

On the calculation of triangle ladder Feynman diagrams

Igor Kondrashuk

UBB, Chillan, Chile

Igor Kondrashuk⁽¹⁾, Eduardo Notte Cuello⁽²⁾, Ivan Parra-Ferrada⁽³⁾

(1) Departamento de Ciencias Basicas, Universidad del Bio-Bio (Chile)

(2) Departamento de Matematica, Universidad de La Serena





(3) Instituto de Fisica y Matematica, Universidad de Talca

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Old results about the calculation of the ladder diagrams is reviewed briefly. The method to reduce the number of loops in the triangle ladder Feynman diagrams in an arbitrary number of space-time dimensions is given

References

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-  Ivan Gonzalez and Igor Kondrashuk, *Box ladders in a non-integer dimension*, Theoretical and Mathematical Physics, **177**, pp. 1515-1540 , 2013.
-  Ivan Gonzalez, Bernd A. Kniehl, Igor Kondrashuk, Eduardo A. Notte-Cuello, Ivan Parra-Ferrada and Marko A. Rojas-Medar, Explicit calculation of multi-fold contour integrals of certain ratios of Euler gamma functions. Part 1, Nuclear Physics B, **925**, pp. 607-614, 2017.

Loop reduction in Feynman diagrams

$$\begin{aligned}
 & \text{Triangle Ladder Diagram} = \left[\text{Term 1} \frac{J}{\epsilon_2 \epsilon_3} + \text{Term 2} \frac{1}{\epsilon_1 \epsilon_2} + \text{Term 3} \frac{J}{\epsilon_1 \epsilon_3} \right]
 \end{aligned}$$

$$\epsilon_1 + \epsilon_2 + \epsilon_3 = 0 \quad \text{always}$$

$$J = \frac{\Gamma(1 - \epsilon_1)\Gamma(1 - \epsilon_2)\Gamma(1 - \epsilon_3)}{\Gamma(1 + \epsilon_1)\Gamma(1 + \epsilon_2)\Gamma(1 + \epsilon_3)}$$

Proof of the loop reduction in d=4 dimensions. Part I

$$\begin{aligned}
 & \text{Diagram 1} = - \frac{\Gamma(\epsilon_2)\Gamma(2+\epsilon_3)\Gamma(1+\epsilon_1)}{\Gamma(1+\epsilon_1)\Gamma(1+\epsilon_2)\Gamma(1+\epsilon_3)} \text{Diagram 2} \\
 & \frac{\Gamma(\epsilon_2)\Gamma(1+\epsilon_3)\Gamma(2+\epsilon_1)}{\Gamma(1+\epsilon_1)\Gamma(1+\epsilon_2)\Gamma(1+\epsilon_3)} \text{Diagram 3} + \frac{\Gamma(1-\epsilon_2)\Gamma(-\epsilon_1)\Gamma(-\epsilon_3)}{\Gamma(1+\epsilon_1)\Gamma(1+\epsilon_2)\Gamma(1+\epsilon_3)} \text{Diagram 4} \\
 & = - \frac{\Gamma(\epsilon_2)\Gamma(2+\epsilon_3)\Gamma(1+\epsilon_1)}{\Gamma(1+\epsilon_1)\Gamma(1+\epsilon_2)\Gamma(1+\epsilon_3)} \frac{\Gamma(-\epsilon_3)\Gamma(1-\epsilon_1)\Gamma(1-\epsilon_2)}{\Gamma(2+\epsilon_3)\Gamma(1+\epsilon_1)\Gamma(1+\epsilon_2)} \text{Diagram 5} \\
 & - \frac{\Gamma(\epsilon_2)\Gamma(1+\epsilon_3)\Gamma(2+\epsilon_1)}{\Gamma(1+\epsilon_1)\Gamma(1+\epsilon_2)\Gamma(1+\epsilon_3)} \frac{\Gamma(1-\epsilon_1)\Gamma(1-\epsilon_3)\Gamma(2-\epsilon_2)}{\Gamma(1+\epsilon_1)\Gamma(1+\epsilon_3)\Gamma(\epsilon_2)} \text{Diagram 6} \\
 & + \frac{\Gamma(1-\epsilon_2)\Gamma(-\epsilon_1)\Gamma(-\epsilon_3)}{\Gamma(1+\epsilon_1)\Gamma(1+\epsilon_2)\Gamma(1+\epsilon_3)} \text{Diagram 7} = J \frac{1}{\epsilon_2\epsilon_3} \text{Diagram 8} \\
 & - J \frac{1}{\epsilon_1\epsilon_2\epsilon_3} \frac{1}{1+\epsilon_3} \text{Diagram 9} + J \frac{1}{\epsilon_1\epsilon_3} \text{Diagram 10} =
 \end{aligned}$$

