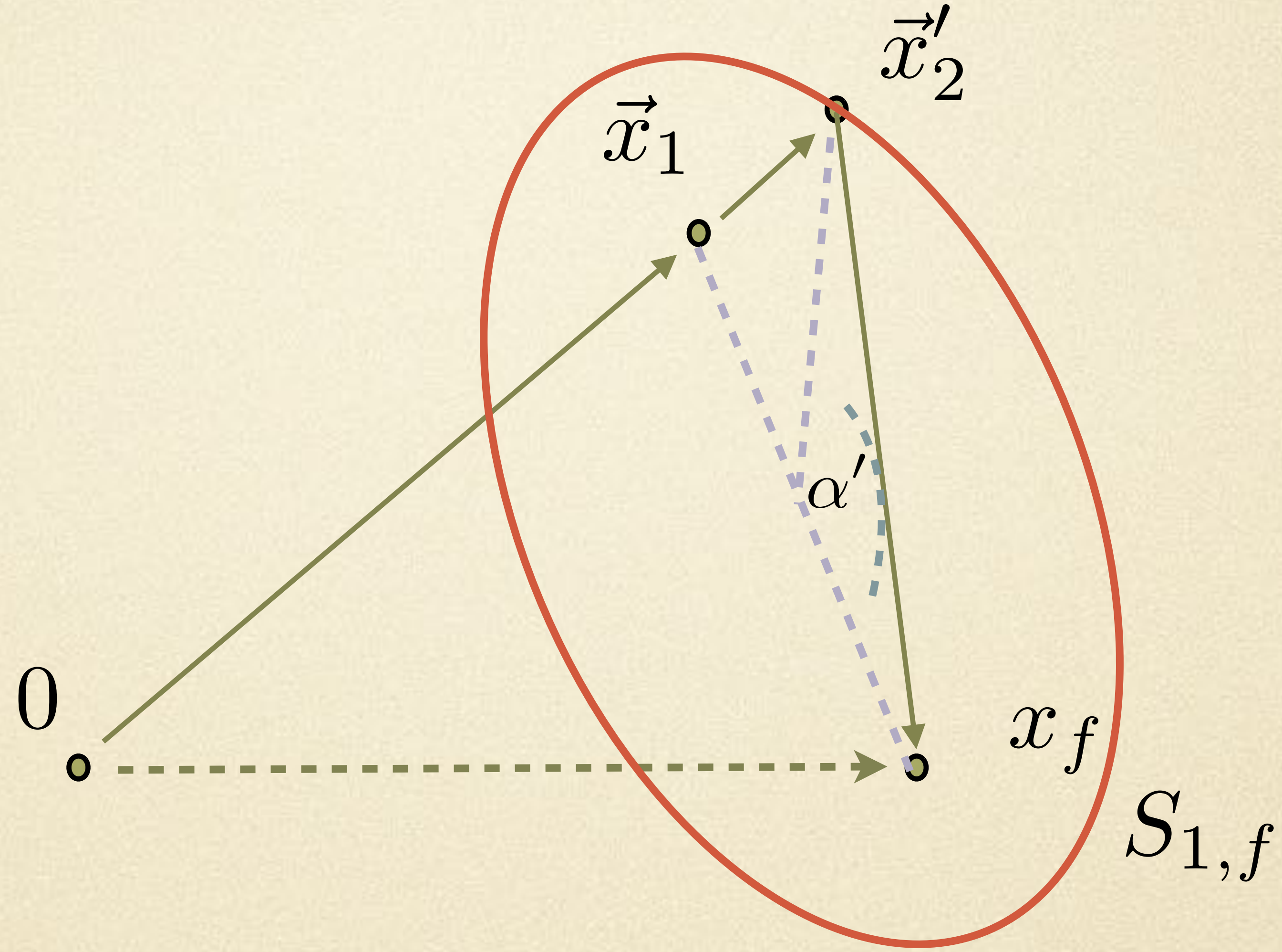

