On the calculation of triangle ladder Feynman diagrams

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Marcelo Loewe's Fest

Abstract

Old results about the calculation of the ladder diagrams is reviewed briefly. The method to reduce the number of loops in the triangle ladder Feynman diagrams in an arbitrary number of space-time dimensions is given

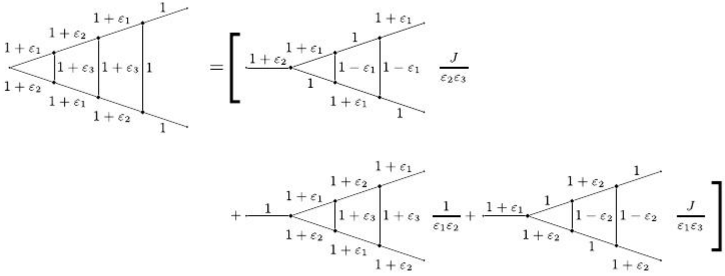
References

Pedro Allendes, Bernd Kniehl, Igor Kondrashuk, Eduardo A. Notte Cuello, Marko Rojas Medar, *Solution to Bethe-Salpeter equation via Mellin-Barnes transform* , Nuclear Physics B, **870**, pp. 243-277, 2013.

Bernd Kniehl, Igor Kondrashuk, Eduardo A. Notte Cuello, Ivan Parra Ferrada, Marko Rojas Medar, *Two-fold Mellin-Barnes transforms of Usyukina-Davy-dychev functions*, Nuclear Physics B, **876**, pp. 322-333, 2013.

Ivan Gonzalez and Igor Kondrashuk, *Box ladders in a non-integer dimension*, Theoretical and Mathematical Physics, **177**, pp. 1515-1540 , 2013.

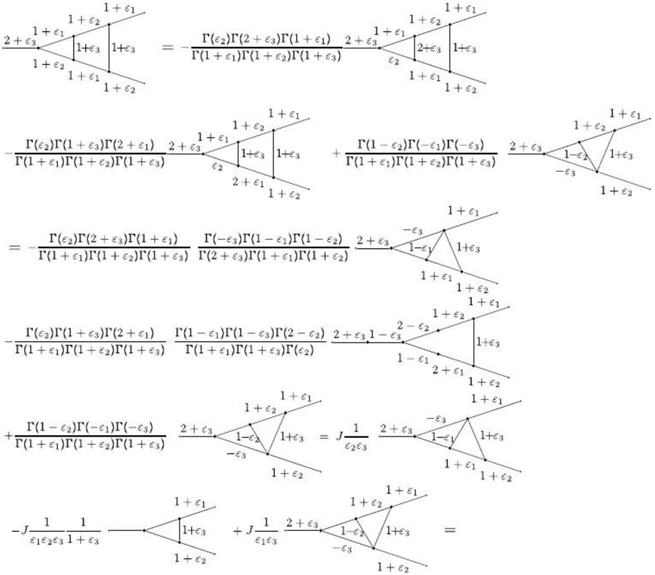
Ivan Gonzalez, Bernd A. Kniehl, Igor Kondrashuk, Eduardo A. Notte-Cuello, Ivan Parra-Ferrada and Marko A. Rojas-Medar, Explicit calculation of multi-fold contour integrals of certain ratios of Euler gamma functions. Part 1, Nuclear Physics B, **925**, pp. 607-614, 2017.



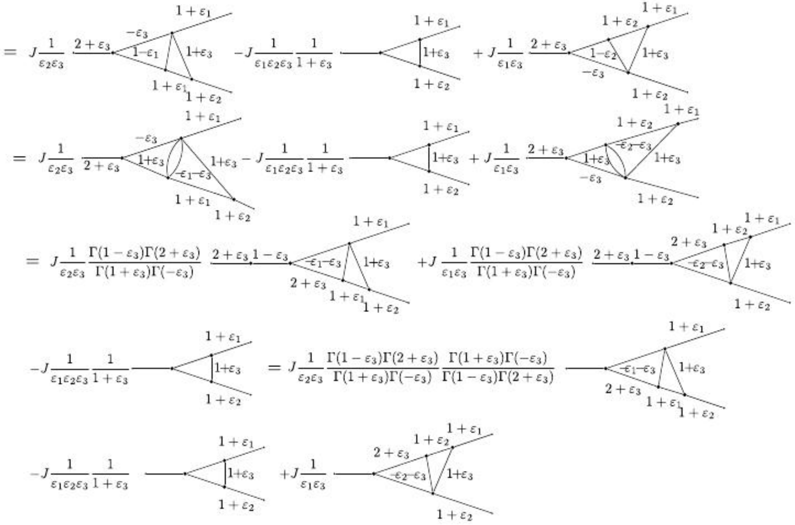
$$\varepsilon_1 + \varepsilon_2 + \varepsilon_3 = 0 \quad \text{always}$$

$$J = \frac{\Gamma(1-\varepsilon_1)\Gamma(1-\varepsilon_2)\Gamma(1-\varepsilon_3)}{\Gamma(1+\varepsilon_1)\Gamma(1+\varepsilon_2)\Gamma(1+\varepsilon_3)}$$