Proof of the loop reduction in d=4 dimensions. Part II

$$\begin{aligned}
 &= J \frac{1}{\varepsilon_2 \varepsilon_3} \frac{2+\varepsilon_3}{1-\varepsilon_1} \begin{array}{c} 1+\varepsilon_1 \\ -\varepsilon_3 \\ \diagdown \quad \diagup \\ 1+\varepsilon_3 \\ \diagup \quad \diagdown \\ 1+\varepsilon_1 \quad 1+\varepsilon_2 \end{array} - J \frac{1}{\varepsilon_1 \varepsilon_2 \varepsilon_3} \frac{1}{1+\varepsilon_3} \begin{array}{c} 1+\varepsilon_1 \\ \diagdown \quad \diagup \\ 1+\varepsilon_3 \\ \diagup \quad \diagdown \\ 1+\varepsilon_2 \end{array} + J \frac{1}{\varepsilon_1 \varepsilon_3} \frac{2+\varepsilon_3}{1-\varepsilon_2} \begin{array}{c} 1+\varepsilon_1 \\ 1+\varepsilon_2 \\ \diagdown \quad \diagup \\ 1+\varepsilon_3 \\ \diagup \quad \diagdown \\ 1+\varepsilon_2 \quad 1+\varepsilon_1 \end{array} \\
 &= J \frac{1}{\varepsilon_2 \varepsilon_3} \frac{2+\varepsilon_3}{1+\varepsilon_3} \begin{array}{c} 1+\varepsilon_1 \\ -\varepsilon_3 \\ \diagdown \quad \diagup \\ 1+\varepsilon_3 \\ \diagup \quad \diagdown \\ 1+\varepsilon_1 \quad 1+\varepsilon_2 \end{array} - J \frac{1}{\varepsilon_1 \varepsilon_2 \varepsilon_3} \frac{1}{1+\varepsilon_3} \begin{array}{c} 1+\varepsilon_1 \\ \diagdown \quad \diagup \\ 1+\varepsilon_3 \\ \diagup \quad \diagdown \\ 1+\varepsilon_2 \end{array} + J \frac{1}{\varepsilon_1 \varepsilon_3} \frac{2+\varepsilon_3}{1+\varepsilon_3} \begin{array}{c} 1+\varepsilon_1 \\ 1+\varepsilon_2 \\ \diagdown \quad \diagup \\ 1+\varepsilon_3 \\ \diagup \quad \diagdown \\ 1+\varepsilon_2 \quad 1+\varepsilon_1 \end{array} \\
 &= J \frac{1}{\varepsilon_2 \varepsilon_3} \frac{\Gamma(1-\varepsilon_3)\Gamma(2+\varepsilon_3)}{\Gamma(1+\varepsilon_3)\Gamma(-\varepsilon_3)} \frac{2+\varepsilon_3, 1-\varepsilon_3}{2+\varepsilon_3} \begin{array}{c} 1+\varepsilon_1 \\ -\varepsilon_1-\varepsilon_3 \\ \diagdown \quad \diagup \\ 1+\varepsilon_3 \\ \diagup \quad \diagdown \\ 1+\varepsilon_1 \quad 1+\varepsilon_2 \end{array} + J \frac{1}{\varepsilon_1 \varepsilon_3} \frac{\Gamma(1-\varepsilon_3)\Gamma(2+\varepsilon_3)}{\Gamma(1+\varepsilon_3)\Gamma(-\varepsilon_3)} \frac{2+\varepsilon_3, 1-\varepsilon_3}{2+\varepsilon_3} \begin{array}{c} 1+\varepsilon_1 \\ 1+\varepsilon_2 \\ \diagdown \quad \diagup \\ 1+\varepsilon_3 \\ \diagup \quad \diagdown \\ 1+\varepsilon_2 \quad 1+\varepsilon_1 \end{array} \\
 &- J \frac{1}{\varepsilon_1 \varepsilon_2 \varepsilon_3} \frac{1}{1+\varepsilon_3} \begin{array}{c} 1+\varepsilon_1 \\ \diagdown \quad \diagup \\ 1+\varepsilon_3 \\ \diagup \quad \diagdown \\ 1+\varepsilon_2 \end{array} = J \frac{1}{\varepsilon_2 \varepsilon_3} \frac{\Gamma(1-\varepsilon_3)\Gamma(2+\varepsilon_3)}{\Gamma(1+\varepsilon_3)\Gamma(-\varepsilon_3)} \frac{\Gamma(1+\varepsilon_3)\Gamma(-\varepsilon_3)}{\Gamma(1-\varepsilon_3)\Gamma(2+\varepsilon_3)} \begin{array}{c} 1+\varepsilon_1 \\ -\varepsilon_1-\varepsilon_3 \\ \diagdown \quad \diagup \\ 1+\varepsilon_3 \\ \diagup \quad \diagdown \\ 1+\varepsilon_1 \quad 1+\varepsilon_2 \end{array} \\
 &- J \frac{1}{\varepsilon_1 \varepsilon_2 \varepsilon_3} \frac{1}{1+\varepsilon_3} \begin{array}{c} 1+\varepsilon_1 \\ \diagdown \quad \diagup \\ 1+\varepsilon_3 \\ \diagup \quad \diagdown \\ 1+\varepsilon_2 \end{array} + J \frac{1}{\varepsilon_1 \varepsilon_3} \begin{array}{c} 1+\varepsilon_1 \\ 1+\varepsilon_2 \\ \diagdown \quad \diagup \\ 2+\varepsilon_3 \\ \diagup \quad \diagdown \\ 1+\varepsilon_3 \end{array}
 \end{aligned}$$

Proof of the loop reduction in d=4 dimensions. Part III

$$\begin{aligned}
 &= J \frac{1}{\varepsilon_2 \varepsilon_3} \frac{\Gamma(1-\varepsilon_1)\Gamma(2+\varepsilon_1+\varepsilon_3)\Gamma(1-\varepsilon_3)}{\Gamma(1+\varepsilon_1)\Gamma(-\varepsilon_1-\varepsilon_3)\Gamma(1+\varepsilon_3)} \text{---} \text{triangle} \text{---} -J \frac{1}{\varepsilon_1 \varepsilon_2 \varepsilon_3} \frac{1}{1+\varepsilon_3} \text{---} \text{triangle} \text{---} \\
 &+ J \frac{1}{\varepsilon_1 \varepsilon_3} \frac{\Gamma(1-\varepsilon_2)\Gamma(2-\varepsilon_1)\Gamma(1-\varepsilon_3)}{\Gamma(1+\varepsilon_2)\Gamma(\varepsilon_1)\Gamma(1+\varepsilon_3)} \text{---} \text{triangle} \text{---} \\
 &= -J^2 \frac{1}{\varepsilon_2 \varepsilon_3^2} \frac{1}{1+\varepsilon_3} \text{---} \text{triangle} \text{---} -J \frac{1}{\varepsilon_1 \varepsilon_2 \varepsilon_3} \frac{1}{1+\varepsilon_3} \text{---} \text{triangle} \text{---} -J^2 \frac{1}{\varepsilon_1 \varepsilon_3^2} \frac{1}{1+\varepsilon_3} \text{---} \text{triangle} \text{---} \\
 &= -\frac{1}{(1+\varepsilon_3)\varepsilon_3} J \left[\frac{J}{\varepsilon_2 \varepsilon_3} \text{---} \text{triangle} \text{---} + \frac{1}{\varepsilon_1 \varepsilon_2} \text{---} \text{triangle} \text{---} + \frac{J}{\varepsilon_1 \varepsilon_3} \text{---} \text{triangle} \text{---} \right]
 \end{aligned}$$