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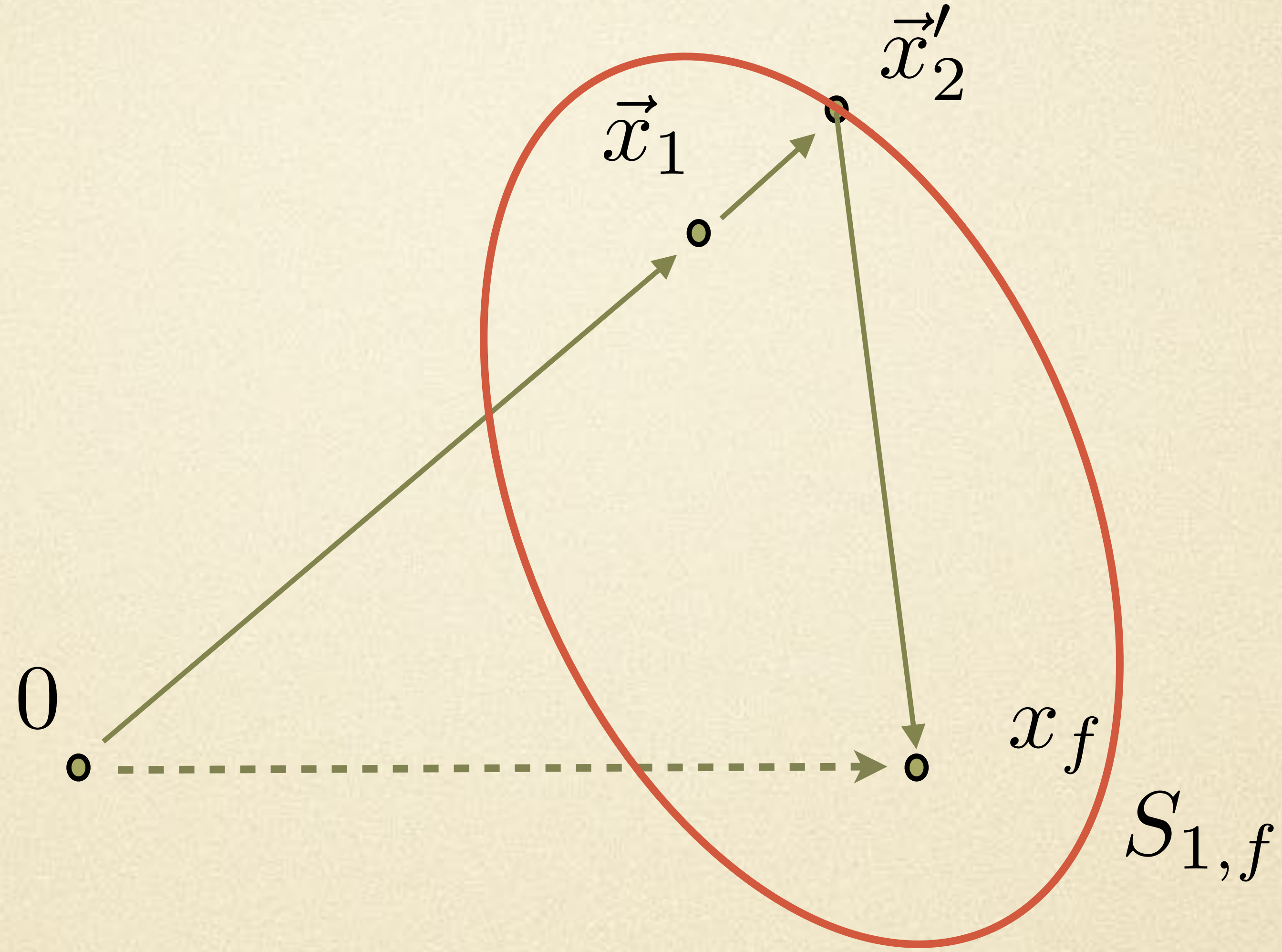