Proof of the loop reduction in d=4 dimensions. Part I

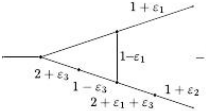


Proof of the loop reduction in d=4 dimensions. Part II

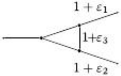


Proof of the loop reduction in d=4 dimensions. Part III

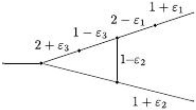
$$= J\frac{1}{\varepsilon_2\varepsilon_3}\frac{\Gamma(1-\varepsilon_1)\Gamma(2+\varepsilon_1+\varepsilon_3)\Gamma(1-\varepsilon_3)}{\Gamma(1+\varepsilon_1)\Gamma(-\varepsilon_1-\varepsilon_3)\Gamma(1+\varepsilon_3)}$$



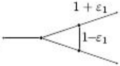
$$-J\frac{1}{\varepsilon_1\varepsilon_2\varepsilon_3}\frac{1}{1+\varepsilon_3}$$



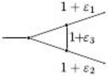
$$+J\frac{1}{\varepsilon_1\varepsilon_3}\frac{\Gamma(1-\varepsilon_2)\Gamma(2-\varepsilon_1)\Gamma(1-\varepsilon_3)}{\Gamma(1+\varepsilon_2)\Gamma(\varepsilon_1)\Gamma(1+\varepsilon_3)}$$



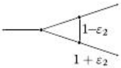
$$= -J^2\frac{1}{\varepsilon_2\varepsilon_3^2}\frac{1}{1+\varepsilon_3}$$



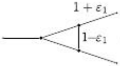
$$-J\frac{1}{\varepsilon_1\varepsilon_2\varepsilon_3}\frac{1}{1+\varepsilon_3}$$



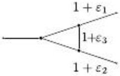
$$-J^2\frac{1}{\varepsilon_1\varepsilon_3^2}\frac{1}{1+\varepsilon_3}$$



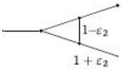
$$= -\frac{1}{(1+\varepsilon_3)\varepsilon_3}J\Bigg[\frac{J}{\varepsilon_2\varepsilon_3}$$



$$+\frac{1}{\varepsilon_1\varepsilon_2}$$

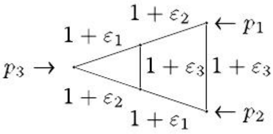


$$+\frac{J}{\varepsilon_1\varepsilon_3}$$

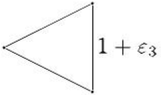


$$\Bigg]$$

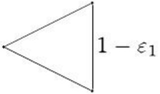
Loop reduction in the momentum space



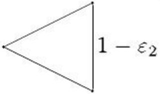
$$= J\frac{\pi^2}{(p_3^2)^{1-\varepsilon_3}}\left[\frac{1}{\varepsilon_1\varepsilon_2}\right.$$



$$+ \; (p_2^2)^{\varepsilon_2}\frac{1}{\varepsilon_2\varepsilon_3}$$



$$+ \; (p_1^2)^{\varepsilon_1}\frac{1}{\varepsilon_1\varepsilon_3}$$

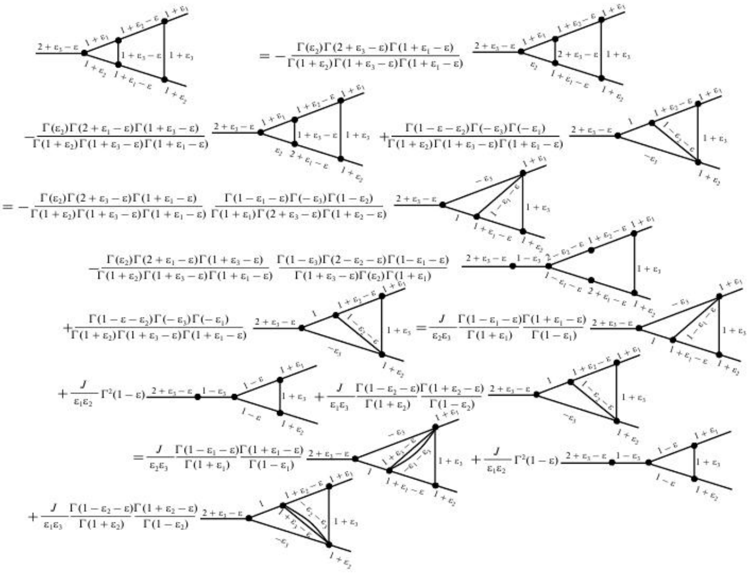


$$\left.\right]$$

$$\int d^dp\; e^{ipx}\frac{1}{(p^2)^\alpha} = \pi^{d/2}\frac{\Gamma(d/2-\alpha)}{\Gamma(\alpha)}\left(\frac{4}{x^2}\right)^{d/2-\alpha} \Longrightarrow$$