Loop reduction in the momentum space

$$\begin{aligned}
 & p_3 \rightarrow \left[\text{triangle diagram with external momenta } p_1, p_2, p_3 \text{ and internal lines } 1+\epsilon_1, 1+\epsilon_2, 1+\epsilon_3 \right] = J \frac{\pi^2}{(p_3^2)^{1-\epsilon_3}} \left[\frac{1}{\epsilon_1 \epsilon_2} \left[\text{triangle diagram with external momenta } p_1, p_2 \text{ and internal lines } 1+\epsilon_3 \right] + \right. \\
 & \left. + (p_2^2)^{\epsilon_2} \frac{1}{\epsilon_2 \epsilon_3} \left[\text{triangle diagram with external momenta } p_1, p_3 \text{ and internal lines } 1-\epsilon_1 \right] + (p_1^2)^{\epsilon_1} \frac{1}{\epsilon_1 \epsilon_3} \left[\text{triangle diagram with external momenta } p_2, p_3 \text{ and internal lines } 1-\epsilon_2 \right] \right]
 \end{aligned}$$

$$\int d^d p \, e^{ipx} \frac{1}{(p^2)^\alpha} = \pi^{d/2} \frac{\Gamma(d/2 - \alpha)}{\Gamma(\alpha)} \left(\frac{4}{x^2} \right)^{d/2 - \alpha} \Rightarrow$$

$$\frac{1}{(x^2)^\alpha} = \pi^{-d/2} 4^{-\alpha} \frac{\Gamma(d/2 - \alpha)}{\Gamma(\alpha)} \int d^d p \, e^{ipx} \frac{1}{(p^2)^{d/2 - \alpha}}.$$

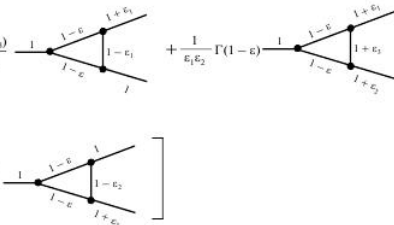
Loop reduction in non-integer dimensions $d = 4 - 2\varepsilon$

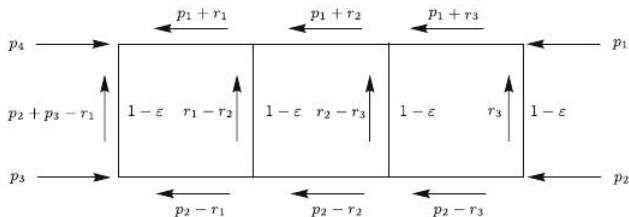
$$\begin{aligned}
 & \text{Diagram 1} = - \frac{\Gamma(\varepsilon_2)\Gamma(2+\varepsilon_3-\varepsilon)\Gamma(1+\varepsilon_1-\varepsilon)}{\Gamma(1+\varepsilon_2)\Gamma(1+\varepsilon_3-\varepsilon)\Gamma(1+\varepsilon_1-\varepsilon)} \text{Diagram 2} \\
 & \frac{\Gamma(\varepsilon_2)\Gamma(2+\varepsilon_1-\varepsilon)\Gamma(1+\varepsilon_1-\varepsilon)}{\Gamma(1+\varepsilon_2)\Gamma(1+\varepsilon_3-\varepsilon)\Gamma(1+\varepsilon_1-\varepsilon)} \text{Diagram 3} + \frac{\Gamma(1-\varepsilon-\varepsilon_2)\Gamma(-\varepsilon_3)\Gamma(-\varepsilon_1)}{\Gamma(1+\varepsilon_2)\Gamma(1+\varepsilon_3-\varepsilon)\Gamma(1+\varepsilon_1-\varepsilon)} \text{Diagram 4} \\
 & = - \frac{\Gamma(\varepsilon_2)\Gamma(2+\varepsilon_3-\varepsilon)\Gamma(1+\varepsilon_1-\varepsilon)}{\Gamma(1+\varepsilon_2)\Gamma(1+\varepsilon_3-\varepsilon)\Gamma(1+\varepsilon_1-\varepsilon)} \frac{\Gamma(1-\varepsilon_1-\varepsilon)\Gamma(-\varepsilon_3)\Gamma(1-\varepsilon_2)}{\Gamma(1+\varepsilon_1)\Gamma(2+\varepsilon_3-\varepsilon)\Gamma(1+\varepsilon_2-\varepsilon)} \text{Diagram 5} \\
 & \quad - \frac{\Gamma(\varepsilon_2)\Gamma(2+\varepsilon_1-\varepsilon)\Gamma(1+\varepsilon_3-\varepsilon)}{\Gamma(1+\varepsilon_2)\Gamma(1+\varepsilon_3-\varepsilon)\Gamma(1+\varepsilon_1-\varepsilon)} \frac{\Gamma(1-\varepsilon_3)\Gamma(2-\varepsilon_2-\varepsilon)\Gamma(1-\varepsilon_1-\varepsilon)}{\Gamma(1+\varepsilon_3-\varepsilon)\Gamma(\varepsilon_2)\Gamma(1+\varepsilon_1)} \text{Diagram 6} \\
 & \quad + \frac{\Gamma(1-\varepsilon-\varepsilon_2)\Gamma(-\varepsilon_3)\Gamma(-\varepsilon_1)}{\Gamma(1+\varepsilon_2)\Gamma(1+\varepsilon_3-\varepsilon)\Gamma(1+\varepsilon_1-\varepsilon)} \text{Diagram 7} = \frac{J}{\varepsilon_2\varepsilon_3} \frac{\Gamma(1-\varepsilon_1-\varepsilon)\Gamma(1+\varepsilon_1-\varepsilon)}{\Gamma(1+\varepsilon_1)} \frac{2+\varepsilon_1-\varepsilon}{\Gamma(1-\varepsilon_1)} \text{Diagram 8} \\
 & \quad + \frac{J}{\varepsilon_1\varepsilon_2} \Gamma^2(1-\varepsilon) \text{Diagram 9} + \frac{J}{\varepsilon_1\varepsilon_3} \frac{\Gamma(1-\varepsilon_2-\varepsilon)\Gamma(1+\varepsilon_2-\varepsilon)}{\Gamma(1+\varepsilon_2)} \frac{2+\varepsilon_1-\varepsilon}{\Gamma(1-\varepsilon_2)} \text{Diagram 10} \\
 & = \frac{J}{\varepsilon_1\varepsilon_3} \frac{\Gamma(1-\varepsilon_1-\varepsilon)\Gamma(1+\varepsilon_1-\varepsilon)}{\Gamma(1+\varepsilon_1)} \frac{2+\varepsilon_1-\varepsilon}{\Gamma(1-\varepsilon_1)} \text{Diagram 11} + \frac{J}{\varepsilon_1\varepsilon_2} \Gamma^2(1-\varepsilon) \text{Diagram 12} \\
 & \quad + \frac{J}{\varepsilon_1\varepsilon_3} \frac{\Gamma(1-\varepsilon_2-\varepsilon)\Gamma(1+\varepsilon_2-\varepsilon)}{\Gamma(1+\varepsilon_2)} \frac{2+\varepsilon_1-\varepsilon}{\Gamma(1-\varepsilon_2)} \text{Diagram 13}
 \end{aligned}$$

