Numerically Computing QCD Laplace Sum-Rules Using pySecDec

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QCD sum-rules is a methodology for computing hadron properties.

Properties → Masses, widths, mixing angles, decay rates,...

Hadrons → Mesons, baryons, glueballs, hybrids, diquarks, tetraquarks, pentaquarks,...

Citations →

- M.A. Shifman (ed), *Vacuum Structure and QCD Sum-Rules,* North-Holland (1992)
- S. Narison, *QCD as a Theory of Hadrons,* Cambridge University Press (2004)

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QCD correlation functions satisfy dispersion relations.

hadron spectral function

$$\Pi(q^2) = \frac{q^8}{\pi} \int_{t_0}^{\infty} \frac{\mathrm{Im}\Pi(t)}{t^4(t-q^2)}\, \mathrm{d}t + \cdots$$

hadron threshold

subtraction constants

Quark-hadron duality!

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The Borel transform suppresses contributions from excited states.

Borel transform

Borel parameter

$$\hat{\mathcal{B}} = \lim_{N,\, q^2 \to \infty} \frac{q^{2N}}{\Gamma(N)} \left(\frac{\mathrm{d}}{\mathrm{d}q^2} \right)^N \quad \text{where } \tau \equiv -\frac{N}{q^2}$$

$$\mathcal{R}_k(\tau) \equiv \frac{1}{\tau} \hat{\mathcal{B}} \left\{ q^{2k} \Pi\left(q^2\right) \right\}$$

$$= \int_{t_0}^{\infty} t^k \mathrm{e}^{-t\tau} \frac{1}{\pi} \mathrm{Im}\Pi(t)\, \mathrm{d}t$$

unsubtracted Laplace sum-rule (LSR)

exponential kernel

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The spectral function is split into resonance and continuum intervals.

resonances → $\rho(t)$

continuum threshold parameter → s_0

$$\mathrm{Im}\Pi(t) = \rho(t) + \theta(t - s_0)\mathrm{Im}\Pi^{\mathrm{QCD}}(t)$$

$$\mathcal{R}_k(\tau,\, s_0) = \mathcal{R}_k(\tau) - \int_{s_0}^{\infty} t^k \mathrm{e}^{-t\tau} \frac{1}{\pi} \mathrm{Im}\Pi^{\mathrm{QCD}}(t)\, \mathrm{d}t$$

$$= \int_{t_0}^{s_0} t^k \mathrm{e}^{-t\tau} \frac{1}{\pi} \rho(t)\, \mathrm{d}t$$

subtracted Laplace sum-rule (LSR) → $\mathcal{R}_k(\tau,\, s_0)$

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A single narrow resonance is often used to model the ground state.

$$\rho(t) = \pi f_H^2 \delta(t - m_H^2)$$

ground state hadron mass

$$M(\tau,\, s_0) \equiv \sqrt{\frac{\mathcal{R}_1(\tau,\, s_0)}{\mathcal{R}_0(\tau,\, s_0)}} = m_H$$

Predictions for s_0 and m_H extracted as best-fit parameters.

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The correlator is computed using the operator product expansion (OPE).