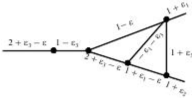
$$\frac{1}{(x^2)^\alpha} = \pi^{-d/2}4^{-\alpha}\frac{\Gamma(d/2-\alpha)}{\Gamma(\alpha)}\int d^dp\; e^{ipx}\frac{1}{(p^2)^{d/2-\alpha}}.$$

Loop reduction in non-integer dimensions $d = 4 - 2\varepsilon$

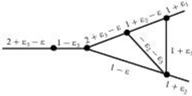


Loop reduction in non-integer dimensions. Part II

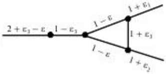
$$= \frac{J}{\varepsilon_2\varepsilon_3}\frac{\Gamma(1-\varepsilon_1-\varepsilon)}{\Gamma(1+\varepsilon_1)}\frac{\Gamma(1+\varepsilon_1-\varepsilon)}{\Gamma(1-\varepsilon_1)}\frac{\Gamma(1-\varepsilon_3)\Gamma(1-\varepsilon)\Gamma(2+\varepsilon_3-\varepsilon)}{\Gamma(1+\varepsilon_3-\varepsilon)\Gamma(1)\Gamma(-\varepsilon_3)}$$



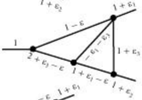
$$+ \frac{J}{\varepsilon_1\varepsilon_3}\frac{\Gamma(1-\varepsilon_2-\varepsilon)}{\Gamma(1+\varepsilon_2)}\frac{\Gamma(1+\varepsilon_2-\varepsilon)}{\Gamma(1-\varepsilon_2)}\frac{\Gamma(1-\varepsilon_3)\Gamma(1-\varepsilon)\Gamma(2+\varepsilon_3-\varepsilon)}{\Gamma(1+\varepsilon_3-\varepsilon)\Gamma(1)\Gamma(-\varepsilon_3)}$$



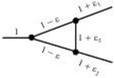
$$+ \frac{J}{\varepsilon_1\varepsilon_2}\Gamma^2(1-\varepsilon)$$



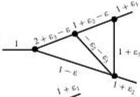
$$= \frac{J}{\varepsilon_2\varepsilon_3}\frac{\Gamma(1-\varepsilon_1-\varepsilon)}{\Gamma(1+\varepsilon_1)}\frac{\Gamma(1+\varepsilon_1-\varepsilon)}{\Gamma(1-\varepsilon_1)}$$



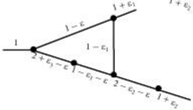
$$- \frac{J}{\varepsilon_1\varepsilon_2\varepsilon_3}\frac{\Gamma(1-\varepsilon)}{1+\varepsilon_3-\varepsilon}$$



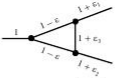
$$+ \frac{J}{\varepsilon_1\varepsilon_3}\frac{\Gamma(1-\varepsilon_2-\varepsilon)}{\Gamma(1+\varepsilon_2)}\frac{\Gamma(1+\varepsilon_2-\varepsilon)}{\Gamma(1-\varepsilon_2)}$$



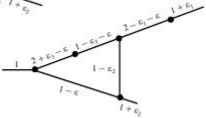
$$= \frac{J}{\varepsilon_2\varepsilon_3}\frac{\Gamma(1-\varepsilon_1-\varepsilon)}{\Gamma(1+\varepsilon_1)}\frac{\Gamma(1+\varepsilon_1-\varepsilon)}{\Gamma(1-\varepsilon_1)}\frac{\Gamma(1-\varepsilon_1)\Gamma(1-\varepsilon_3-\varepsilon)\Gamma(2-\varepsilon_2-\varepsilon)}{\Gamma(1+\varepsilon_1-\varepsilon)\Gamma(1+\varepsilon_3)\Gamma(\varepsilon_2)}$$



$$- \frac{J}{\varepsilon_1\varepsilon_2\varepsilon_3}\frac{\Gamma(1-\varepsilon)}{1+\varepsilon_3-\varepsilon}$$

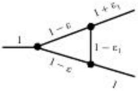


$$+ \frac{J}{\varepsilon_1\varepsilon_3}\frac{\Gamma(1-\varepsilon_2-\varepsilon)}{\Gamma(1+\varepsilon_2)}\frac{\Gamma(1+\varepsilon_2-\varepsilon)}{\Gamma(1-\varepsilon_2)}\frac{\Gamma(1-\varepsilon_2)\Gamma(1-\varepsilon_3-\varepsilon)\Gamma(2-\varepsilon_1-\varepsilon)}{\Gamma(1+\varepsilon_2-\varepsilon)\Gamma(1+\varepsilon_3)\Gamma(\varepsilon_1)}$$

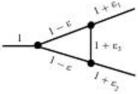


Loop reduction in non-integer dimensions. Part III

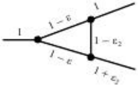
$$= -\frac{J}{\varepsilon_3(1+\varepsilon_3-\varepsilon)}\Bigg[\ \frac{1}{\varepsilon_2\varepsilon_3}\frac{\Gamma(1-\varepsilon_1-\varepsilon)\Gamma(1-\varepsilon_2-\varepsilon)\Gamma(1-\varepsilon_3)}{\Gamma(1+\varepsilon_1)\Gamma(1+\varepsilon_2)\Gamma(1+\varepsilon_3-\varepsilon)}$$