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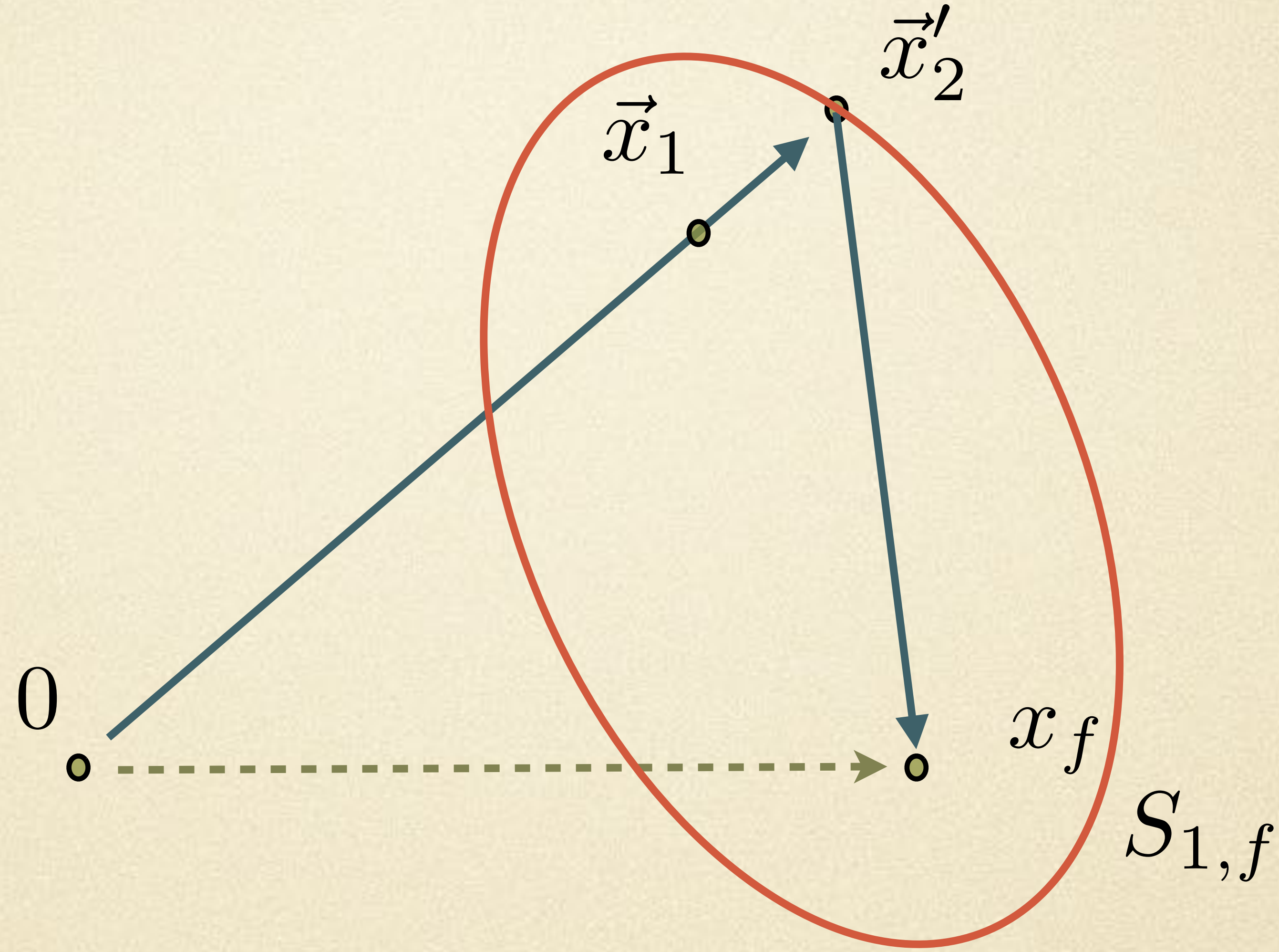

