Precision QFT in 2, 3 and 4d

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recent work with John Gracey, Ian Jack,
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and earlier work with
J. Möller, C. Studerus

LoeweFest 2017; PUC Santiago

Motivation

- LHC: era of precision QFT
  - ▷ large SM backgrounds
  - ▷ dominant effects: mainly QCD
  - ▷ high precision required for BSM searches
- theory working horse: renormalizable QFTs
  - ▷ perturbative expansions $\Rightarrow$ Feynman diagrams
  - ▷ higher-loop effects important (e.g. $g_\mu - 2$; m_q; H production; ...)
- lots of machinery developed recently
  - ▷ high automatization of complicated perturbative calculations
  - ▷ algebraic handling of Feynman diagrams
  - ▷ reduction of Feynman integrals to masters
  - ▷ numerical and/or algebraic determination of masters

Strategy

- for higher-loop precision, need to regularize and renormalize theory
  - ▷ work in dimensional reg. $d^4x \to d^dx$ and in $\overline{\text{MS}}$ scheme
  - ▷ evaluate all independent RCs: absorb *divergences*
  - ▷ e.g. QCD fields: $\psi_b = \sqrt{Z_2}\psi_r$, $A_b = \sqrt{Z_3}A_r$, $c_b = \sqrt{Z_3^c}c_r$
  - ▷ e.g. QCD couplings: $m_b = Z_m m_r$, $g_b = \mu^\epsilon Z_g g_r$
  - ▷ equivalently, def anomalous dimensions: $\gamma_i = -\partial_{\ln\mu^2} \ln Z_i$

- keep gauge group general

- generate all Fy diagrams; perform algebra: group-, Lorentz-, . . . [Nogueira QGRAF; Vermaseren FORM]

- project all Fy integrals to unique set of scalar massive vacuum ints
  - ▷ exact decomposition of propagators ($m \in \{0, m, M\}$) [Chetyrkin/Misiak/Münz 1998]
  - ▷ $\frac{1}{(k-p)^2+m^2} = \frac{1}{k^2+M^2} + \frac{2kp-p^2+M^2-m^2}{(k^2+M^2)((k-p)^2+m^2)}$
  - ▷ recursively lower degree of UV div
  - ▷ other IR regularization schemes: e.g. (local/global) R*

- map all integrals to minimal set: IBP [Chetyrkin/Tkachov 1981; Laporta 2000]

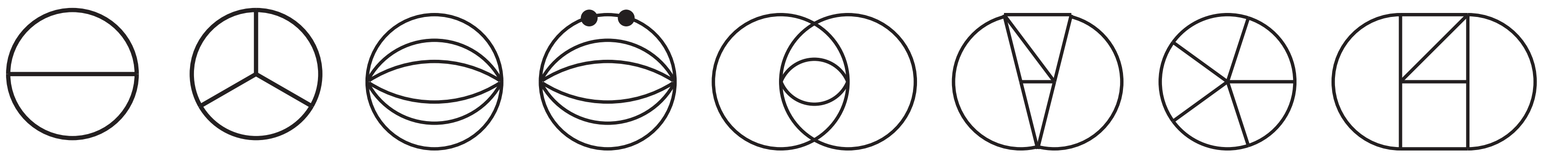
- evaluate this minimal set (analytically/numerically to high accuracy) [with Luthe since 2011]

Machinery

- perform IBP reduction with symbolic power x on one line
- derive difference equation for generalized master $I(x) \equiv \int \frac{1}{D_1^x D_2 \dots D_N}$
  - ▷ generic form: $\sum_{j=0}^{R} p_j(x) I(x+j) = F(x)$
- typically, want $I(1)$; solve the difference equation
  - ▷ explicitly (if 1st order)
  - ▷ numerically (very general setup) [Laporta 2000]
- solve via factorial series $I(x) = I_0(x) + \sum_{j=1}^{R} I_j(x)$,
  - ▷ where $I_j(x) = \mu_j^x \sum_{s=0}^{\infty} a_j(s) \frac{\Gamma(x+1)}{\Gamma(x+1+s-K_j)}$
- need boundary condition for fixing, say, $a_j(0)$: use decoupling at large x
  - ▷ $I(x) = \int_{k_1} g(k_1)/(k_1^2+1)^x \;\Rightarrow\; I(x) \sim (1)^x x^{-d/2} g(0)$
- deep expansions (ϵ^{20}) at 5 loops (132 masters) at high precision (>250 digits)

[with Luthe 2016]