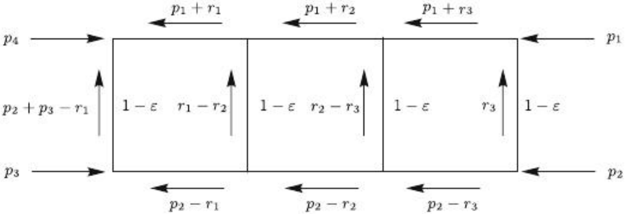
$$+\frac{1}{\varepsilon_1\varepsilon_2}\Gamma(1-\varepsilon)$$



$$+\frac{1}{\varepsilon_1\varepsilon_3}\frac{\Gamma(1-\varepsilon_1-\varepsilon)\Gamma(1-\varepsilon_2-\varepsilon)\Gamma(1-\varepsilon_3)}{\Gamma(1+\varepsilon_1)\Gamma(1+\varepsilon_2)\Gamma(1+\varepsilon_3-\varepsilon)}$$



$$\Bigg]$$



Formulas to prove

$$\oint_C dz_2 dz_3 \; D^{(u,v)}[1+\varepsilon_1 - z_3, 1+\varepsilon_2 - z_2, 1+\varepsilon_3] \times$$
$$\times D^{(z_2,z_3)}[1+\varepsilon_2, 1+\varepsilon_1, 1+\varepsilon_3] =$$
$$J\left[\frac{D^{(u,v-\varepsilon_2)}[1-\varepsilon_1]}{\varepsilon_2\varepsilon_3} + \frac{D^{(u,v)}[1+\varepsilon_3]}{\varepsilon_1\varepsilon_2} + \frac{D^{(u-\varepsilon_1,v)}[1-\varepsilon_2]}{\varepsilon_1\varepsilon_3}\right],$$

$$D^{(z_2,z_3)}[\nu_1,\nu_2,\nu_3] = \frac{\Gamma(-z_2)\Gamma(-z_3)\Gamma(-z_2-\nu_2-\nu_3+d/2)}{\Pi_i\Gamma(\nu_i)}$$
$$\times\frac{\Gamma(-z_3-\nu_1-\nu_3+d/2)\Gamma(z_2+z_3+\nu_3)\Gamma(\Sigma\nu_i - d/2 + z_3 + z_2)}{\Gamma(d-\Sigma_i\nu_i)},$$

$$\varepsilon_1+\varepsilon_2+\varepsilon_3 = 0 \quad \text{and} \quad D^{(u,v)}[1+\nu] \equiv D^{(u,v)}[1,1,1+\nu].$$

Barnes Lemmas

$$\oint_C dz\, \Gamma(\lambda_1 + z)\Gamma(\lambda_2 + z)\Gamma(\lambda_3 - z)\Gamma(\lambda_4 - z) = \frac{\Gamma(\lambda_1 + \lambda_3)\Gamma(\lambda_1 + \lambda_4)\Gamma(\lambda_2 + \lambda_3)\Gamma(\lambda_2 + \lambda_4)}{\Gamma(\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4)},$$

$$\oint_C dz\, \frac{\Gamma(\lambda_1 + z)\Gamma(\lambda_2 + z)\Gamma(\lambda_3 + z)\Gamma(\lambda_4 - z)\Gamma(\lambda_5 - z)}{\Gamma(\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 + \lambda_5 + z)} = \frac{\Gamma(\lambda_1 + \lambda_4)\Gamma(\lambda_2 + \lambda_4)\Gamma(\lambda_3 + \lambda_4)\Gamma(\lambda_1 + \lambda_5)\Gamma(\lambda_2 + \lambda_5)\Gamma(\lambda_3 + \lambda_5)}{\Gamma(\lambda_1 + \lambda_2 + \lambda_4 + \lambda_5)\Gamma(\lambda_1 + \lambda_3 + \lambda_4 + \lambda_5)\Gamma(\lambda_2 + \lambda_3 + \lambda_4 + \lambda_5)}$$

The Integrant

$$D^{(z_2,z_3)}[1+\varepsilon_2, 1+\varepsilon_1, 1+\varepsilon_3] = \frac{\Gamma(-z_2)\Gamma(-z_3)\Gamma(-z_2+\varepsilon_2)\Gamma(-z_3+\varepsilon_1)\Gamma(1+z_2+z_3)}{\Gamma(1+\varepsilon_1)\Gamma(1+\varepsilon_2)\Gamma(1+\varepsilon_3)} \times \Gamma(1+z_2+z_3+\varepsilon_3),$$

$$D^{(u,v)}[1+\varepsilon_1-z_3, 1+\varepsilon_2-z_2, 1+\varepsilon_3] = \frac{\Gamma(-u)\Gamma(-v)\Gamma(-u+\varepsilon_1+z_2)\Gamma(-v+\varepsilon_2+z_3)}{\Gamma(1+\varepsilon_1-z_3)\Gamma(1+\varepsilon_2-z_2)\Gamma(1+\varepsilon_3)\Gamma(1+z_2+z_3)} \times \Gamma(1+u+v+\varepsilon_3)\Gamma(1-z_2-z_3+u+v),$$

