# Three-dimensional Newtonian gravity with cosmological constant and torsion

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# Outline

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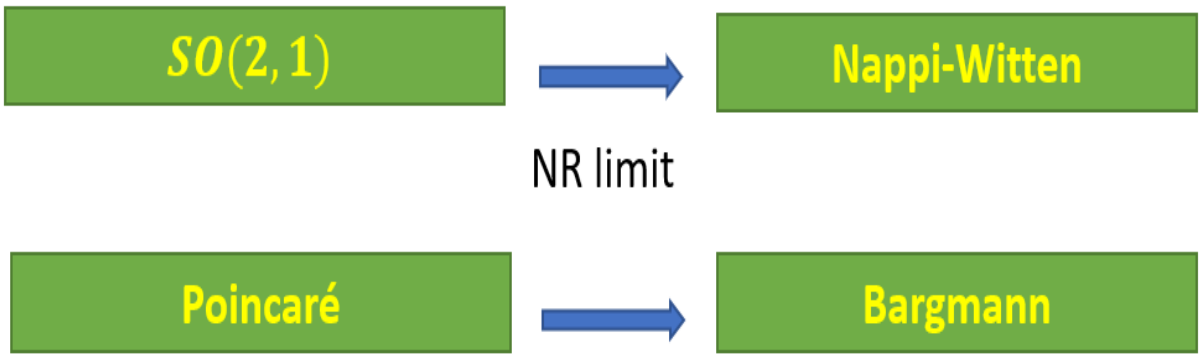
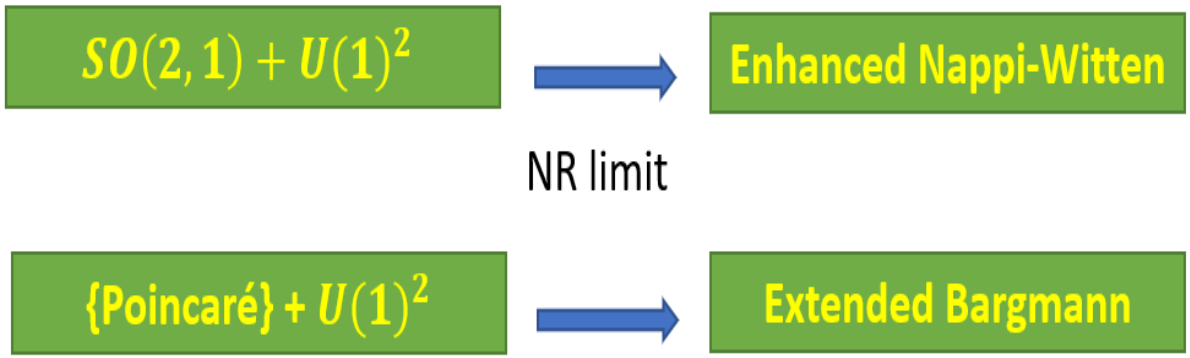
# Introduction

- Newtonian gravity has received a growing interest these last years.
- The non-relativistic (NR) theories have been useful describe of the quantum Hall effect and to approach strongly coupled condensed matter systems as well as NR effective field theories.
- A NR theory can be obtained by a suitable **limiting process** from a relativistic theory. We will understand by NR limit as $c \to \infty$.