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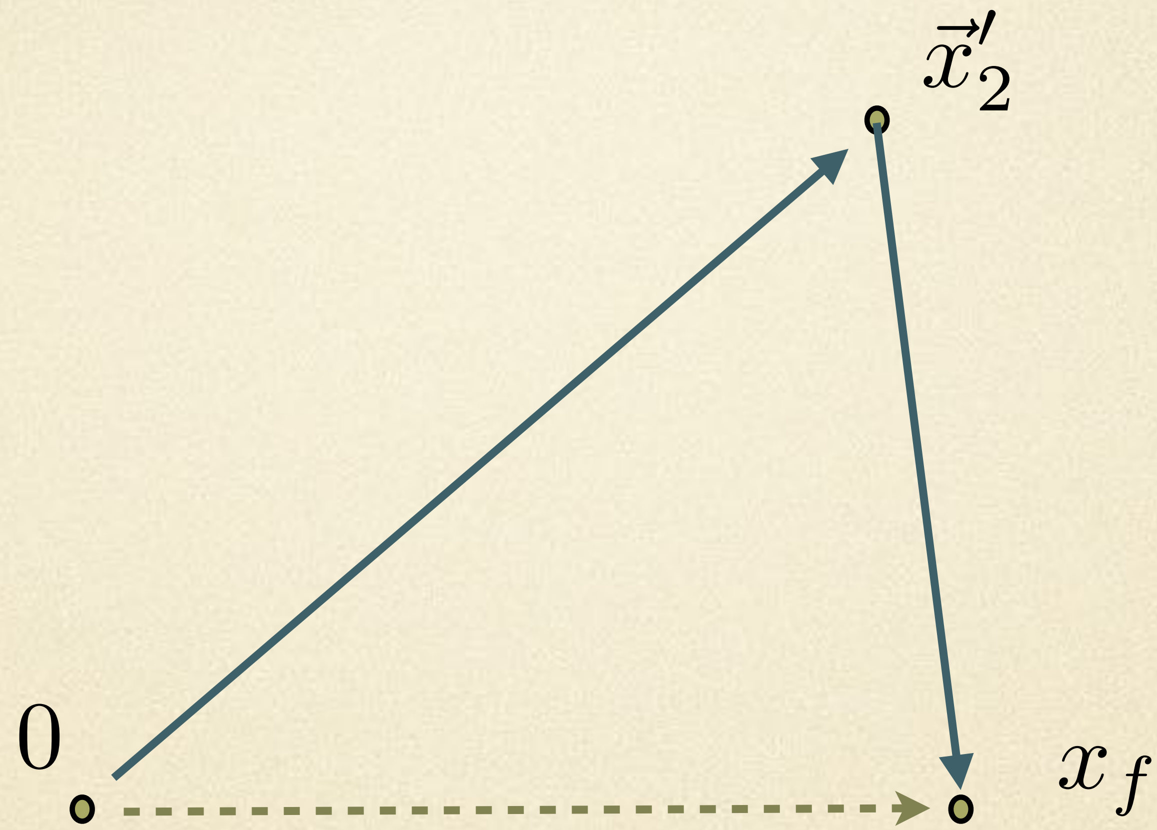

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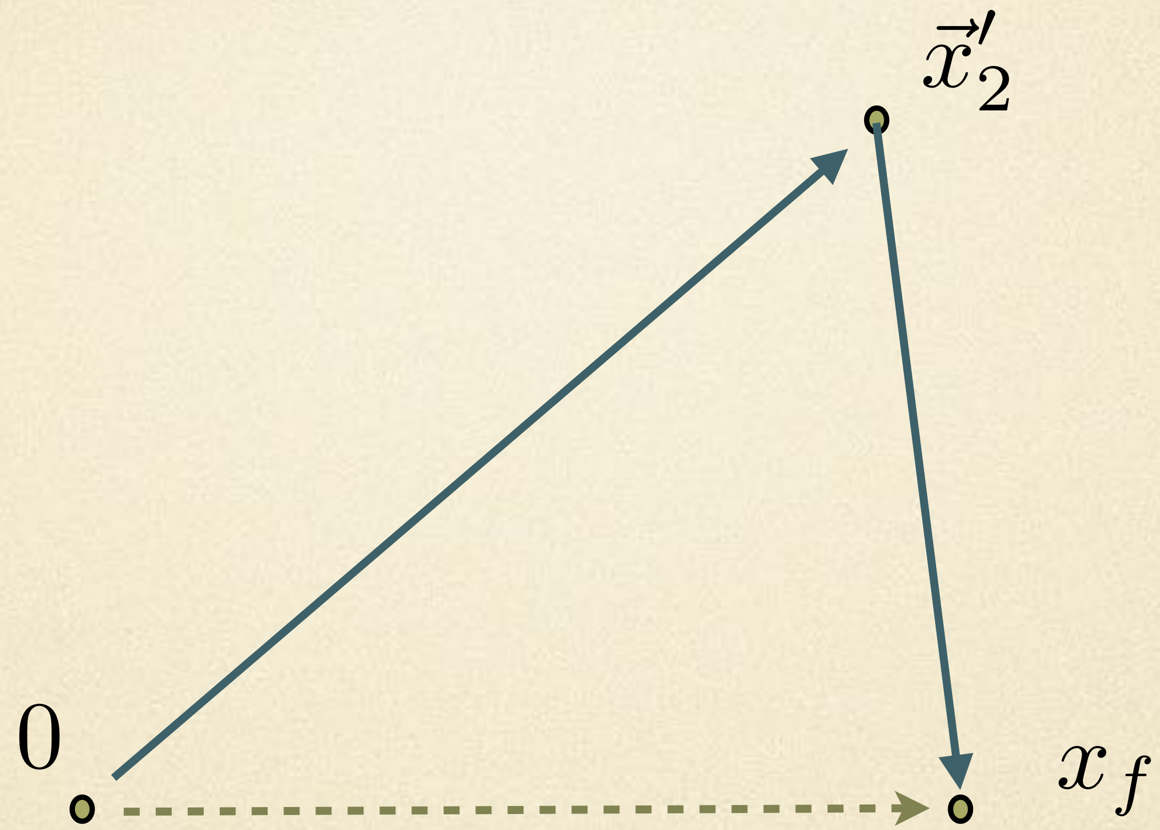