$$D^{(u,v)}[1+\varepsilon_1-z_3, 1+\varepsilon_2-z_2, 1+\varepsilon_3]D^{(z_2,z_3)}[1+\varepsilon_2, 1+\varepsilon_1, 1+\varepsilon_3] = \frac{\Gamma(-u)\Gamma(-v)\Gamma(1+u+v+\varepsilon_3)}{\Gamma(1+\varepsilon_1)\Gamma(1+\varepsilon_2)\Gamma^2(1+\varepsilon_3)} \frac{1}{\varepsilon_1-z_3}\frac{1}{\varepsilon_2-z_2}\Gamma(-z_2)\Gamma(-z_3) \times \Gamma(1+z_2+z_3+\varepsilon_3)\Gamma(-u+\varepsilon_1+z_2)\Gamma(-v+\varepsilon_2+z_3) \times \Gamma(1-z_2-z_3+u+v).$$

The term $D^{(u,v)}[1+\varepsilon_3]$

$$\frac{D^{(u,v)}[1+\varepsilon_3]}{\varepsilon_1\varepsilon_2} =$$

$$\frac{1}{\varepsilon_1\varepsilon_2}\frac{\Gamma(-u)\Gamma(-v)\Gamma(-\varepsilon_3-u)\Gamma(-\varepsilon_3-v)\Gamma^2(1+\varepsilon_3+u+v)}{\Gamma(1-\varepsilon_3)\Gamma(1+\varepsilon_3)},$$

$$\frac{D^{(u,v-\varepsilon_2)}[1-\varepsilon_1]}{\varepsilon_2\varepsilon_3} =$$

$$\frac{1}{\varepsilon_2\varepsilon_3}\frac{\Gamma(-u)\Gamma(\varepsilon_2-v)\Gamma(\varepsilon_1-u)\Gamma(-\varepsilon_3-v)\Gamma^2(1+\varepsilon_3+u+v)}{\Gamma(1-\varepsilon_1)\Gamma(1+\varepsilon_1)},$$

$$\frac{D^{(u-\varepsilon_1,v)}[1-\varepsilon_2]}{\varepsilon_1\varepsilon_3} =$$

$$\frac{1}{\varepsilon_1\varepsilon_3}\frac{\Gamma(\varepsilon_1-u)\Gamma(-v)\Gamma(-\varepsilon_3-u)\Gamma(\varepsilon_2-v)\Gamma^2(1+\varepsilon_3+u+v)}{\Gamma(1-\varepsilon_2)\Gamma(1+\varepsilon_2)}$$

Simple trick

$$
\frac{1}{z_3-\varepsilon_1}\frac{1}{z_2-\varepsilon_2}\Gamma(-z_2)\Gamma(-z_3)\Gamma(1+z_2+z_3+\varepsilon_3)\Gamma(-u+\varepsilon_1+z_2)\times
$$
$$
\times\Gamma(-v+\varepsilon_2+z_3)\Gamma(1-z_2-z_3+u+v)=
$$
$$
=\frac{z_2+z_2+\varepsilon_3}{(z_3-\varepsilon_1)(z_2-\varepsilon_2)}\Gamma(-z_2)\Gamma(-z_3)\Gamma(z_2+z_3+\varepsilon_3)\Gamma(-u+\varepsilon_1+z_2)\times
$$
$$
\times\Gamma(-v+\varepsilon_2+z_3)\Gamma(1-z_2-z_3+u+v)=
$$
$$
=\left(\frac{1}{z_3-\varepsilon_1}+\frac{1}{z_2-\varepsilon_2}\right)\Gamma(-z_2)\Gamma(-z_3)\Gamma(z_2+z_3+\varepsilon_3)\Gamma(-u+\varepsilon_1+z_2)\times
$$
$$
\times\Gamma(-v+\varepsilon_2+z_3)\Gamma(1-z_2-z_3+u+v).
$$

$$
\oint_C dz_2\frac{1}{z_2-\varepsilon_2}\Gamma(-z_2)\Gamma(-u+\varepsilon_1+z_2)\oint_C dz_3\Gamma(-z_3)\Gamma(z_2+z_3+\varepsilon_3)\times
$$
$$
\Gamma(-v+\varepsilon_2+z_3)\Gamma(1-z_2-z_3+u+v).
$$