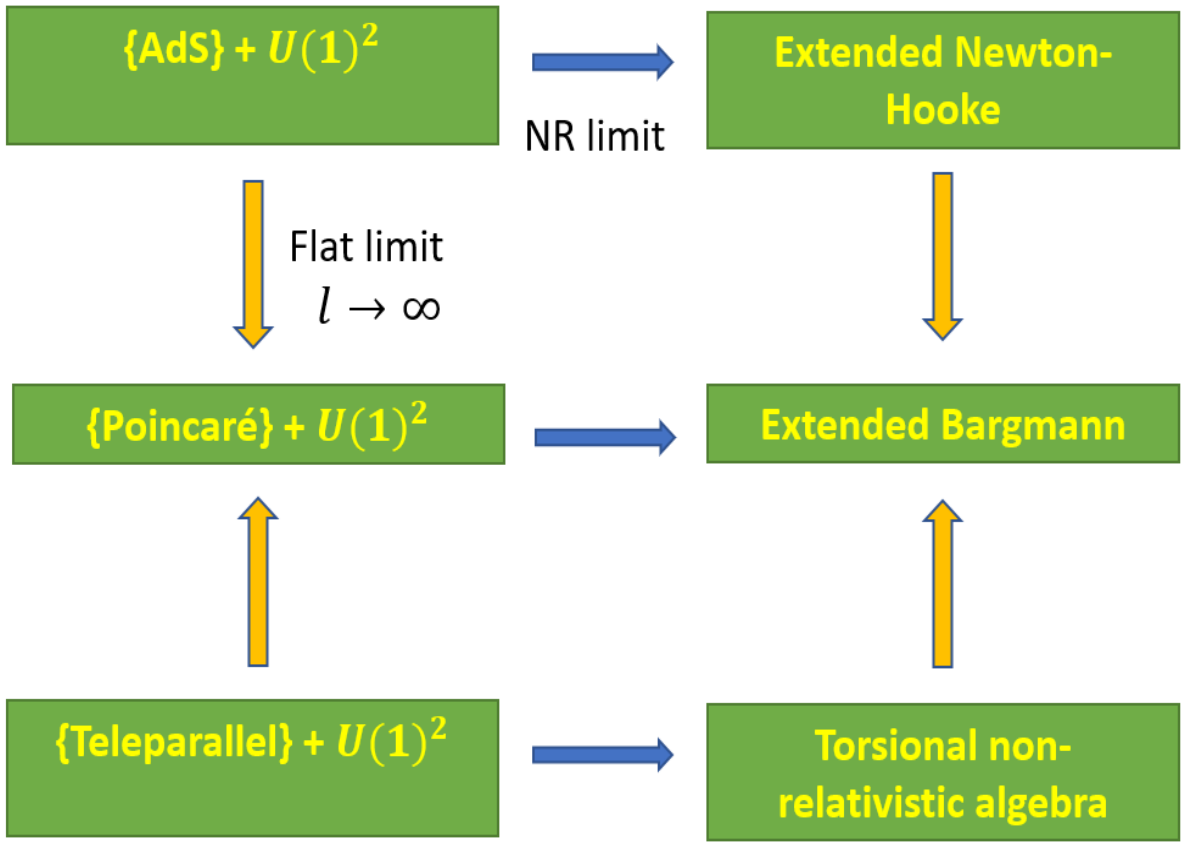
# Introduction

- In the limit $c \to \infty$ there might appear infinities in the contraction of the original Lagrangian.
- In $D = 2+1$, the Chern-Simons (CS) formalism allows us to construct NR gravity actions whose underlying symmetry can be obtained as a NR limit of a relativistic algebra. The infinities and degeneracy can be avoided by considering additional $\mathfrak{u}(1)$ generators.



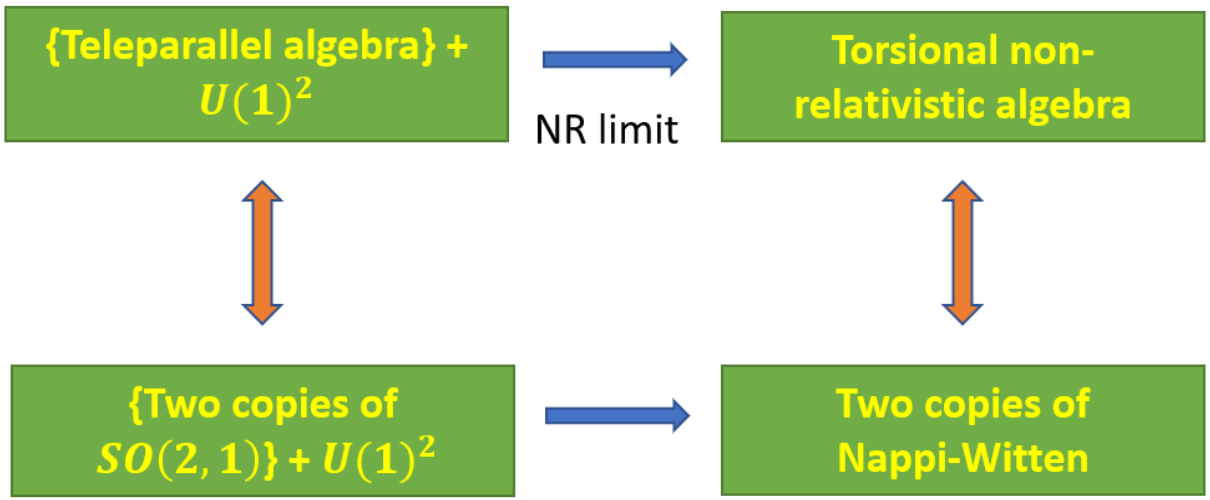
# Introduction

- In the flat limit $l \to \infty$ with the vanishing cosmological constant is recover the Galilean gravity from Newton-Hooke algebra. [H. Bacry, J. Levy-Leblond (1968)],
- An alternative cosmological extension of the three-dimensional Galilean CS gravity is formulated by the switching on the torsion.



# Introduction

- The teleparallel algebra can be rewritten as two copies of the $SO(2,1)$ algebra.
- In an analogous way the torsional non-relativistic algebra corresponds to two copies of the Nappi-Witten algebra. [C.R. Nappi, E. Witten (1993)]



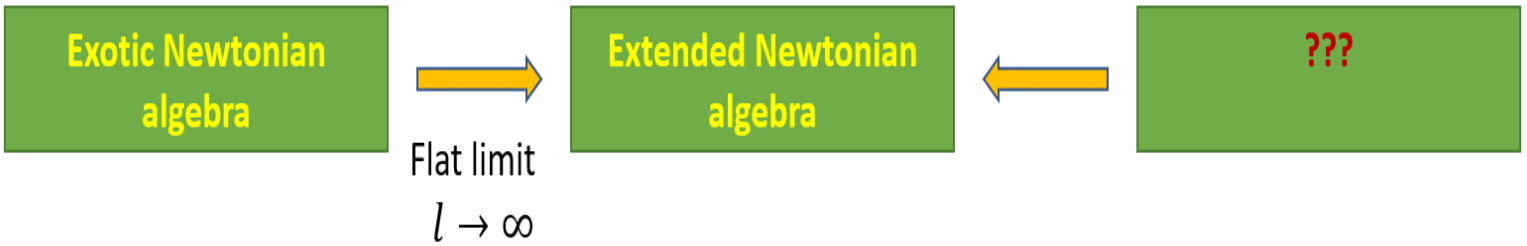

- The cosmological constant can be interpreted as a source for the torsion in the teleparallel CS gravity. [R. Caroca, P. Concha, D. Peñafiel, E. Rodríguez (2021)], [P. Concha, L. Ravera, E. Rodríguez (2022)],

# Introduction

- An action principle for Newtonian gravity requires considering a diverse geometry based on a novel non-relativistic symmetry, which is an extension of the Bargmann algebra called extended Newtonian algebra.