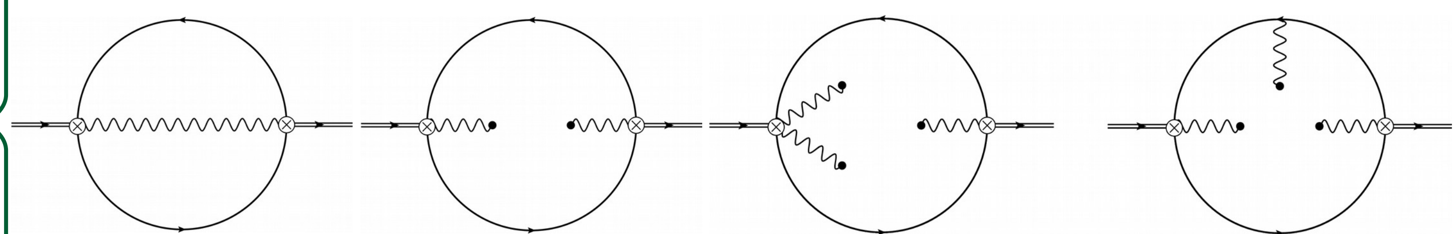
perturbation theory

Wilson coefficients

$$\Pi^{\rm QCD}(q^2) = \mathcal{I}(q^2) + \mathcal{C}_I(q^2)\langle \alpha G^2 \rangle + \mathcal{C}_{II}(q^2)\langle g^3 G^3 \rangle + \cdots$$

condensates

charmonium hybrid



R. Berg, D. Harnett, R.T. Kleiv, T.G. Steele, Phys. Rev. D86 (2012) 034002

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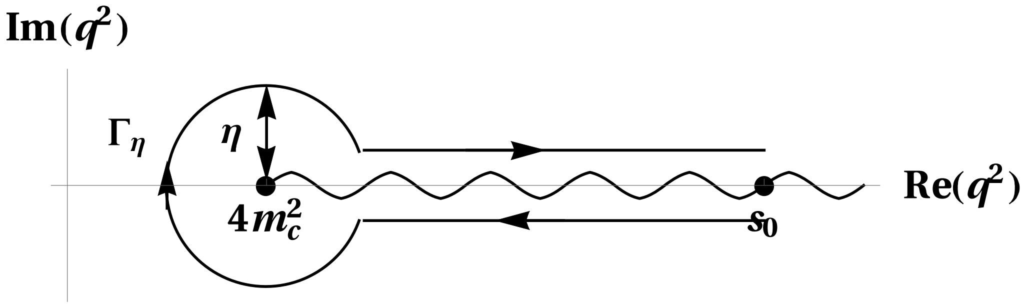
The Borel transform has an integral representation.



$$\mathcal{R}_k(\tau,\, s_0) = \int_{4m_c^2(1+\eta)}^{s_0} t^k \mathrm{e}^{-t\tau} \frac{1}{\pi} \mathrm{Im}\Pi^{\mathrm{QCD}}(t)\,\mathrm{d}t + \int_{\Gamma_\eta} q^{2k} \mathrm{e}^{-q^2\tau} \Pi^{\mathrm{QCD}}(q^2)\,\mathrm{d}q^2$$

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Divergent loop integrals are handled through dimensional regularization.

$$\int \frac{\mathrm{d}^4 p}{(2\pi)^4} \to \int \frac{\mathrm{d}^D p}{(2\pi)^D} \text{ where } D = 4 + 2\epsilon$$

Dimensionally regularized integrals can be difficult to compute due to:

- Number of external lines
- Number of loops
- Number of distinct masses (scales)

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pySecDec numerically calculates dimensionally regularized integrals.

http://secdec.hepforge.org

1. Loop integrands written in terms of Feynman parameters.
2. Sector decomposition used to isolate divergences.
3. Monte Carlo integration used to numerically evaluate Feynman parameter integrals.

- S. Borowka et al., Comput. Phys. Commun. 222 (2018) 313
- G. Heinrich, Int. J. Mod. Phys. A23 (2008) 1457