Loop reduction in non-integer dimensions. Part II

$$\begin{aligned}
 &= \frac{J}{\varepsilon_2 \varepsilon_3} \frac{\Gamma(1-\varepsilon_1-\varepsilon)\Gamma(1+\varepsilon_1-\varepsilon)\Gamma(1-\varepsilon_3)\Gamma(1-\varepsilon)\Gamma(2+\varepsilon_3-\varepsilon)}{\Gamma(1+\varepsilon_1)\Gamma(1-\varepsilon_1)\Gamma(1+\varepsilon_3-\varepsilon)\Gamma(1)\Gamma(-\varepsilon_3)} \text{Diagram 1} \\
 &+ \frac{J}{\varepsilon_1 \varepsilon_3} \frac{\Gamma(1-\varepsilon_2-\varepsilon)\Gamma(1+\varepsilon_2-\varepsilon)}{\Gamma(1+\varepsilon_2)\Gamma(1-\varepsilon_2)} \frac{\Gamma(1-\varepsilon_3)\Gamma(1-\varepsilon)\Gamma(2+\varepsilon_3-\varepsilon)}{\Gamma(1+\varepsilon_3-\varepsilon)\Gamma(1)\Gamma(-\varepsilon_3)} \text{Diagram 2} \\
 &+ \frac{J}{\varepsilon_1 \varepsilon_2} \Gamma^2(1-\varepsilon) \text{Diagram 3} = \frac{J}{\varepsilon_2 \varepsilon_3} \frac{\Gamma(1-\varepsilon_1-\varepsilon)\Gamma(1+\varepsilon_1-\varepsilon)}{\Gamma(1+\varepsilon_1)\Gamma(1-\varepsilon_1)} \text{Diagram 4} \\
 &- \frac{J}{\varepsilon_1 \varepsilon_2 \varepsilon_3} \frac{\Gamma(1-\varepsilon)}{1+\varepsilon_3-\varepsilon} \text{Diagram 5} + \frac{J}{\varepsilon_1 \varepsilon_3} \frac{\Gamma(1-\varepsilon_2-\varepsilon)\Gamma(1+\varepsilon_2-\varepsilon)}{\Gamma(1+\varepsilon_2)\Gamma(1-\varepsilon_2)} \frac{1}{\Gamma(1-\varepsilon_3)} \text{Diagram 6} \\
 &= \frac{J}{\varepsilon_2 \varepsilon_3} \frac{\Gamma(1-\varepsilon_1-\varepsilon)\Gamma(1+\varepsilon_1-\varepsilon)}{\Gamma(1+\varepsilon_1)\Gamma(1-\varepsilon_1)} \frac{\Gamma(1-\varepsilon_3)\Gamma(1-\varepsilon)\Gamma(2-\varepsilon_3-\varepsilon)}{\Gamma(1+\varepsilon_3-\varepsilon)\Gamma(1+\varepsilon_3)\Gamma(\varepsilon_3)} \text{Diagram 7} \\
 &- \frac{J}{\varepsilon_1 \varepsilon_2 \varepsilon_3} \frac{\Gamma(1-\varepsilon)}{1+\varepsilon_3-\varepsilon} \text{Diagram 8} + \frac{J}{\varepsilon_1 \varepsilon_3} \frac{\Gamma(1-\varepsilon_2-\varepsilon)\Gamma(1+\varepsilon_2-\varepsilon)\Gamma(1-\varepsilon_3)\Gamma(1-\varepsilon)\Gamma(2-\varepsilon_3-\varepsilon)}{\Gamma(1+\varepsilon_2)\Gamma(1-\varepsilon_2)\Gamma(1+\varepsilon_3-\varepsilon)\Gamma(1+\varepsilon_3)\Gamma(\varepsilon_3)} \text{Diagram 9}
 \end{aligned}$$

Loop reduction in non-integer dimensions. Part III

$$= -\frac{J}{\varepsilon_3(1+\varepsilon_3-\varepsilon)} \left[\frac{1}{\varepsilon_2\varepsilon_3} \frac{\Gamma(1-\varepsilon_1-\varepsilon)\Gamma(1-\varepsilon_2-\varepsilon)\Gamma(1-\varepsilon_3)}{\Gamma(1+\varepsilon_1)\Gamma(1+\varepsilon_2)\Gamma(1+\varepsilon_3-\varepsilon)} \text{Diagram 1} + \frac{1}{\varepsilon_1\varepsilon_2} \Gamma(1-\varepsilon) \text{Diagram 2} \right. \\ \left. + \frac{1}{\varepsilon_1\varepsilon_3} \frac{\Gamma(1-\varepsilon_1-\varepsilon)\Gamma(1-\varepsilon_2-\varepsilon)\Gamma(1-\varepsilon_3)}{\Gamma(1+\varepsilon_1)\Gamma(1+\varepsilon_2)\Gamma(1+\varepsilon_3-\varepsilon)} \text{Diagram 3} \right]$$