  [D. Hansen, J. Hartong, N. A. Obers (2019)], [N. Ozdemir, M. Ozkan, O. Tunca, and U. Zorba (2019)],

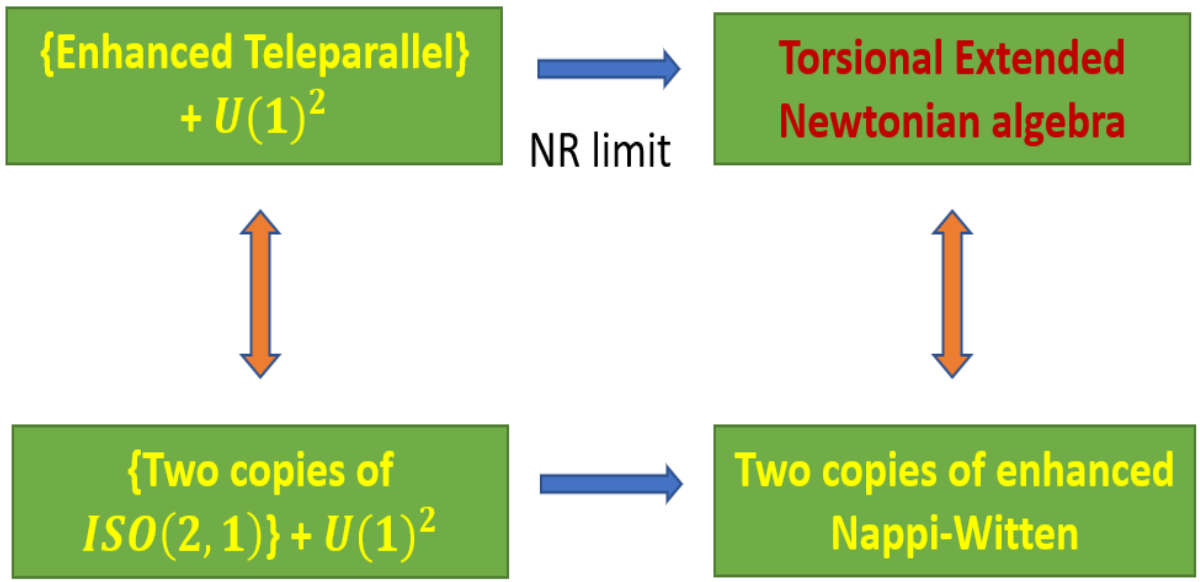
- The inclusion of a cosmological constant in the extended Newtonian gravity is describe by the exotic Newtonian algebra. [P. Concha, L. Ravera, E. Rodríguez (2020)], [J. Gomis, A. Kleinschmidt, J. Palmkvist, and P. Salgado-Rebolledo (2020)],



# Outline

# NR limit of enhanced teleparallel algebra

- A nobel relativistic algebra called enhanced teleparallel algebra, which is an extension of the teleparallel algebra. [P. Concha, E. Rodríguez, G.Rubio, P. Yañez (2022), arXiv:2204.11763 [hep-th]],
- An alternative inclusion of a cosmological constant in the extended Newtonian gravity is describe by the NR limit of this nobel algebra.



# Enhanced teleparallel algebra

The generators of the enhanced teleparallel algebra satisfy following non-vanishing commutation relations, where $A, B, C = 0, 1, 2$.

## Enhanced teleparallel algebra

$$\left[\tilde{J}_A, \tilde{J}_B\right] = \epsilon_{ABC}\tilde{J}^C \quad \left[\tilde{J}_A, \tilde{P}_B\right] = \epsilon_{ABC}\tilde{P}^C \quad \left[\tilde{P}_A, \tilde{P}_B\right] = -\frac{1}{l^2}\epsilon_{ABC}\tilde{P}^C$$
$$\left[\tilde{J}_A, \tilde{S}_B\right] = \epsilon_{ABC}\tilde{S}^C \quad \left[\tilde{J}_A, \tilde{T}_B\right] = \epsilon_{ABC}\tilde{T}^C \quad \left[\tilde{S}_A, \tilde{P}_B\right] = \epsilon_{ABC}\tilde{T}^C$$
$$\left[\tilde{T}_A, \tilde{P}_B\right] = -\frac{1}{l^2}\epsilon_{ABC}\tilde{S}^C$$

one can notice that the enhanced teleparallel algebra can be written as two copies of the $ISO(2,1)$ algebra by considering the following redefinition of the generators,

$$\tilde{J}_A = \tilde{J}^+_A - \tilde{J}^-_A \quad \tilde{P}_A = -\frac{1}{2}\tilde{J}^-_A$$
$$\tilde{S}_A = \tilde{P}^+_A - \tilde{P}^-_A \quad \tilde{T}_A = -\frac{1}{2}\tilde{P}^-_A$$

# Enhanced teleparallel algebra

The redefinition of the generators of the enhanced teleparallel algebra satisfy following non-vanishing commutation relations:

**Enhanced teleparallel algebra**

$$\left[\tilde{J}^{\pm}_{A}, \tilde{J}^{\pm}_{B}\right] = \epsilon_{ABC}\tilde{J}^{C,\pm} \qquad \left[\tilde{J}\pm_{A}, \tilde{P}\pm_{B}\right] = \epsilon_{ABC}\tilde{P}^{C,\pm}$$

in order to obtain a different torsional NR symmetry we require to consider $\mathfrak{u}(1)$ generators, $Y_1$ and $Y_2$. In this way $\{\text{enhanced teleparallel}\}+\mathfrak{u}(1)^2$ algebra admits the following non-vanishing components of a invariant tensor:

$$\left\langle \tilde{J}_A \tilde{J}_B \right\rangle = \tilde{\alpha_0}\eta_{AB} \quad \left\langle \tilde{J}_A \tilde{P}_B \right\rangle = \tilde{\alpha_1}\eta_{AB} \quad \left\langle \tilde{P}_A \tilde{P}_B \right\rangle = -\tfrac{2}{l}\tilde{\alpha_1}\eta_{AB}$$

$$\left\langle \tilde{J}_A \tilde{S}_B \right\rangle = \tilde{\beta_0}\eta_{AB} \quad \left\langle \tilde{P}_A \tilde{S}_B \right\rangle = \tilde{\beta_1}\eta_{AB} \quad \left\langle \tilde{P}_A \tilde{T}_B \right\rangle = -\tfrac{2}{l}\tilde{\beta_1}\eta_{AB}$$

$$\left\langle \tilde{J}_A \tilde{T}_B \right\rangle = \tilde{\beta_1}\eta_{AB} \quad \left\langle \tilde{Y}_1 \tilde{Y}_1 \right\rangle = \tilde{\alpha_0} \quad \left\langle \tilde{Y}_1 \tilde{Y}_2 \right\rangle = \tilde{\alpha_1} \quad \left\langle \tilde{Y}_2 \tilde{Y}_2 \right\rangle = -\tfrac{2}{l}\tilde{\alpha_1}$$

# Torsional extended Newtonian algebra

A non-relativistic version of the $\{\text{enhanced teleparallel}\}+\mathfrak{u}(1)^2$ algebra appears after considering an Inönü-Wigner contraction.

$$
\begin{aligned}
\tilde{J}_0 &= \frac{J}{2} - \xi^4 Z & \tilde{J}_a &= \frac{\xi}{2} G_a - \frac{\xi^3}{2} B_a \\
\tilde{P}_0 &= \frac{H}{2} - \xi^4 Y & \tilde{P}_a &= \frac{\xi}{2} P_a - \frac{\xi^3}{2} T_a \\
\tilde{S}_0 &= -\xi^2 S - \xi^4 Z & \tilde{S}_a &= -\xi G_a - \xi^3 B_a \\
\tilde{T}_0 &= -\xi^2 H - \xi^4 Y & \tilde{T}_a &= -\xi P_a - \xi^3 T_a \\
\tilde{Y}_1 &= \frac{J}{2} + \xi^4 Z & \tilde{Y}_2 &= \frac{H}{2} + \xi^4 Y
\end{aligned}
\tag{1}
$$

Then, in the limit $\xi \to \infty$, the non-relativistic generators satisfy the commutation relations of the extended Newtonian algebra along with a set the commutators proportional to cosmological constant.

# Torsional extended Newtonian algebra

The nobel non-relativistic symmetry is called torsional extended Newtonian (TEN) algebra, where $a, b = 1, 2$.

## Torsional extended Newtonian algebra

$$
\begin{aligned}
[J, G_a] &= \epsilon_{ab} G_b & [G_a, G_b] &= -\epsilon_{ab} S & [H, G_a] &= \epsilon_{ab} P_b \\
[J, P_a] &= \epsilon_{ab} P_b & [G_a, P_b] &= -\epsilon_{ab} M & [H, B_a] &= \epsilon_{ab} T_b \\
[J, B_a] &= \epsilon_{ab} B_b & [G_a, B_b] &= -\epsilon_{ab} Z & [S, G_a] &= \epsilon_{ab} B_b \\
[J, T_a] &= \epsilon_{ab} T_b & [G_a, T_b] &= -\epsilon_{ab} Y & [S, P_a] &= \epsilon_{ab} T_b \\
[M, G_a] &= \epsilon_{ab} T_b & [P_a, B_b] &= -\epsilon_{ab} Y &
\end{aligned}
$$

$$
\begin{aligned}
[H, P_a] &= -\frac{2}{l^2} \epsilon_{ab} T_b & [P_a, P_b] &= \frac{2}{l^2} \epsilon_{ab} M & [H, T_a] &= -\frac{2}{l^2} \epsilon_{ab} T_b \\
[M, P_a] &= -\frac{2}{l^2} \epsilon_{ab} T_b & [P_a, T_b] &= \frac{2}{l^2} \epsilon_{ab} Y &
\end{aligned}
$$

# Torsional extended Newtonian algebra

The TEN algebra can be written as two copies of the enhanced Nappi-Witten algebra by considering the following redefinition of the generators,

$$J = J^+ + J^- \qquad G_a = G_a^+ + G_a^- \qquad H = -\frac{2}{l} J^-$$

$$S = S^+ + S^- \qquad B_a = B_a^+ + B_a^- \qquad M = -\frac{2}{l} S^-$$

$$Z = Z^+ + Z^- \qquad P_a = -\tfrac{2}{l} G_a^- \qquad T_a = -\tfrac{2}{l} B_a^- \qquad Y = -\frac{2}{l} B^-$$

## Torsional extended Newtonian algebra

$$[J^\pm, G_a^\pm] = \epsilon_{ab} G_b^\pm \qquad \left[G_a^\pm, G_b^\pm\right] = -\epsilon_{ab} S^\pm \qquad [S^\pm, G_a^\pm] = \epsilon_{ab} B_b^\pm$$

$$[J^\pm, B_a^\pm] = \epsilon_{ab} B_b^\pm \qquad \left[G_a^\pm, B_b^\pm\right] = -\epsilon_{ab} B^\pm$$