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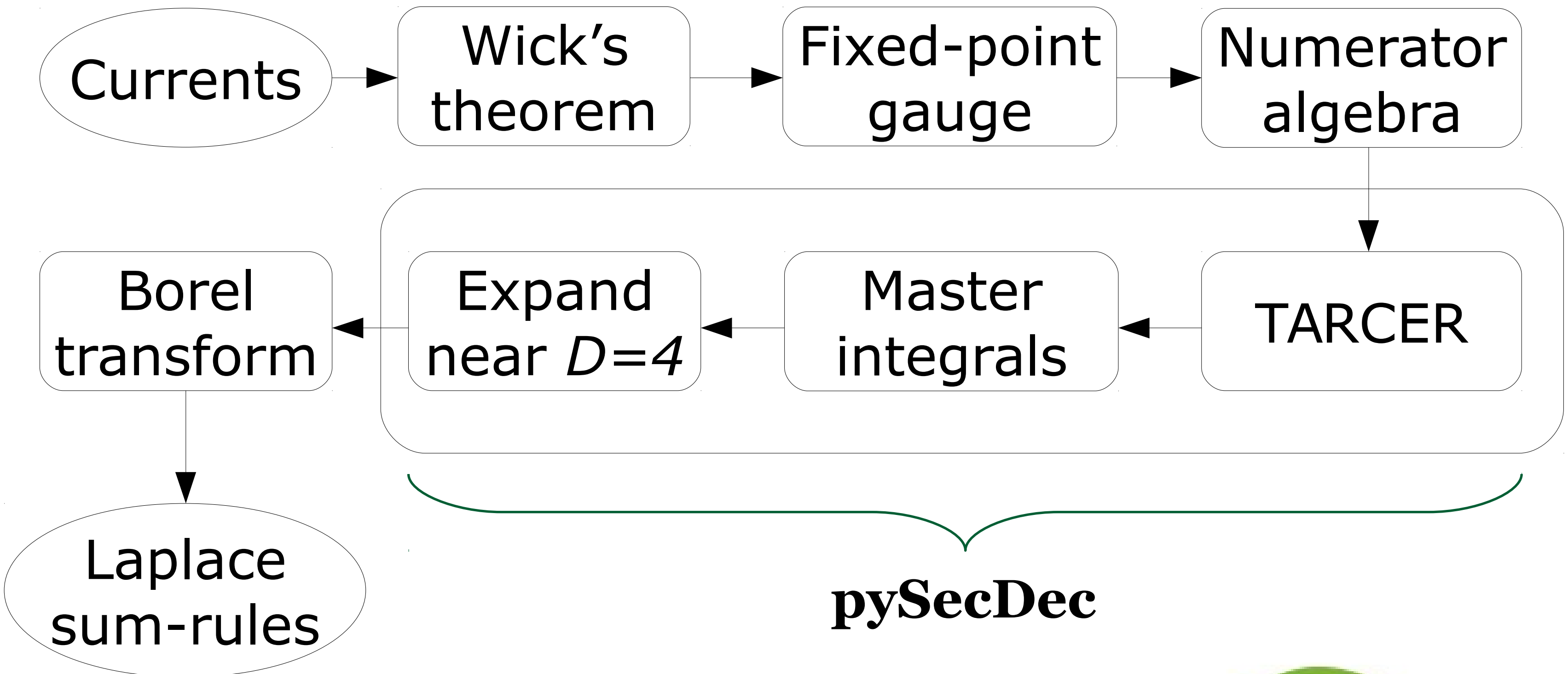
pySecDec has a number of useful features.

- No limits on numbers of external lines, loops, or quarks masses.
- No restrictions on external momenta.
- Computes finite and divergent parts.
- Provides error estimates.
- Works on scalar or tensor integrands.
- Fully open source.

Can pySecDec be used to numerically compute QCD Laplace sum-rules?

- Is the output of pySecDec accurate enough to reliably compute the contour integral needed to formulate QCD Laplace sum-rules?
- Are run-times reasonable, i.e., can we actually apply the QCD sum-rules analysis methodology?
- What sort of computing power is needed?

pySecDec collapses three analytic steps into one numerical step.



As a test, we consider the 0^{-+} charmonium hybrid correlator.

$$\Pi(z) = \frac{m_c^6 \alpha_s}{270\pi^3}\Big(9(4z^3 - 25z^2 + 31z - 10)\,{}_3F_2\big(1,1,1;\tfrac{3}{2},3;z\big)$$
$$+ z(8z^3 + 8z^2 + 29z - 10)\,{}_3F_2\big(1,1,2;\tfrac{5}{2},3;z\big)\Big)$$
$$+ \frac{m_c^2}{18\pi} z(2z+1)\,{}_2F_1\big(1,1;\tfrac{5}{2};z\big)\langle \alpha G^2\rangle$$
$$+ \frac{1}{384\pi^2(z-1)}\Big((2z^2 - 2z + 1)\,{}_2F_1\big(1,1;\tfrac{5}{2};z\big)$$
$$+ (10z^2 - 20z + 7)\Big)\langle g^3 G^3\rangle$$

where $z = \frac{q^2}{4m_c^2}$

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Laplace sum-rules inherit uncertainties from QCD parameters.