Formulas to prove

$$\oint_C dz_2 dz_3 D^{(u,v)}[1 + \varepsilon_1 - z_3, 1 + \varepsilon_2 - z_2, 1 + \varepsilon_3] \times \\ \times D^{(z_2, z_3)}[1 + \varepsilon_2, 1 + \varepsilon_1, 1 + \varepsilon_3] = \\ J \left[\frac{D^{(u, v - \varepsilon_2)}[1 - \varepsilon_1]}{\varepsilon_2 \varepsilon_3} + \frac{D^{(u, v)}[1 + \varepsilon_3]}{\varepsilon_1 \varepsilon_2} + \frac{D^{(u - \varepsilon_1, v)}[1 - \varepsilon_2]}{\varepsilon_1 \varepsilon_3} \right],$$

$$D^{(z_2, z_3)}[\nu_1, \nu_2, \nu_3] = \frac{\Gamma(-z_2) \Gamma(-z_3) \Gamma(-z_2 - \nu_2 - \nu_3 + d/2)}{\prod_i \Gamma(\nu_i)} \\ \times \frac{\Gamma(-z_3 - \nu_1 - \nu_3 + d/2) \Gamma(z_2 + z_3 + \nu_3) \Gamma(\sum \nu_i - d/2 + z_3 + z_2)}{\Gamma(d - \sum_i \nu_i)},$$

$$\varepsilon_1 + \varepsilon_2 + \varepsilon_3 = 0 \quad \text{and} \quad D^{(u, v)}[1 + \nu] \equiv D^{(u, v)}[1, 1, 1 + \nu].$$

$$\oint_C dz \frac{\Gamma(\lambda_1 + z)\Gamma(\lambda_2 + z)\Gamma(\lambda_3 - z)\Gamma(\lambda_4 - z)}{\Gamma(\lambda_1 + \lambda_3)\Gamma(\lambda_1 + \lambda_4)\Gamma(\lambda_2 + \lambda_3)\Gamma(\lambda_2 + \lambda_4)} =$$

$$\oint_C dz \frac{\Gamma(\lambda_1 + z)\Gamma(\lambda_2 + z)\Gamma(\lambda_3 + z)\Gamma(\lambda_4 - z)\Gamma(\lambda_5 - z)}{\Gamma(\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 + \lambda_5 + z)} =$$

$$\frac{\Gamma(\lambda_1 + \lambda_4)\Gamma(\lambda_2 + \lambda_4)\Gamma(\lambda_3 + \lambda_4)\Gamma(\lambda_1 + \lambda_5)\Gamma(\lambda_2 + \lambda_5)\Gamma(\lambda_3 + \lambda_5)}{\Gamma(\lambda_1 + \lambda_2 + \lambda_4 + \lambda_5)\Gamma(\lambda_1 + \lambda_3 + \lambda_4 + \lambda_5)\Gamma(\lambda_2 + \lambda_3 + \lambda_4 + \lambda_5)}$$

$$\begin{aligned}
 & D^{(z_2, z_3)}[1 + \varepsilon_2, 1 + \varepsilon_1, 1 + \varepsilon_3] = \\
 & \frac{\Gamma(-z_2)\Gamma(-z_3)\Gamma(-z_2 + \varepsilon_2)\Gamma(-z_3 + \varepsilon_1)\Gamma(1 + z_2 + z_3)}{\Gamma(1 + \varepsilon_1)\Gamma(1 + \varepsilon_2)\Gamma(1 + \varepsilon_3)} \times \\
 & \quad \times \Gamma(1 + z_2 + z_3 + \varepsilon_3), \\
 & D^{(u, v)}[1 + \varepsilon_1 - z_3, 1 + \varepsilon_2 - z_2, 1 + \varepsilon_3] = \\
 & \frac{\Gamma(-u)\Gamma(-v)\Gamma(-u + \varepsilon_1 + z_2)\Gamma(-v + \varepsilon_2 + z_3)}{\Gamma(1 + \varepsilon_1 - z_3)\Gamma(1 + \varepsilon_2 - z_2)\Gamma(1 + \varepsilon_3)\Gamma(1 + z_2 + z_3)} \times \\
 & \quad \times \Gamma(1 + u + v + \varepsilon_3)\Gamma(1 - z_2 - z_3 + u + v), \\
 & D^{(u, v)}[1 + \varepsilon_1 - z_3, 1 + \varepsilon_2 - z_2, 1 + \varepsilon_3] D^{(z_2, z_3)}[1 + \varepsilon_2, 1 + \varepsilon_1, 1 + \varepsilon_3] \\
 = & \frac{\Gamma(-u)\Gamma(-v)\Gamma(1 + u + v + \varepsilon_3)}{\Gamma(1 + \varepsilon_1)\Gamma(1 + \varepsilon_2)\Gamma^2(1 + \varepsilon_3)} \frac{1}{\varepsilon_1 - z_3} \frac{1}{\varepsilon_2 - z_2} \Gamma(-z_2)\Gamma(-z_3) \times \\
 & \quad \times \Gamma(1 + z_2 + z_3 + \varepsilon_3)\Gamma(-u + \varepsilon_1 + z_2)\Gamma(-v + \varepsilon_2 + z_3) \times \\
 & \quad \times \Gamma(1 - z_2 - z_3 + u + v).
 \end{aligned}$$

The term $D^{(u,v)}[1 + \varepsilon_3]$

$$\begin{aligned} & \frac{D^{(u,v)}[1 + \varepsilon_3]}{\varepsilon_1 \varepsilon_2} = \\ & \frac{1}{\varepsilon_1 \varepsilon_2} \frac{\Gamma(-u)\Gamma(-v)\Gamma(-\varepsilon_3 - u)\Gamma(-\varepsilon_3 - v)\Gamma^2(1 + \varepsilon_3 + u + v)}{\Gamma(1 - \varepsilon_3)\Gamma(1 + \varepsilon_3)}, \\ & \frac{D^{(u,v-\varepsilon_2)}[1 - \varepsilon_1]}{\varepsilon_2 \varepsilon_3} = \\ & \frac{1}{\varepsilon_2 \varepsilon_3} \frac{\Gamma(-u)\Gamma(\varepsilon_2 - v)\Gamma(\varepsilon_1 - u)\Gamma(-\varepsilon_3 - v)\Gamma^2(1 + \varepsilon_3 + u + v)}{\Gamma(1 - \varepsilon_1)\Gamma(1 + \varepsilon_1)}, \\ & \frac{D^{(u-\varepsilon_1,v)}[1 - \varepsilon_2]}{\varepsilon_1 \varepsilon_3} = \\ & \frac{1}{\varepsilon_1 \varepsilon_3} \frac{\Gamma(\varepsilon_1 - u)\Gamma(-v)\Gamma(-\varepsilon_3 - u)\Gamma(\varepsilon_2 - v)\Gamma^2(1 + \varepsilon_3 + u + v)}{\Gamma(1 - \varepsilon_2)\Gamma(1 + \varepsilon_2)} \end{aligned}$$