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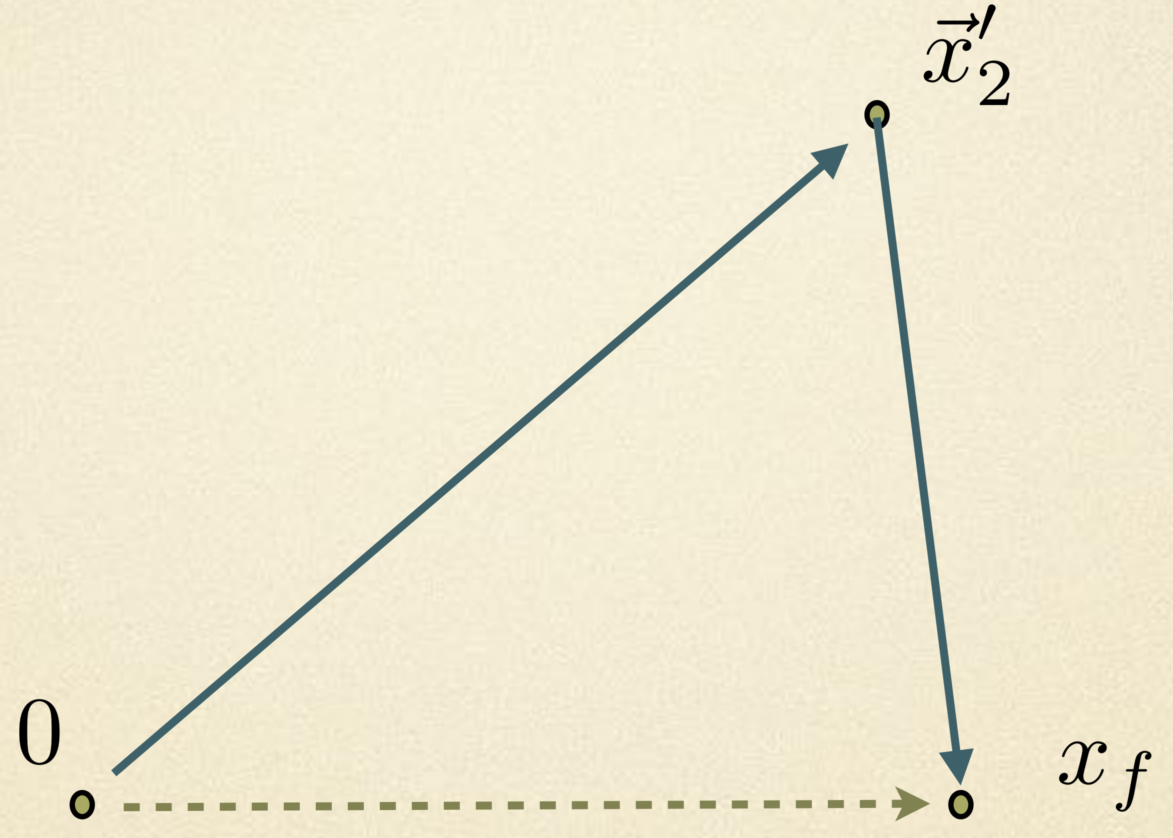