$$\alpha_s(\mu) = \frac{\alpha_s(M_\tau)}{1 + \frac{25\alpha_s(M_\tau)}{12\pi}\log\left(\frac{\mu^2}{M_\tau^2}\right)} \text{ where } \alpha_s(M_\tau) = 0.330 \pm 0.014$$

$$m_c(\mu) = \overline{m}_c\left(\frac{\alpha_s(\mu)}{\alpha_s(\overline{m}_c)}\right)^{\frac{12}{25}} \text{ where } \overline{m}_c = (1.275 \pm 0.025)\ \text{GeV}$$

$$\langle \alpha G^2 \rangle = (0.075 \pm 0.02)\ \text{GeV}^4$$

$$\langle g^3 G^3 \rangle = ((8.2 \pm 1.0)\ \text{GeV}^2)\langle \alpha G^2 \rangle$$

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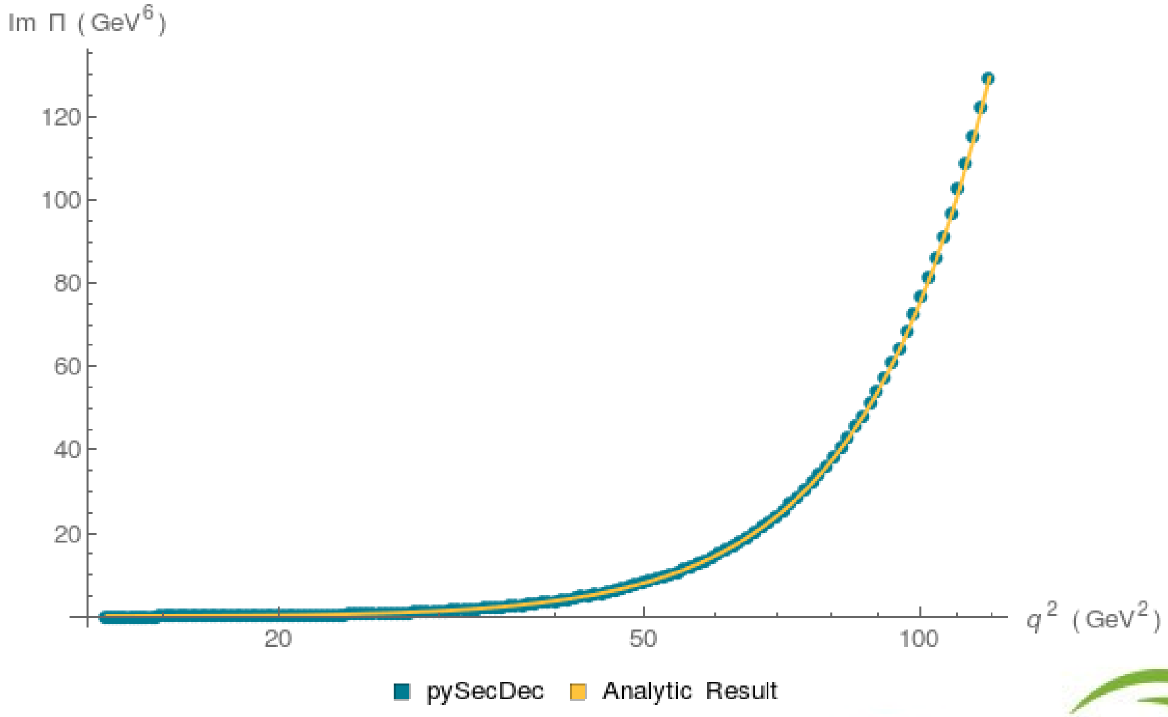
For Im Π, the two computational methods are in excellent agreement.