Simple trick

$$\begin{aligned} & \frac{1}{z_3 - \varepsilon_1} \frac{1}{z_2 - \varepsilon_2} \Gamma(-z_2) \Gamma(-z_3) \Gamma(1 + z_2 + z_3 + \varepsilon_3) \Gamma(-u + \varepsilon_1 + z_2) \times \\ & \quad \times \Gamma(-v + \varepsilon_2 + z_3) \Gamma(1 - z_2 - z_3 + u + v) = \\ & = \frac{z_2 + z_2 + \varepsilon_3}{(z_3 - \varepsilon_1)(z_2 - \varepsilon_2)} \Gamma(-z_2) \Gamma(-z_3) \Gamma(z_2 + z_3 + \varepsilon_3) \Gamma(-u + \varepsilon_1 + z_2) \times \\ & \quad \times \Gamma(-v + \varepsilon_2 + z_3) \Gamma(1 - z_2 - z_3 + u + v) = \\ & = \left(\frac{1}{z_3 - \varepsilon_1} + \frac{1}{z_2 - \varepsilon_2} \right) \Gamma(-z_2) \Gamma(-z_3) \Gamma(z_2 + z_3 + \varepsilon_3) \Gamma(-u + \varepsilon_1 + z_2) \times \\ & \quad \times \Gamma(-v + \varepsilon_2 + z_3) \Gamma(1 - z_2 - z_3 + u + v). \end{aligned}$$

$$\oint_C dz_2 \frac{1}{z_2 - \varepsilon_2} \Gamma(-z_2) \Gamma(-u + \varepsilon_1 + z_2) \oint_C dz_3 \Gamma(-z_3) \Gamma(z_2 + z_3 + \varepsilon_3) \times \\ \Gamma(-v + \varepsilon_2 + z_3) \Gamma(1 - z_2 - z_3 + u + v).$$

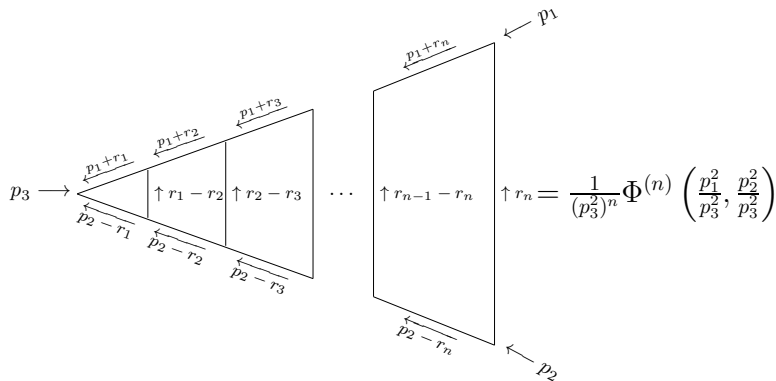
Application of the Barnes lemmas

$$\begin{aligned}
 & \oint_C dz_2 \frac{1}{z_2 - \varepsilon_2} \Gamma(-z_2) \Gamma(-u + \varepsilon_1 + z_2) \oint_C dz_3 \Gamma(-z_3) \Gamma(z_2 + z_3 + \varepsilon_3) \times \\
 & \quad \Gamma(-v + \varepsilon_2 + z_3) \Gamma(1 - z_2 - z_3 + u + v) = \\
 & \quad \oint_C dz_2 \frac{1}{z_2 - \varepsilon_2} \Gamma(-z_2) \Gamma(-u + \varepsilon_1 + z_2) \times \\
 & \quad \times \frac{\Gamma(z_2 + \varepsilon_3) \Gamma(-v + \varepsilon_2) \Gamma(1 + \varepsilon_3 + u + v) \Gamma(1 + \varepsilon_2 + u - z_2)}{\Gamma(1 + u - \varepsilon_1)} = \\
 & = \frac{\Gamma(-v + \varepsilon_2) \Gamma(1 + \varepsilon_3 + u + v)}{\Gamma(1 + u - \varepsilon_1)} \oint_C dz_2 \frac{1}{z_2 - \varepsilon_2} \Gamma(z_2 + \varepsilon_3) \Gamma(-z_2) \times \\
 & \quad \times \Gamma(-u + \varepsilon_1 + z_2) \Gamma(1 + \varepsilon_2 + u - z_2).
 \end{aligned}$$

$$\begin{aligned}
 & \frac{\Gamma(-v + \varepsilon_2) \Gamma(1 + \varepsilon_3 + u + v)}{\Gamma(1 + u - \varepsilon_1)} \oint_C dz_2 \frac{1}{z_2 - \varepsilon_2} \Gamma(z_2 + \varepsilon_3) \Gamma(-z_2) \times \\
 & \quad \times \Gamma(-u + \varepsilon_1 + z_2) \Gamma(1 + \varepsilon_2 + u - z_2) = \\
 & - \frac{\Gamma(-v + \varepsilon_2) \Gamma(1 + \varepsilon_3 + u + v)}{\Gamma(1 + u - \varepsilon_1)} \oint_C dz_2 \frac{1}{z_2 + \varepsilon_2} \Gamma(-z_2 + \varepsilon_3) \Gamma(z_2) \times \\
 & \quad \times \Gamma(-u + \varepsilon_1 - z_2) \Gamma(1 + \varepsilon_2 + u + z_2) = \\
 & - \frac{\Gamma(-v + \varepsilon_2) \Gamma(1 + \varepsilon_3 + u + v)}{\Gamma(1 + u - \varepsilon_1)} \oint_C dz_2 \frac{\Gamma(z_2 + \varepsilon_2)}{\Gamma(1 + z_2 + \varepsilon_2)} \Gamma(-z_2 + \varepsilon_3) \Gamma(z_2) \times \\
 & \quad \times \Gamma(-u + \varepsilon_1 - z_2) \Gamma(1 + \varepsilon_2 + u + z_2) =
 \end{aligned}$$