Sample results (4d)



$$I_{28686.1.1} = +(-3)\,\epsilon^0 + (-\frac{3}{2})\,\epsilon^1 + (\frac{13}{24})\,\epsilon^2 + (-\frac{1267}{1440})\,\epsilon^3 + (-\frac{4193}{3456})\,\epsilon^4 + $$
$$+135.95072868792871461956492733702218574897992953584\,\epsilon^5 + \ldots$$

$$I_{28686.1.3} = +(0)\,\epsilon^0 + (\frac{3}{2})\,\epsilon^1 + (-\frac{1}{2})\,\epsilon^2 + (-\frac{443}{360})\,\epsilon^3 + (\frac{95}{216})\,\epsilon^4 + $$
$$-38.292059175062436961881799538284449799148385376441\,\epsilon^5 + \ldots$$

$$I_{30862.1.1} = +(-\frac{3}{5})\,\epsilon^0 + (-\frac{27}{10})\,\epsilon^1 + (-\frac{4\zeta_3}{5} - \frac{421}{60})\,\epsilon^2 + (-\frac{12\zeta_2^2}{25} + \frac{24\zeta_3}{5} + \frac{211}{24})\,\epsilon^3 + (\frac{72\zeta_2^2}{25} - 98\zeta_3 + \frac{32\zeta_5}{5} + \frac{12959}{48})\,\epsilon^4 +$$
$$+1143.1838307558764599466030303839590323268318605888\,\epsilon^5 + \ldots$$

$$I_{30231.1.1} = +(0)\,\epsilon^0 + (0)\,\epsilon^1 + (\frac{3\zeta_3}{5})\,\epsilon^2 + (\frac{9\zeta_2^2}{25} + \frac{21\zeta_3}{5} + 3\zeta_5)\,\epsilon^3 + (-36H_2\zeta_3 + \frac{12\zeta_2^3}{7} + \frac{63\zeta_2^2}{25} - \frac{21\zeta_3^2}{5} + 27\zeta_3 - \frac{24\zeta_5}{5})\,\epsilon^4$$
$$-531.32391547725635267943444561495368318398901378435\,\epsilon^5 + \ldots$$

$$I_{32596.1.1} = +(0)\,\epsilon^0 + (0)\,\epsilon^1 + (0)\,\epsilon^2 + (0)\,\epsilon^3 + (-14\zeta_7)\,\epsilon^4 + $$
[Wheel: Broadhurst 1985]
$$+235.07729596783467131454388080950411779239347239580\,\epsilon^5 + \ldots$$

$$I_{32279.3.1} = +(0)\,\epsilon^0 + (0)\,\epsilon^1 + (0)\,\epsilon^2 + (0)\,\epsilon^3 + (-\frac{441\zeta_7}{40})\,\epsilon^4 + $$
[Zigzag: Broadhurst/Kreimer 1995; Brown/Schnetz 2012]
$$+181.78223928612340820790788236018642961198741994209\,\epsilon^5 + \ldots$$

d=2: Gross-Neveu model

- SU(N) Gross-Neveu model: $\bar{\psi}(i\partial\!\!\!/ - m)\psi + g(\bar{\psi}\psi)^2$
  - ▷ renormalizable in 2d (not 4d)
  - ▷ cave: evanescent operators!

- laboratory for insights
  - ▷ $\approx$ QCD: asy. free; dyn. symmetry breaking $\rightarrow m_{fer}$
  - ▷ $\approx$ CM theory: e.g. $N \rightarrow 0 \leftrightarrow$ Ising model
  - ▷ $\approx$ AdS/CFT: $1/N$ expansion

- renormalize the model
  - ▷ $\psi_b = \sqrt{Z_\psi}\psi_r$, $m_b = Z_m m_r$, $g_b = Z_g g_r$
  - ▷ evaluate Z_ψ, Z_m, Z_g in $d = 2 - 2\epsilon$ (4loop $\surd$) [with Luthe/Gracey 2016]

- applications: e.g. Wilson-Fisher fixed point
  - ▷ at $N = 4$, GN related to electronic phase transition in graphene
  - ▷ determine critical exponents (at $d = 3$!)
  - ▷ e.g. $\frac{1}{\nu} = -\beta'(g_c) = (0.86, 0.78, 0.93)$ at $(2, 3, 4)$ loops, vs 1.0 from MC estimate

d=3: N=2 SUSY Chern-Simons theory

- take $\mathcal{L}_{SUSY} + \mathcal{L}_{GF}$
  - ▷ renormalize vector superfield $V_b = \sqrt{Z_V} V_r$
  - ▷ renormalize χ matter superfield $\Phi_b = \sqrt{Z_\Phi}\Phi_r$
  - ▷ (quartic) superpotential $W(\Phi)$
  - ▷ non-renormalization thm (topol) $\Rightarrow$ couplings $Y_b^{ijkl} \sim$ "Z_Φ"Y_r^{ijkl}

- status: Zamolodchikov's 2d c-theorem [Zamolodchikov 1986]
  - ▷ analogue in 4d? [Cardy 1988]
  - ▷ a-theorem in even dims [...]
  - ▷ Weyl anomaly important in proof of c-thm $\longrightarrow$ odd dims?