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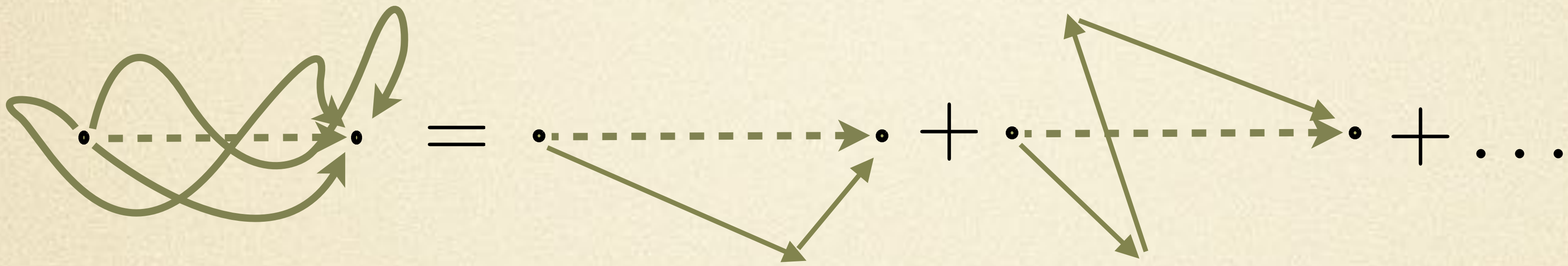

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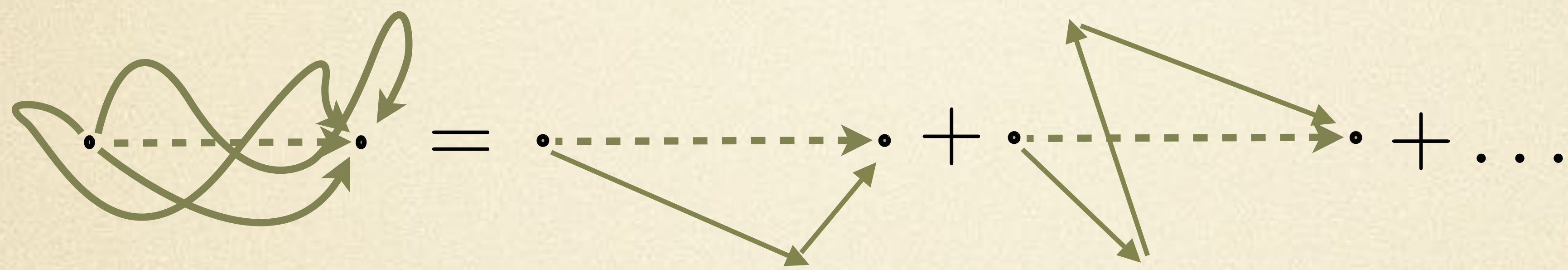

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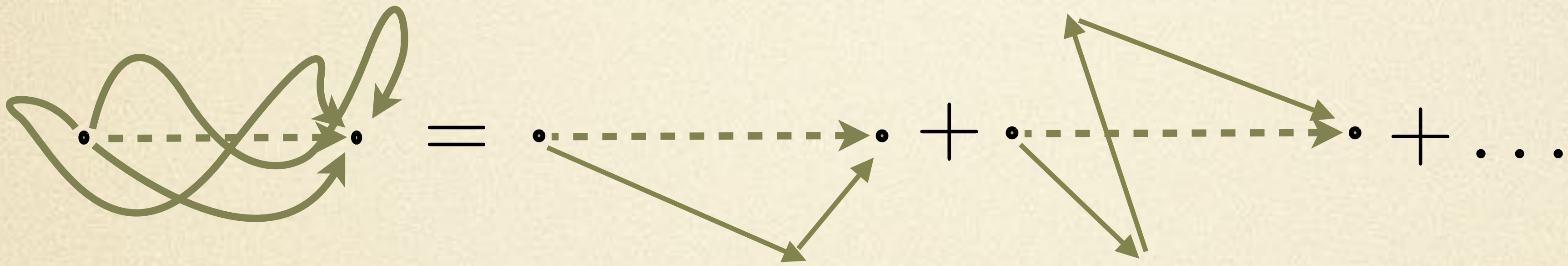