$$\begin{aligned}
 & - \frac{\Gamma(-\nu + \varepsilon_2) \Gamma(1 + \varepsilon_3 + u + \nu)}{\Gamma(1 + u - \varepsilon_1)} \Gamma(\varepsilon_3) \Gamma(-\varepsilon_1) \Gamma(1 + u - \varepsilon_1) \times \\
 & \qquad \frac{\Gamma(-u - \varepsilon_3) \Gamma(-u + \varepsilon_1) \Gamma(1 - \varepsilon_3)}{\Gamma(1 + \varepsilon_2) \Gamma(-u)} = \\
 & \qquad \frac{1}{\varepsilon_1 \varepsilon_3} \frac{\Gamma(1 - \varepsilon_1) \Gamma(1 + \varepsilon_3) \Gamma(1 - \varepsilon_3)}{\Gamma(1 + \varepsilon_2)} \times \\
 & \frac{\Gamma(-\nu + \varepsilon_2) \Gamma(1 + \varepsilon_3 + u + \nu) \Gamma(-u - \varepsilon_3) \Gamma(-u + \varepsilon_1)}{\Gamma(-u)} \sim D^{(u - \varepsilon_1, \nu)} [1 - \varepsilon_2]
 \end{aligned}$$

Ladder diagrams



Explicit form of Usykina-Davydychev functions

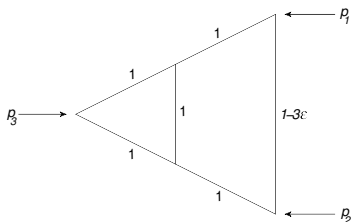
$$\Phi^{(n)}(x, y) = \oint dz_2 dz_3 x^{z_2} y^{z_3} \mathcal{M}^{(n)}(z_2, z_3)$$

The explicit form of the function is given in Davydychev and Usykina papers:

$$\Phi^{(n)}(x, y) = -\frac{1}{n! \lambda} \sum_{j=n}^{2n} \frac{(-1)^j j! \ln^{2n-j}(y/x)}{(j-n)!(2n-j)!} \left[\text{Li}_j\left(-\frac{1}{\rho x}\right) - \text{Li}_j(-\rho y) \right], \quad (1)$$

$$\rho = \frac{2}{1-x-y+\lambda}, \quad \lambda = \sqrt{(1-x-y)^2 - 4xy}.$$

Diagram which we calculate by the same trick in
 $d = 4 - 2\varepsilon$

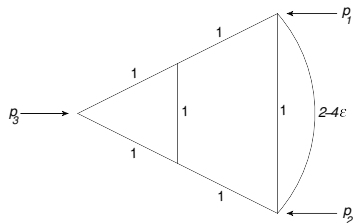


$$\oint_C dz_2 dz_3 D^{(u,v)}[1 - z_3, 1 - z_2, 1 - 3\varepsilon] \times D^{(z_2, z_3)}[1, 1, 1] =$$

$$\frac{1}{\varepsilon^2} \frac{\Gamma^2(1 - \varepsilon)\Gamma(1 + \varepsilon)}{\Gamma(1 - 2\varepsilon)} D^{(u,v)}[1 - 3\varepsilon]$$

$$- \frac{1}{2\varepsilon^2} \frac{\Gamma(1 - \varepsilon)\Gamma(1 + 2\varepsilon)\Gamma(1 - 2\varepsilon)}{\Gamma(1 - 3\varepsilon)\Gamma(1 + \varepsilon)} \left(D^{(u-\varepsilon, v)}[1 - 2\varepsilon] + D^{(u, v-\varepsilon)}[1 - 2\varepsilon] \right)$$

Integral Equation for this diagram

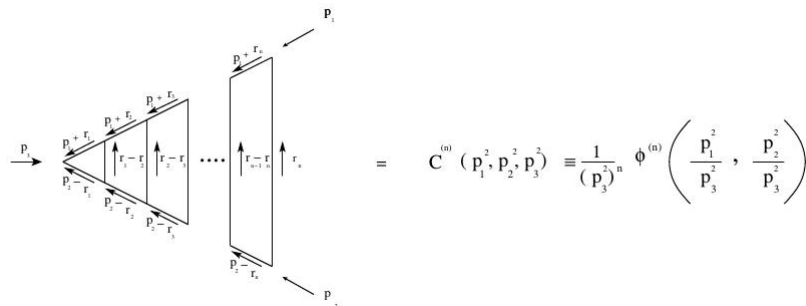


$$\oint_C dz_2 dz_3 D^{(u,v)}[-z_3, -z_2, 2-4\epsilon] \times M(z_2, z_3) =$$

$$\frac{1}{\epsilon^2} \frac{\Gamma^2(1-\epsilon)\Gamma(1+\epsilon)}{\Gamma(1-2\epsilon)} D^{(u,v)}[1-3\epsilon]$$

$$-\frac{1}{2\epsilon^2} \frac{\Gamma(1-\epsilon)\Gamma(1+2\epsilon)\Gamma(1-2\epsilon)}{\Gamma(1-3\epsilon)\Gamma(1+\epsilon)} \left(D^{(u-\epsilon,v)}[1-2\epsilon] + D^{(u,v-\epsilon)}[1-2\epsilon] \right)$$

Ladder diagrams



$$C^{(n)}(p_1^2, p_2^2, p_3^2) \equiv \frac{1}{(p_3^2)^n} \phi^{(n)}\left(\frac{p_1^2}{p_3^2}, \frac{p_2^2}{p_3^2}\right)$$



V. V. Belokurov and N. I. Usyukina, "Calculation Of Ladder Diagrams In Arbitrary Order," J. Phys. A **16** (1983) 2811.






N. I. Usyukina and A. I. Davydychev, "An Approach to the evaluation of three and four point ladder diagrams," Phys. Lett. B **298** (1993) 363.







N. I. Usyukina and A. I. Davydychev, "Exact results for three and four point ladder diagrams with an arbitrary number of rungs," Phys. Lett. B **305** (1993) 136.