- idea: construct a-function in odd dims, order by order [Jack/Jones 2015]
  - ▷ have couplings $g^I \longrightarrow$ RG Beta fcts β^I
  - ▷ $\partial_{\ln\mu} a = G_{IJ}\beta^I\beta^J$, with positive definite G_{IJ}

- new: construction of a-function at 4 loops, in 3d [with Gracey/Jack/Poole 2016]
  - ▷ generally, in SUSY theories many divs cancel
  - ▷ to see this, needed above machinery (incl non-planar graph at 4 loops)

d=4: Quantum Chromodynamics

- drive renormalization program of QCD to five loops [with Luthe/Maier/Marquard]
- long-term effort of three independent groups [also: Chetyrkin/Baikov; Ueda/Vermaseren/Vogt]
- notation: strong coupling constant $a = \frac{C_A\, g^2(\mu)}{16\pi^2}$
  - ▷ color: $n_f \equiv \frac{T_F\, N_f}{C_A}$, $c_f \equiv \frac{C_F}{C_A}$, $d_{FF} \equiv \frac{[sTr(T^a T^b T^c T^d)]^2}{N_A\, T_F^2\, C_A^2}$, $\dots$
- result: 5-loop QCD β-function [with Luthe/Maier/Marquard]
  - ▷ $\beta = \partial_{\ln \mu^2}\, a = -a\left[\epsilon + \frac{11-4\,n_f}{3}\, a + b_1\, a^2 + b_2\, a^3 + b_3\, a^4 + b_4\, a^5 + \dots\right]$
  - ▷ $3^5\, b_4 = n_f^4 \left[c_1\, c_f + c_2\right]$ [Gracey 1996]
    $+\, n_f^3 \left[c_3\, c_f^2 + c_4\, c_f + c_5\, d_{FF} + c_6\right]$ [LMMS 2016]
    $+\, n_f^2 \left[\dots\right] + n_f \left[\dots\right] + \left[c_{22}\, d_{AA} + c_{23}\right]$ [Herzog/Ruijl/Ueda/Vermaseren/Vogt 2017]
  - ▷ the n_f^4 term agrees exactly with known result
  - ▷ all c_i in terms of Zeta values,
    e.g. $c_6 = -3(6231 + 9736\zeta_3 - 3024\zeta_4 - 2880\zeta_5)$
  - ▷ last row confirmed [LMMS 2017]

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- result: 5-loop QCD quark mass anomalous dimension [LMMS 2016]

  ▷ $\gamma_m = \partial_{\ln \mu^2} \ln m_q = -a c_f \Big[3 + \gamma_1 a + \gamma_2 a^2 + \gamma_3 a^3 + \gamma_4 a^4 + \dots \Big]$

  ▷ as usual, $\gamma_1 = (9c_f + 97)/6 - 10/3\, n_f$, ... ,

  ▷ and now $\gamma_4 = n_f^4 \Big[c_1\Big] + n_f^3 \Big[c_2\, c_f + c_3\Big] + n_f^2 \Big[c_4\, c_f^2 + c_5\, c_f + c_6\, d_{FF} + c_7\Big] + \dots$

  ▷ know all c_i analytically (Zetas only), for all n_f and color structures

  ▷ the n_f^4 and n_f^3 terms agree exactly with known results [Gracey 1996]

- completion of 5-loop renormalization program

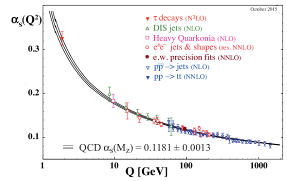
  ▷ besides β and γ_m, have also $Z_{\psi\psi}$, Z_{cc} and Z_{ccg} (Fy gauge + ξ^1) [LMMS 2017]

  ▷ all other RCs follow from these five, due to gauge invariance

  ▷ full gauge dependence now also available [Chetyrkin/Herzog/Falcioni/Vermaseren 2017]

Apparent convergence

- one might be concerned about 'asymptotic series' behavior at 5 loops
  - ▷ e.g. QM: 1d anharmonic oscillator $\hat{H} \sim \hat{p}^2 + \hat{x}^2 + g\hat{x}^4$
  - ▷ ground state energy $E_0 = \sum e_n\, g^n$, with $|\frac{e_{n+1}}{e_n}| \overset{n\gg 1}{\sim} \frac{3n}{4}$, diverges [Bender/Wu 1973]
  - ▷ (however defines unique function $E_0(g)$ via Borel; compare $\frac{1}{1-x} = \sum x^n$)