# Torsional extended Newtonian CS action

The torsional extended Newtonian algebra admits the following non-degenerate components of the invariant tensor

$$\begin{aligned} \langle SS\rangle = \langle JZ\rangle &= -\beta_0 & \langle G_a B_b\rangle &= \beta_0\delta_{ab} \\ \langle G_a T_b\rangle &= \beta_1\delta_{ab} & \langle MS\rangle = \langle HZ\rangle = \langle JY\rangle &= -\beta_1 \\ \langle P_a T_b\rangle &= -\frac{2}{l}\beta_1\delta_{ab} & \langle HY\rangle = \langle MM\rangle &= \frac{2}{l}\beta_1 \end{aligned} \qquad (2)$$

which can be obtained from the relativistic invariants of the enhanced teleparallel algebra by applying the limit $\xi \to \infty$ after considering the contraction (1) of the generators and the rescaling of the relativistic parameters as

$$\tilde{\alpha}_0 = \tilde{\beta}_0 = -\beta_0\xi^4, \quad \tilde{\alpha}_1 = \tilde{\beta}_1 = -\beta_1\xi^4$$

# Torsional extended Newtonian CS action

It is interesting to note that the TEN algebra also admits the bilinear invariant trace of the torsional non-relativistic algebra

$$\langle JS\rangle = -\alpha_0 \qquad \langle HS\rangle = \langle MJ\rangle = -\alpha_1$$
$$\langle G_a G_b\rangle = \alpha_0\delta_{ab} \qquad \langle P_a P_b\rangle = -\frac{2}{l}\alpha_1\delta_{ab}$$
$$\langle G_a P_b\rangle = \alpha_1\delta_{ab} \qquad \langle HM\rangle = \frac{2}{l}\alpha_1$$

where the relativistic parameters obey $\tilde{\alpha}_0 = \tilde{\alpha}_1 = 0$ along the following rescaling:

$$\tilde{\beta}_0 = -\alpha_0\xi^2, \qquad \tilde{\beta}_1 = -\alpha_1\xi^2$$

This components of the invariant tensor are degenerate for the whole TEN algebra although they define non-degenerate invariant trace for the torsional non-relativistic symmetry

# Torsional extended Newtonian CS action

The gauge connection one-form for the TEN algebra reads

$$A = \omega J + \omega^a G_a + \tau H + e^a P_a + mM + sS + b^a B_a + t^a T_a + yY + zZ$$

The corresponding curvature two-forms are given by

$$\begin{aligned}
F(\omega) &= d\omega, & F^a(e^b) &= de^a + \epsilon^{ac}\omega e_c + \epsilon^{ac}\tau\omega_c - \frac{2}{l}\epsilon^{ac}\tau e_c, \\
F(z) &= dz + \frac{1}{2}\epsilon^{ac}\omega_a z_c, & F(\tau) &= d\tau, \\
F(m) &= dm + \epsilon^{ac}e_a\omega_c - \frac{1}{l}\epsilon^{ac}e_a e_c, & F^a(\omega^b) &= d\omega^a + \epsilon^{ac}\omega\omega_c, \\
F^a(b^b) &= d\omega^a + \epsilon^{ac}\omega b_c + \epsilon^{ac}s\omega_c, & F(s) &= ds + \frac{1}{2}\epsilon^{ac}\omega_a\omega_c, \\
F^a(t^b) &= de^a + \epsilon^{ac}\omega t_c + \epsilon^{ac}se_c + \epsilon^{ac}\tau b_c + \epsilon^{ac}m\omega_c - \frac{2}{l}\epsilon^{ac}\tau t_c - \frac{2}{l}\epsilon^{ac}me_c,
\end{aligned}$$

# Torsional extended Newtonian CS action

Since each family of invariant tensor appears by applying different rescaling of the relativistic parameters, in the sequel we shall consider the CS construction by exploiting the components given by (2). We consider the gauge connection $A$ one-form in order to introduce a torsion in three-dimensional CS Newtonian gravity. [P. Concha, E. Rodríguez, G.Rubio, P. Yañez (2022), arXiv:2204.11763 [hep-th]],

$$I_{TEN} = \frac{k}{4\pi}\int \mathfrak{L}_{Enhanced-Bargmann} + \tfrac{2}{l}\beta_1\left[\tau R(y) + yR(\tau) + m(R(m)\right.$$
$$\left. - e_a R^a(t^b) - t_a R^a(e^b) + \tfrac{1}{l}\epsilon^{ab}\left(m e_a e_b + 2\tau e_a t_b\right)\right]$$

where $\mathfrak{L}_{Enhanced-Bargmann}$ corresponds to the Enhanced-Bargmann gravity Lagrangian. The CS action is gauge invariant under the whole TEN algebra and can be seen as a cosmological extension of the Enhanced-Bargmann CS gravity theory diverse to the centrally extended post-Newtonian. [E. Bergshoeff, J. Gomis, and P. Salgado-Rebolledo (2020), [P. Concha, L. Ravera, E. Rodríguez (2020)], [J. Gomis, A. Kleinschmidt, J. Palmkvist, and P. Salgado-Rebolledo (2020)],

# Torsional extended Newtonian CS equations

Indeed, requiring the non-degeneracy of the invariant tensor (2) that is $\beta_1 \neq 0$ and $\beta_0 \neq -\frac{l}{2}\beta_1$. The field equations give the vanishing of the curvatures. In particular, the cosmological constant can be seen as a source for the spatial torsion $T^a\left(e^b\right)$ and for the curvatures $T\left(m\right)$, $T^a\left(t^b\right)$ and $T\left(y\right)$.