Fourier-invariance of UD functions, JHEP08(2008)106



(a) 



(b) ∂_{ω}^2  = $(-4\pi^2)$ 

(c) ∂_{ω}^2  = $(-4\pi^2)$ 

(d) $(\partial_{\omega}^2)^2$  = $(-4\pi^2)^2$ 

(e) $(\partial_{\omega}^2)^2$  = $(-4\pi^2)^2$ 

(f) $(\partial_{\omega}^2)^3$  = $(-4\pi^2)^3$ 

(g) $(\partial_{\omega}^2)^3$  = $(-4\pi^2)^3$ 

Fourier-invariance of UD functions via MB transform

$$\frac{1}{[31]^2} \Phi^{(n)} \left(\frac{[12]}{[31]}, \frac{[23]}{[31]} \right) = \frac{1}{(2\pi)^4} \int d^4 p_1 d^4 p_2 d^4 p_3 \delta(p_1 + p_2 + p_3) \times \\ \times e^{ip_2 x_2} e^{ip_1 x_1} e^{ip_3 x_3} \frac{1}{(p_2^2)^2} \Phi^{(n)} \left(\frac{p_1^2}{p_2^2}, \frac{p_3^2}{p_2^2} \right).$$

The explicit form of the function is given in Davydychev and Usyukina papers:

$$\Phi^{(n)}(x, y) = -\frac{1}{n! \lambda} \sum_{j=n}^{2n} \frac{(-1)^j j! \ln^{2n-j}(y/x)}{(j-n)!(2n-j)!} \left[\text{Li}_j \left(-\frac{1}{\rho x} \right) - \text{Li}_j(-\rho y) \right],$$

$$\rho = \frac{2}{1-x-y+\lambda}, \quad \lambda = \sqrt{(1-x-y)^2 - 4xy}.$$

Fourier-invariance of UD functions via MB transform

Mellin-Barnes transform for the ladder functions:

$$\Phi^{(n)}(x, y) = \oint dz_2 dz_3 x^{z_2} y^{z_3} \mathcal{M}^{(n)}(z_2, z_3)$$

$$\begin{aligned} \frac{1}{[31]^2} \Phi^{(n)}\left(\frac{[12]}{[31]}, \frac{[23]}{[31]}\right) &= \frac{1}{(2\pi)^4} \int d^4 p_1 d^4 p_2 d^4 p_3 \delta(p_1 + p_2 + p_3) \times \\ &\quad \times e^{ip_2 x_2} e^{ip_1 x_1} e^{ip_3 x_3} \frac{1}{(p_2^2)^2} \Phi^{(n)}\left(\frac{p_1^2}{p_2^2}, \frac{p_3^2}{p_2^2}\right) = \\ &= \frac{1}{(2\pi)^8} \int d^4 p_1 d^4 p_2 d^4 p_3 d^4 x_5 e^{ip_2(x_2 - x_5)} e^{ip_1(x_1 - x_5)} e^{ip_3(x_3 - x_5)} \times \\ &\quad \times \frac{1}{(p_2^2)^2} \Phi^{(n)}\left(\frac{p_1^2}{p_2^2}, \frac{p_3^2}{p_2^2}\right) = \end{aligned}$$

Fourier-invariance of UD functions via MB transform

$$\begin{aligned} &= \frac{1}{(2\pi)^8} \int d^4 p_1 d^4 p_2 d^4 p_3 d^4 x_5 \oint dz_2 dz_3 \frac{e^{ip_2(x_2-x_5)} e^{ip_1(x_1-x_5)} e^{ip_3(x_3-x_5)}}{(p_2^2)^{2+z_2+z_3} (p_1^2)^{-z_2} (p_3^2)^{-z_3}} \times \\ &\quad \times \mathcal{M}^{(n)}(z_2, z_3) = \\ &= \frac{(4\pi)^6}{(2\pi)^8} \int d^4 x_5 \oint dz_2 dz_3 \frac{\Gamma(-z_2-z_3)}{\Gamma(2+z_2+z_3)} \frac{\Gamma(2+z_2)}{\Gamma(-z_2)} \frac{\Gamma(2+z_3)}{\Gamma(-z_3)} \\ &\quad \times \frac{2^{2z_2+2z_3-2(2+z_2+z_3)} \mathcal{M}^{(n)}(z_2, z_3)}{(x_2-x_5)^{-z_2-z_3} (x_1-x_5)^{2+z_2} (x_3-x_5)^{2+z_3}} = \\ &= \oint dz_2 dz_3 \frac{\mathcal{M}^{(n)}(z_2, z_3)}{[12]^{-z_3} [23]^{-z_2} [31]^{2+z_2+z_3}} \end{aligned}$$



P. Allendes, N. Guerrero, I. Kondrashuk and E. A. Notte Cuello, "New four-dimensional integrals by Mellin-Barnes transform," J. Math. Phys. **51** (2010) 052304 [arXiv:0910.4805 [hep-th]].

“Orthogonality”

$$\frac{1}{[12]^{\alpha_3}[23]^{\alpha_1}[13]^{\alpha_2}} \sim \oint_C dudv \frac{[13]^u [23]^v}{[12]} \frac{1}{[12]^{\alpha_1+\alpha_2+\alpha_3}} \times \\ \oint_C dz_2 dz_3 D^{(z_2, z_3)}[d/2 - \alpha_1, d/2 - \alpha_2, d/2 - \alpha_3] \times \\ \times D^{(u, v)}[d/2 + z_3, d/2 + z_2, d - z_3 - z_2 - \alpha_1 - \alpha_2 - \alpha_3]$$

Two poles survive $u = -\alpha_2, v = -\alpha_1$

$$\oint_C dz_2 dz_3 D^{(u, v)}[-z_3, -z_2, 2 - 4\varepsilon] \times M(z_2, z_3) = \\ \frac{1}{\varepsilon^2} \frac{\Gamma^2(1 - \varepsilon)\Gamma(1 + \varepsilon)}{\Gamma(1 - 2\varepsilon)} D^{(u, v)}[1 - 3\varepsilon] \\ - \frac{1}{2\varepsilon^2} \frac{\Gamma(1 - \varepsilon)\Gamma(1 + 2\varepsilon)\Gamma(1 - 2\varepsilon)}{\Gamma(1 - 3\varepsilon)\Gamma(1 + \varepsilon)} \left(D^{(u - \varepsilon, v)}[1 - 2\varepsilon] + D^{(u, v - \varepsilon)}[1 - 2\varepsilon] \right)$$