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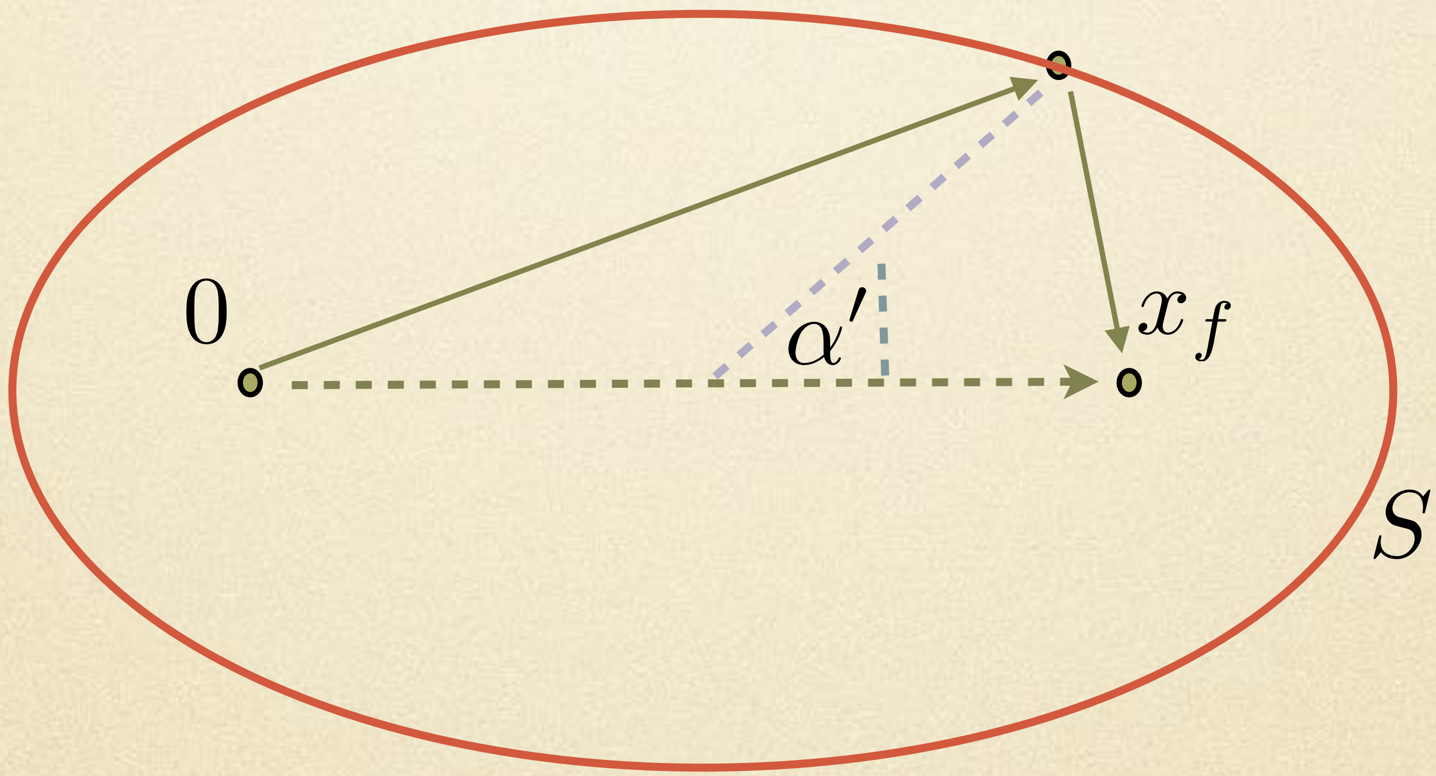
$$\langle 0|x_f\rangle = \mathcal{N} \cdot \langle 0|x_f\rangle_1 = \mathcal{N} \cdot K_0(Mx_f)$$

Concluding Comments

- Generalization to D dimensions
- PI of RPP action can be done, considering I1,I2,I3
- Chapman Kolmogorov becomes „trivial“
- Future work …

Thank You



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