$$T^a\left(e^b\right) = \frac{2}{l}\epsilon^{ac}\tau e_c, \qquad T^a\left(e^b\right) = de^a + \epsilon^{ac}\omega e_c + \epsilon^{ac}\tau\omega_c$$

$$T\left(m\right) = \frac{1}{l}\epsilon^{ac}e_a e_c, \qquad T\left(m\right) = dm + \epsilon^{ac}e_a\omega_c$$

$$T^a\left(t^b\right) = \frac{2}{l}\epsilon^{ac}\left(\tau t_c + m e_c\right), \qquad T^a\left(t^b\right) = dt^a + \epsilon^{ac}\omega t_c + \epsilon^{ac}s e_c + \epsilon^{ac}\tau b_c + \epsilon^{ac}m\omega_c$$

$$T\left(y\right) = \frac{1}{l}\epsilon^{ac}e_a t_c, \qquad T\left(y\right) = dy + \epsilon^{ac}\omega_a t_c + \epsilon^{ac}b_a e_c$$

Hence, the TEN gravity theory is then characterized, on-shell, by vanishing curvatures $R\left(\omega\right) = R^a\left(\omega^b\right) = 0$. Such particular behavior can be seen as the teleparallel analogue of the Newtonian gravity theory. Nevertheless, our result is quite diverse to the usual Newton-Cartan gravity theory with torsion.

# Outline

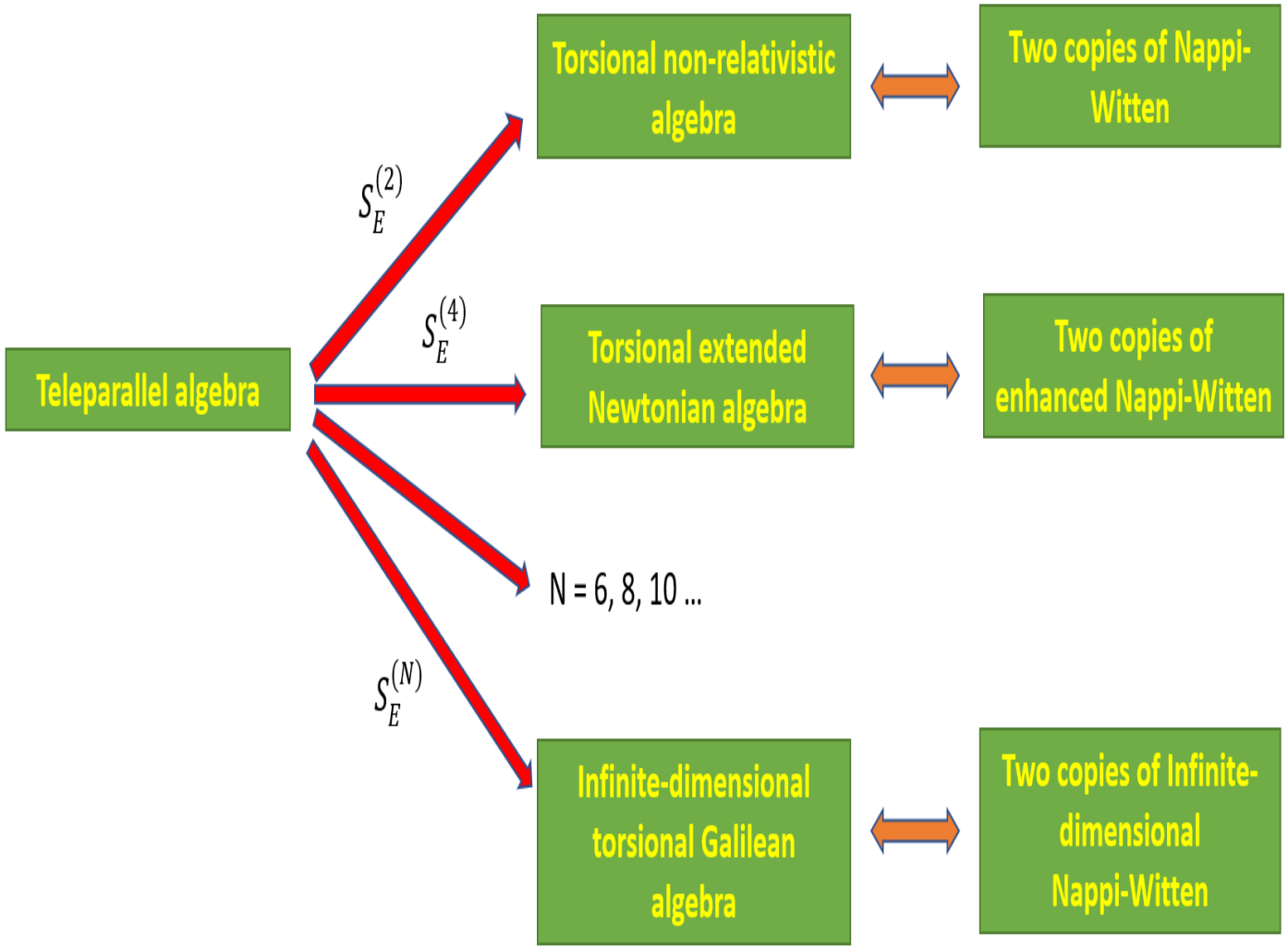
# Infinite-dimensional torsional Galilean algebra and semigroup expansion method