- now QCD. $N_c = 3,\ \alpha_s = \frac{g^2(Q)}{4\pi} =$ 

≈ 0.1 (at EW scale)
  - ▷ Beta function: [see also Baikov/Chetyrkin/Kühn 2016]

$$\beta_{N_f=3} = -0.17\,\alpha_s^2\,[1 + 0.57\,\alpha_s + 0.45\,\alpha_s^2 + 0.68\,\alpha_s^3 + 0.58\,\alpha_s^4 + \ldots]$$
$$\beta_{N_f=4} = -0.16\,\alpha_s^2\,[1 + 0.49\,\alpha_s + 0.31\,\alpha_s^2 + 0.49\,\alpha_s^3 + 0.28\,\alpha_s^4 + \ldots]$$
$$\beta_{N_f=5} = -0.15\,\alpha_s^2\,[1 + 0.40\,\alpha_s + 0.15\,\alpha_s^2 + 0.32\,\alpha_s^3 + 0.08\,\alpha_s^4 + \ldots]$$
$$\beta_{N_f=6} = -0.13\,\alpha_s^2\,[1 + 0.30\,\alpha_s - 0.03\,\alpha_s^2 + 0.18\,\alpha_s^3 + 0.002\,\alpha_s^4 + \ldots]$$

  - ▷ Quark mass anomalous dimension: [see also Baikov/Chetyrkin/Kühn 2014]

$$\gamma_{N_f=3} = -0.32\,\alpha_s\,[1 + 1.21\,\alpha_s + 1.26\,\alpha_s^2 + 1.43\,\alpha_s^3 + 2.04\,\alpha_s^4 + \ldots]$$
$$\gamma_{N_f=4} = -0.32\,\alpha_s\,[1 + 1.16\,\alpha_s + 1.01\,\alpha_s^2 + 0.88\,\alpha_s^3 + 1.15\,\alpha_s^4 + \ldots]$$
$$\gamma_{N_f=5} = -0.32\,\alpha_s\,[1 + 1.12\,\alpha_s + 0.75\,\alpha_s^2 + 0.36\,\alpha_s^3 + 0.43\,\alpha_s^4 + \ldots]$$
$$\gamma_{N_f=6} = -0.32\,\alpha_s\,[1 + 1.07\,\alpha_s + 0.49\,\alpha_s^2 - 0.15\,\alpha_s^3 - 0.10\,\alpha_s^4 + \ldots]$$

Conclusions

- recent advance in methods allows high-loop renormalization of quantum field theories
  - ▷ highly automated computer-algebra approaches
  - ▷ improved IBP algorithms (finite fields, . . .)
  - ▷ new insight into functional content of Feynman integrals
- difference equations (+ lots of CPU) are powerful enough to achieve 5 loops
  - ▷ non-trivial algorithmic fine-tuning
  - ▷ more? memory seems to become an issue
- applications to a host of QFTs in various fields
  - ▷ CM theory / effective models
  - ▷ mathematical physics/ SUSY
  - ▷ phenomenology / QCD+SM
  - ▷ . . .
- applications in various space-time dimensions
  - ▷ 2d, 3d, 4d, . . . , fractional, . . .
- stay tuned for (much) more . . .

Fractional dimensions

- difference eqs carry full information on d
  - expand massive tadpoles e.g. around $d = \frac{10}{3} - 2\epsilon$?
- motivation: renormalization in critical dimension (where [g]=0)
  - e.g. O(N) scalar ϕ^n theory: $\partial_\mu \phi \, \partial^\mu \phi + g\phi^n$ [Gracey 2017]
  - critical dim $D_n = \frac{2n}{n-2} \Rightarrow D_{\{3,4,5,6,7,\dots,\infty\}} = \{6, 4, \frac{10}{3}, 3, \frac{14}{5}, \dots, 2\}$
  - near FP (nontriv zero of Beta fct), RG fcts carry info on phase transitions
- study RG fcts in non-integer dimensions
  - renormalization 'as usual', dim. reg. natural (angular ints?!)
  - fewer diagrams (e.g. ϕ^5: LO 2pt diag has 3 loops)
  - interesting numbers: $\frac{p}{q} \to \Gamma(\frac{p}{q})$ at LO; $\zeta(s) \to$ Dirichlet β fct at NLO [Hager 2002]
- sample results in $d = \frac{10}{3} - 2\epsilon$ [Luthe]
  - as usual we normalize by $1/J^{loop}$; here, $J \sim \Gamma(1 - \frac{d}{2}) = -\frac{3}{2}\Gamma(\frac{1}{3}) + \dots$
  - 2-loop sunset $= -2.4882241714632542542\,\epsilon^0 - 8.6116306893649818141\,\epsilon^1 + \dots$
  - 3-loop merc. $= -0.0305966721641989027\,\epsilon^0 + 0.0149523122719722312\,\epsilon^1 + \dots$
  - 4-loop non-pl $= +0.0006880032418228675\,\epsilon^0 + 0.0023853718027957011\,\epsilon^1 + \dots$
  - etc.