Application of the Barnes lemmas

$$
\oint_C dz_2 \frac{1}{z_2-\varepsilon_2}\Gamma(-z_2)\Gamma(-u+\varepsilon_1+z_2)\oint_C dz_3\Gamma(-z_3)\Gamma(z_2+z_3+\varepsilon_3)\times
$$
$$
\Gamma(-v+\varepsilon_2+z_3)\Gamma(1-z_2-z_3+u+v)=
$$
$$
\oint_C dz_2 \frac{1}{z_2-\varepsilon_2}\Gamma(-z_2)\Gamma(-u+\varepsilon_1+z_2)\times
$$
$$
\times\frac{\Gamma(z_2+\varepsilon_3)\Gamma(-v+\varepsilon_2)\Gamma(1+\varepsilon_3+u+v)\Gamma(1+\varepsilon_2+u-z_2)}{\Gamma(1+u-\varepsilon_1)}=
$$
$$
=\frac{\Gamma(-v+\varepsilon_2)\Gamma(1+\varepsilon_3+u+v)}{\Gamma(1+u-\varepsilon_1)}\oint_C dz_2\frac{1}{z_2-\varepsilon_2}\Gamma(z_2+\varepsilon_3)\Gamma(-z_2)\times
$$
$$
\times\Gamma(-u+\varepsilon_1+z_2)\Gamma(1+\varepsilon_2+u-z_2).
$$

Reflection

$$\frac{\Gamma\left(-v+\varepsilon_2\right)\Gamma\left(1+\varepsilon_3+u+v\right)}{\Gamma\left(1+u-\varepsilon_1\right)}\oint_C dz_2\frac{1}{z_2-\varepsilon_2}\Gamma\left(z_2+\varepsilon_3\right)\Gamma\left(-z_2\right)\times$$
$$\times\Gamma\left(-u+\varepsilon_1+z_2\right)\Gamma\left(1+\varepsilon_2+u-z_2\right)=$$
$$-\frac{\Gamma\left(-v+\varepsilon_2\right)\Gamma\left(1+\varepsilon_3+u+v\right)}{\Gamma\left(1+u-\varepsilon_1\right)}\oint_C dz_2\frac{1}{z_2+\varepsilon_2}\Gamma\left(-z_2+\varepsilon_3\right)\Gamma\left(z_2\right)\times$$
$$\times\Gamma\left(-u+\varepsilon_1-z_2\right)\Gamma\left(1+\varepsilon_2+u+z_2\right)=$$
$$-\frac{\Gamma\left(-v+\varepsilon_2\right)\Gamma\left(1+\varepsilon_3+u+v\right)}{\Gamma\left(1+u-\varepsilon_1\right)}\oint_C dz_2\frac{\Gamma(z_2+\varepsilon_2)}{\Gamma(1+z_2+\varepsilon_2)}\Gamma\left(-z_2+\varepsilon_3\right)\Gamma\left(z_2\right)\times$$
$$\times\Gamma\left(-u+\varepsilon_1-z_2\right)\Gamma\left(1+\varepsilon_2+u+z_2\right)=$$

Result

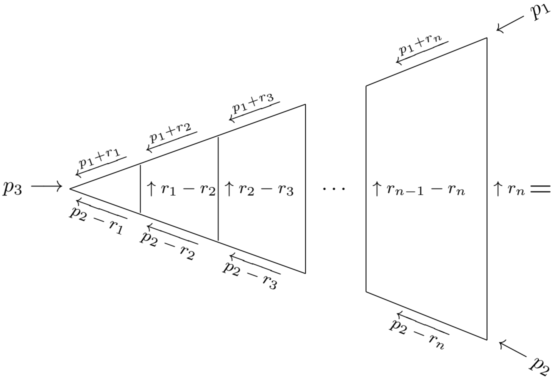
$$
-\frac{\Gamma(-v+\varepsilon_2)\Gamma(1+\varepsilon_3+u+v)}{\Gamma(1+u-\varepsilon_1)}\Gamma(\varepsilon_3)\Gamma(-\varepsilon_1)\Gamma(1+u-\varepsilon_1)\times
$$

$$
\frac{\Gamma(-u-\varepsilon_3)\Gamma(-u+\varepsilon_1)\Gamma(1-\varepsilon_3)}{\Gamma(1+\varepsilon_2)\Gamma(-u)} =
$$

$$
\frac{1}{\varepsilon_1\varepsilon_3}\frac{\Gamma(1-\varepsilon_1)\Gamma(1+\varepsilon_3)\Gamma(1-\varepsilon_3)}{\Gamma(1+\varepsilon_2)}\times
$$

$$
\frac{\Gamma(-v+\varepsilon_2)\Gamma(1+\varepsilon_3+u+v)\Gamma(-u-\varepsilon_3)\Gamma(-u+\varepsilon_1)}{\Gamma(-u)} \sim D^{(u-\varepsilon_1,v)}[1-\varepsilon_2]
$$

Ladder diagrams



$$= \frac{1}{(p_3^2)^n}\Phi^{(n)}\left(\frac{p_1^2}{p_3^2},\frac{p_2^2}{p_3^2}\right)$$

Explicit form of Usykina-Davydychev functions

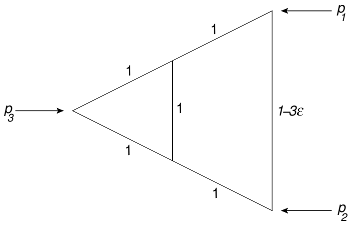
$$\Phi^{(n)}(x,y) = \oint dz_2 dz_3 x^{z_2} y^{z_3} \mathcal{M}^{(n)}(z_2,z_3)$$