Let us note that the field equations for the torsional non-relativistic gravity theory $R(\omega) = R^a(\omega^b) = R(s) = R(\tau) = R(m) = R^a(e^b) = 0$ are contained in the TEN gravity model. Such behavior may suggests that there exists an infinite dimensional non-relativistic symmetry that contains the torsional non-relativistic and the TEN gravity theories as particular cases. [P. Concha, E. Rodríguez, G.Rubio, P. Yañez (2022), arXiv:2204.11763 [hep-th]],

- An infinite-dimensional torsional Galilean algebra denoted as $t\mathfrak{G}^{(N)}$ can be obtained by applying the Semigroup expansion (S-expansion) method to the teleparallel algebra. [F. Izaurieta, E. Rodríguez, P. Salgado (2006),
- Although one could explore its construction by considering a non-relativistic limit, it would require starting from an infinite-dimensional relativistic algebra.
- Before applying the S-expansion to the relativistic teleparallel symmetry we require to consider a particular subspace decomposition of the original algebra.

# Infinite-dimensional torsional Galilean algebra

In the following scheme we can summarize the results found for the infinite-dimensional algebra $\mathfrak{tG}^{(N)}$ through of the S-expansion method.



# Semigroup expansion method

The S-expansion procedure consists in obtaining a new Lie algebra $\mathfrak{G}$, by combining the elements of a semigroup S with the structure constants of a Lie algebra $\mathfrak{g}$.

$$\mathfrak{G} = S \times \mathfrak{g}$$

## Advantages of the S-expansion procedure

- It not only provides us with the commutation relations of the expanded algebra, but also allows us to compute the non-vanishing components of the invariant tensor of the expanded algebra in terms of the original ones.
- It can reproduce the Maurer-Cartan forms power series expansion for a particular choice of the semigroup S.

# Non-relativistic expansion

A NR algebra can be obtained by expanding its relativistic counterpart with a particular semigroup. To this end, we require to consider a $\mathbb{Z}_2$-graded subspace decomposition

$$[V_0, V_0] \subset V_0 \qquad [V_0, V_1] \subset V_1 \qquad [V_1, V_1] \subset V_0$$

Let $S_E^{(N)} = \{\lambda_0, \lambda_1, \ldots, \lambda_{N+1}\}$ be the relevant semigroup whose elements satisfy the following multiplication law:

$$\lambda_\alpha \lambda_\beta = \begin{cases} \lambda_{\alpha+\beta} & \text{if} \quad \alpha+\beta \leq N+1\,, \\ \lambda_{N+1} & \text{if} \quad \alpha+\beta > N+1\,, \end{cases}$$

where $\lambda_{N+1} = 0_s$ is the zero element of the semigroup such that $0_s \lambda_\alpha = 0_s$. Then, the NR version of the Lie original algebra $\mathfrak{g}$ is obtained by considering a resonant $S_E^{(N)}$-expansion of the relativistic algebra $\mathfrak{g}$.

# Non-relativistic expansion of the teleparallel algebra

A infinite-dimensional NR version of the teleparallel algebra $\mathfrak{Tel}$ requires to consider its no vanishing commutation relations:

**Teleparallel algebra**

$$\begin{aligned}\left[\tilde{J}_A, \tilde{J}_B\right] &= \epsilon_{ABC}\tilde{J}^C \\ \left[\tilde{J}_A, \tilde{P}_B\right] &= \epsilon_{ABC}\tilde{P}^C \\ \left[\tilde{P}_A, \tilde{P}_B\right] &= -\frac{2}{l}\epsilon_{ABC}\tilde{P}^C\end{aligned}$$

with $l \to \infty$ is obtained the Poincaré algebra. The non-relativistic algebras appears as a $0_s$-reduction of the expanded resonant algebra $S_E^{(N)} \times \mathfrak{Tel}$, where the expanded generators are expressed in terms of the Teleparallel ones through the semigroup elements as

$$\begin{aligned} J^{(m)} &= \lambda_{2m}\tilde{J}_0, & G_a^{(m)} &= \lambda_{2m+1}\tilde{J}_a, \\ P^{(m)} &= \lambda_{2m}\tilde{P}_0, & P_a^{(m)} &= \lambda_{2m+1}\tilde{P}_a, \end{aligned}$$

[R.Caroca, P.Concha, D.Peñafiel, E.Rodríguez (2021)]

# Non-relativistic expansion of the teleparallel algebra

A resonant descomposition of $S_E^{(N)} = S_0 \cup S_1$ is given by

$$
\begin{aligned}
S_0 &= \left\{ \lambda_{2m}, \text{ with } m = 0, \ldots, \left[ \frac{N}{2} \right] \right\} \cup \{\lambda_{N+1}\}, \\
S_1 &= \left\{ \lambda_{2m+1}, \text{ with } m = 0, \ldots, \left[ \frac{N+1}{2} \right] \right\} \cup \{\lambda_{N+1}\}.
\end{aligned}
$$

Here $[x]$ denotes the integer part of $x$. The Teleparallel algebra is the direct sum $\mathfrak{Tel} = V_0 \oplus V_1$, with $V_0 = \{\tilde{J}_0, \tilde{P}_0\}$ and $V_1 = \{\tilde{J}_a, \tilde{P}_a\}$. The algebra $S_E^{(N)} \times \mathfrak{Tel}$ satisfy the following commutation relations