Analytic and pySecDec computations of Im Π



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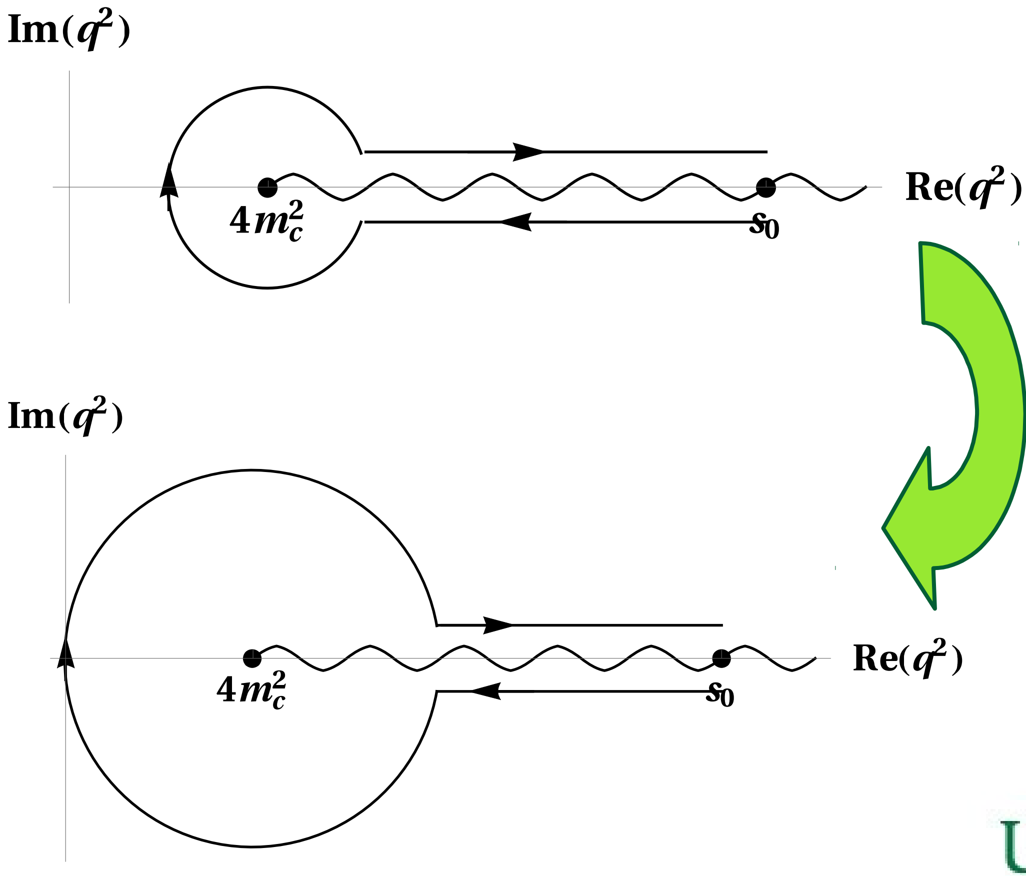
Divergences near the branch point are avoided by contour deformation.



Enlarge the keyhole.