The explicit form of the function is given in Davydychev and Usyukina papers:

$$\Phi^{(n)}(x,y) = -\frac{1}{n!\lambda}\sum_{j=n}^{2n}\frac{(-1)^j j!\ln^{2n-j}(y/x)}{(j-n)!(2n-j)!}\left[\mathrm{Li_j}\left(-\frac{1}{\rho x}\right) - \mathrm{Li}_j(-\rho y)\right], \quad (1)$$

$$\rho = \frac{2}{1-x-y+\lambda}, \qquad \lambda = \sqrt{(1-x-y)^2 - 4xy}.$$

Diagram which we calculate by the same trick in $d = 4 - 2\varepsilon$

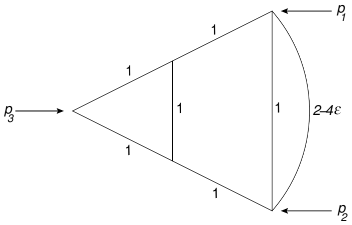


$$\oint_C dz_2 dz_3 \; D^{(u,v)}[1-z_3, 1-z_2, 1-3\varepsilon] \times D^{(z_2,z_3)}[1,1,1] =$$

$$\frac{1}{\varepsilon^2}\frac{\Gamma^2(1-\varepsilon)\Gamma(1+\varepsilon)}{\Gamma(1-2\varepsilon)} D^{(u,v)}[1-3\varepsilon]$$

$$-\frac{1}{2\varepsilon^2}\frac{\Gamma(1-\varepsilon)\Gamma(1+2\varepsilon)\Gamma(1-2\varepsilon)}{\Gamma(1-3\varepsilon)\Gamma(1+\varepsilon)}\left(D^{(u-\varepsilon,v)}[1-2\varepsilon] + D^{(u,v-\varepsilon)}[1-2\varepsilon]\right)$$

Integral Equation for this diagram

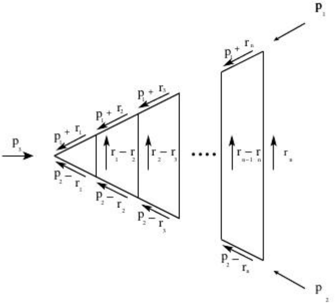


$$\oint_C dz_2 dz_3 \; D^{(u,v)}[-z_3, -z_2, 2-4\varepsilon] \times M(z_2,z_3) =$$

$$\frac{1}{\varepsilon^2}\frac{\Gamma^2(1-\varepsilon)\Gamma(1+\varepsilon)}{\Gamma(1-2\varepsilon)} D^{(u,v)}[1-3\varepsilon]$$

$$-\frac{1}{2\varepsilon^2}\frac{\Gamma(1-\varepsilon)\Gamma(1+2\varepsilon)\Gamma(1-2\varepsilon)}{\Gamma(1-3\varepsilon)\Gamma(1+\varepsilon)}\left(D^{(u-\varepsilon,v)}[1-2\varepsilon] + D^{(u,v-\varepsilon)}[1-2\varepsilon]\right)$$

Ladder diagrams



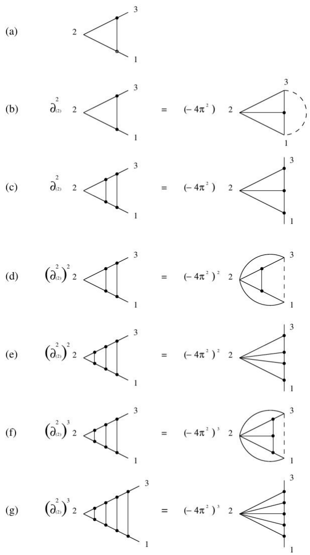
$$= C^{(n)}(p_1^2, p_2^2, p_3^2) \equiv \frac{1}{(p_3^2)^n}\, \phi^{(n)}\left(\frac{p_1^2}{p_3^2}, \frac{p_2^2}{p_3^2}\right)$$

V. V. Belokurov and N. I. Usyukina, "Calculation Of Ladder Diagrams In Arbitrary Order," J. Phys. A **16** (1983) 2811.

N. I. Usyukina and A. I. Davydychev, "An Approach to the evaluation of three and four point ladder diagrams," Phys. Lett. B **298** (1993) 363.

N. I. Usyukina and A. I. Davydychev, "Exact results for three and four point ladder diagrams with an arbitrary number of rungs," Phys. Lett. B **305** (1993) 136.