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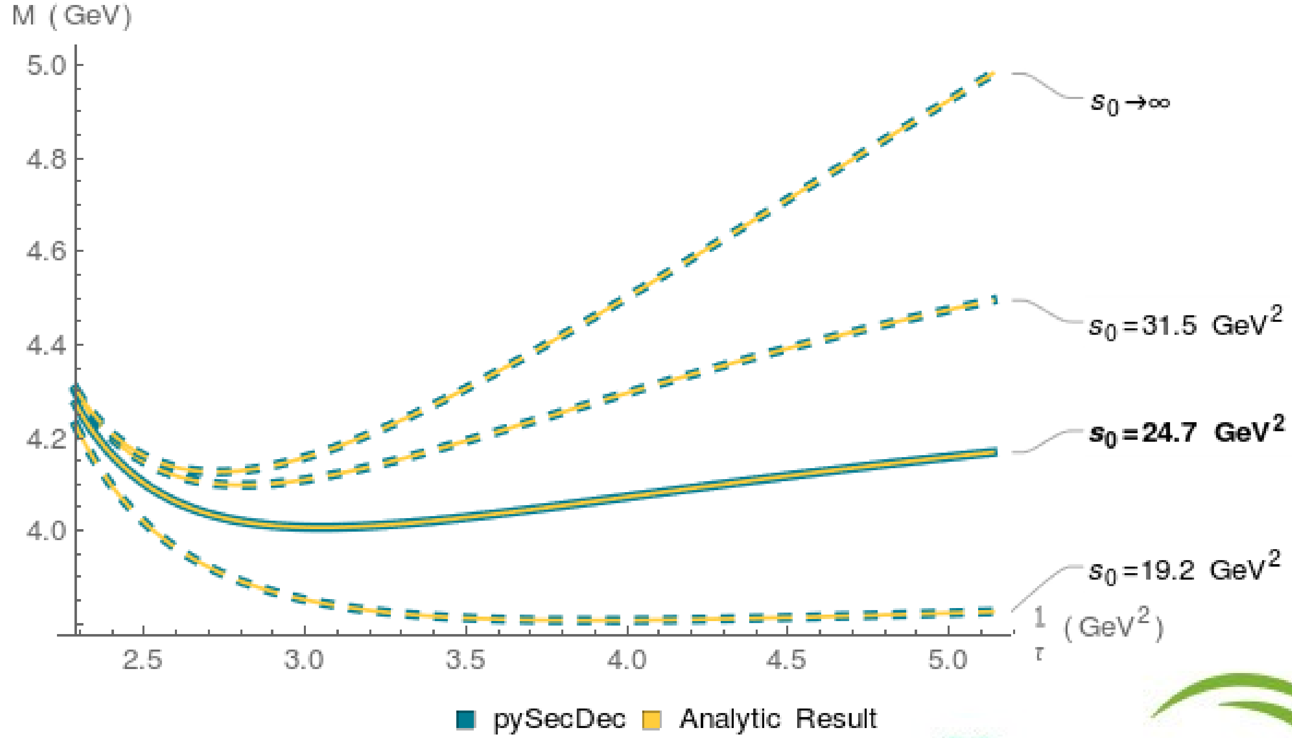
For $M(\tau,s_o)$, the two computational methods are in excellent agreement.

Analytic and pySecDec Computations of $M(\tau,s_o)$



The s_o to ∞ limit can be computed through adaptive step-sizing.

$$\int_{4m_c^2(1+\eta)}^{\infty} t^k \mathrm{e}^{-t\tau}\frac{1}{\pi}\mathrm{Im}\Pi(t)\,\mathrm{d}t \approx \sum_{n=0}^{144} t_n^k \mathrm{e}^{-t_n\tau}\frac{1}{\pi}\mathrm{Im}\Pi(t_n)\Delta t_n$$

s_o to ∞ well-approximated by s_o=120 GeV²

where

$$t_n = \eta + 4m_c^2(1.02^n)$$
$$\Delta t = t_{n+1} - t_n$$

finer grid for small *t* than for large *t*

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Run-times needed to formulate QCD Laplace sum-rules were reasonable.

- Calculations were run on a laptop.
- Calculations were completed in a matter of hours (overnight?).

However, run-times did increase significantly with increased integrand "complexity."

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Computing Laplace sum-rules using pySecDec is convenient.

- Uncertainty due to pySecDec negligible compared to that from QCD parameters.
- Calculations can be run on a PC or laptop.
- Run-times measured in hours (not weeks!).
- In principle, we can consider higher loop diagrams, more external lines (i.e., 3-point functions), and hadrons containing both a charm and bottom quark.

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Acknowledgements

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