Fourier-invariance of UD functions, JHEP08(2008)106



Fourier-invariance of UD functions via MB transform

$$\frac{1}{[31]^2}\Phi^{(n)}\left(\frac{[12]}{[31]},\frac{[23]}{[31]}\right) = \frac{1}{(2\pi)^4}\int d^4p_1 d^4p_2 d^4p_3 \ \delta(p_1+p_2+p_3)\times$$
$$\times e^{ip_2x_2}e^{ip_1x_1}e^{ip_3x_3}\frac{1}{(p_2^2)^2}\Phi^{(n)}\left(\frac{p_1^2}{p_2^2},\frac{p_3^2}{p_2^2}\right).$$

The explicit form of the function is given in Davydychev and Usyukina papers:

$$\Phi^{(n)}(x,y) = -\frac{1}{n!\lambda}\sum_{j=n}^{2n}\frac{(-1)^j j!\ln^{2n-j}(y/x)}{(j-n)!(2n-j)!}\left[\mathrm{Li_j}\left(-\frac{1}{\rho x}\right) - \mathrm{Li}_j(-\rho y)\right],$$

$$\rho = \frac{2}{1-x-y+\lambda}, \qquad \lambda = \sqrt{(1-x-y)^2-4xy}.$$

Fourier-invariance of UD functions via MB transform

Mellin-Barnes transform for the ladder functions:

$$\Phi^{(n)}(x,y) = \oint dz_2 dz_3 x^{z_2} y^{z_3} \mathcal{M}^{(n)}(z_2,z_3)$$

$$\frac{1}{[31]^2}\Phi^{(n)}\left(\frac{[12]}{[31]},\frac{[23]}{[31]}\right) = \frac{1}{(2\pi)^4}\int d^4p_1 d^4p_2 d^4p_3\ \delta(p_1+p_2+p_3)\times$$

$$\times e^{ip_2x_2}e^{ip_1x_1}e^{ip_3x_3}\frac{1}{(p_2^2)^2}\Phi^{(n)}\left(\frac{p_1^2}{p_2^2},\frac{p_3^2}{p_2^2}\right) =$$

$$= \frac{1}{(2\pi)^8}\int d^4p_1 d^4p_2 d^4p_3 d^4x_5 e^{ip_2(x_2-x_5)}e^{ip_1(x_1-x_5)}e^{ip_3(x_3-x_5)}\times$$

$$\times\frac{1}{(p_2^2)^2}\Phi^{(n)}\left(\frac{p_1^2}{p_2^2},\frac{p_3^2}{p_2^2}\right) =$$

Fourier-invariance of UD functions via MB transform

$$= \frac{1}{(2\pi)^8} \int d^4p_1 d^4p_2 d^4p_3 d^4x_5 \oint dz_2 dz_3 \frac{e^{ip_2(x_2-x_5)} e^{ip_1(x_1-x_5)} e^{ip_3(x_3-x_5)}}{(p_2^2)^{2+z_2+z_3}(p_1^2)^{-z_2}(p_3^2)^{-z_3}} \times$$

$$\times \mathcal{M}^{(n)}(z_2, z_3) =$$

$$= \frac{(4\pi)^6}{(2\pi)^8} \int d^4x_5 \oint dz_2 dz_3 \frac{\Gamma(-z_2 - z_3)}{\Gamma(2+z_2+z_3)} \frac{\Gamma(2+z_2)}{\Gamma(-z_2)} \frac{\Gamma(2+z_3)}{\Gamma(-z_3)}$$

$$\times \frac{2^{2z_2+2z_3-2(2+z_2+z_3)} \mathcal{M}^{(n)}(z_2, z_3)}{(x_2 - x_5)^{-z_2-z_3}(x_1 - x_5)^{2+z_2}(x_3 - x_5)^{2+z_3}} =$$

$$= \oint dz_2 dz_3 \frac{\mathcal{M}^{(n)}(z_2, z_3)}{[12]^{-z_3}[23]^{-z_2}[31]^{2+z_2+z_3}}$$

P. Allendes, N. Guerrero, I. Kondrashuk and E. A. Notte Cuello, "New four-dimensional integrals by Mellin-Barnes transform," J. Math. Phys. **51** (2010) 052304 [arXiv:0910.4805 [hep-th]].

"Orthogonality"

$$\frac{1}{[12]^{\alpha_3}[23]^{\alpha_1}[13]^{\alpha_2}} \sim \oint_C dudv \frac{[13]}{[12]}^u \frac{[23]}{[12]}^v \frac{1}{[12]^{\alpha_1+\alpha_2+\alpha_3}} \times$$
$$\oint_C dz_2dz_3 \; D^{(z_2,z_3)}[d/2-\alpha_1, d/2-\alpha_2, d/2-\alpha_3] \times$$
$$\times D^{(u,v)}[d/2+z_3, d/2+z_2, d-z_3-z_2-\alpha_1-\alpha_2-\alpha_3]$$

Two poles survive $u = -\alpha_2, v = -\alpha_1$

$$\oint_C dz_2dz_3 \; D^{(u,v)}[-z_3,-z_2,2-4\varepsilon] \times M(z_2,z_3) =$$
$$\frac{1}{\varepsilon^2}\frac{\Gamma^2(1-\varepsilon)\Gamma(1+\varepsilon)}{\Gamma(1-2\varepsilon)} D^{(u,v)}[1-3\varepsilon]$$
$$-\frac{1}{2\varepsilon^2}\frac{\Gamma(1-\varepsilon)\Gamma(1+2\varepsilon)\Gamma(1-2\varepsilon)}{\Gamma(1-3\varepsilon)\Gamma(1+\varepsilon)}\left(D^{(u-\varepsilon,v)}[1-2\varepsilon] + D^{(u,v-\varepsilon)}[1-2\varepsilon]\right)$$