**Infinite-dimensional torsional Galilean algebra**

$$
\begin{aligned}
&[J^{(m)}, G_a^{(n)}] = \epsilon_{ab} G_b^{(m+n)}, \qquad [G_a^{(m)}, G_b^{(n)}] = -\epsilon_{ab} J^{(m+n+1)}, \\
&[J^{(m)}, P_a^{(n)}] = \epsilon_{ab} P_b^{(m+n)}, \qquad [P^{(m)}, G_a^{(n)}] = \epsilon_{ab} P_b^{(m+n)}, \\
&[G_a^{(m)}, P_b^{(n)}] = -\epsilon_{ab} P^{(m+n+1)}, \qquad [P^{(m)}, P_a^{(n)}] = -\frac{2}{l} \epsilon_{ab} P_b^{(m+n)}, \\
&[P_a^{(m)}, P_b^{(n)}] = \frac{2}{l} \epsilon_{ab} P^{(m+n+1)}
\end{aligned} \tag{3}
$$

# Non-relativistic expansion of the teleparallel algebra

The S-expansion method provides us with the non-vanishing components of the invariant tensor of the expanded algebra in terms of the original ones. In particular, the non-vanishing components of the invariant tensor for the $\mathfrak{Tel}$ algebra are given by

$$\left\langle \tilde{J}_A \tilde{J}_B \right\rangle = \tilde{\alpha}_0 \eta_{AB}, \qquad \left\langle \tilde{J}_A \tilde{P}_B \right\rangle = \tilde{\alpha}_1 \eta_{AB}, \qquad \left\langle \tilde{P}_A \tilde{P}_B \right\rangle = -\frac{2\tilde{\alpha}_1}{l} \eta_{AB}$$

Thus, one can show that the S-expanded algebra admits the following invariant tensor,

$$\begin{aligned}
\left\langle J^{(m)} J^{(n)} \right\rangle &= -\beta_{m+n}, & \left\langle G_a^{(m)} G_b^{(n)} \right\rangle &= \beta_{m+n+1} \delta_{ab}, \\
\left\langle J^{(m)} P^{(n)} \right\rangle &= -\gamma_{m+n}, & \left\langle G_a^{(m)} P_b^{(n)} \right\rangle &= \gamma_{m+n+1}\, \delta_{ab}, \\
\left\langle P^{(m)} P^{(n)} \right\rangle &= \tfrac{2}{l}\gamma_{m+n}, & \left\langle P_a^{(m)} P_b^{(n)} \right\rangle &= -\frac{2}{l}\gamma_{m+n+1}\, \delta_{ab},
\end{aligned} \tag{4}$$

where $\beta$'s and $\gamma$'s are arbitrary constants and are related to the $\mathfrak{Tel}$ constant as

$$\beta_{m+n} = \lambda_{2(m+n)} \tilde{\alpha}_0, \qquad \gamma_{m+n} = \lambda_{2(m+n)} \tilde{\alpha}_1,$$

# Non-relativistic expansion of the teleparallel algebra

Let us remark that the infinite dimensional torsional Galilean algebra $\mathfrak{tG}^{(\mathrm{N})}$ can be rewritten as two copies of an infinite-dimensional extension of the Nappi-Witten algebra, which we have denoted as $\mathfrak{nw}(\mathrm{N})$. [E. Bergshoeff, J. Gomis, and P. Salgado-Rebolledo (2020)]

**Infinite-dimensional torsional Galilean algebra**

$$\left[J^{(m)\pm}, G_a^{(n)\pm}\right] = \epsilon_{ab} G_b^{(m+n)\pm} \qquad \left[G_a^{(m)\pm}, G_b^{(n)\pm}\right] = -\epsilon_{ab} J^{(m+n+1)\pm}$$

The two copies of $\mathfrak{nw}(\mathrm{N})$ appear after considering the following redefinition of the $\mathfrak{tG}^{(\mathrm{N})}$ generators

$$J^{(m)} = J^{(m)+} + J^{(m)-}, \qquad H = -\frac{2}{l} J^{(m)-},$$
$$G_a^{(m)} = G_a^{(m)+} + G_a^{(m)-}, \qquad P_a^{(m)} = -\frac{2}{l} G_a^{(m)-}$$

In particular, $\mathfrak{nw}(2)$ and $\mathfrak{nw}(4)$ are the usual Nappi-Witten and enhanced Nappi-Wittenalgebras, r espectively. The torsional NR algebra is $\mathfrak{tG}^{(2)}$ and TEN corresponds to $\mathfrak{tG}^{(4)}$.

# Infinite-dimensional torsional Galilean gravity action

In order to construct a Chern-Simons action, we define a connection one-form taking values on (3)

$$A=\sum_{m=0}^{\left[\frac{N}{2}\right]}\left(\omega^{(m)}J^{(m)}+\tau^{(m)}H^{(m)}\right)+\sum_{m=0}^{\left[\frac{N+1}{2}\right]}\left(\omega^{(m)a}G_a^{(m)}+e^{(m)a}P_a^{(m)}\right) \tag{5}$$

The corresponding curvature two-forms are given by

$$R\left(\omega^{(k)}\right)=d\omega^{(k)}+\frac{1}{2}\sum_{m,n=1}^{N/2}\epsilon^{ac}\omega_a^{(m)}\omega_c^{(n)}\delta^k_{m+n+1}$$

$$R^a\left(\omega^{b(k)}\right)=d\omega^{a(k)}+\sum_{m,n=1}^{N/2}\epsilon^{ac}\omega^{(m)}\omega_c^{(n)}\delta^k_{m+n}$$

$$R\left(\tau^{(k)}\right)=d\tau^{(k)}+\sum_{m,n=1}^{N/2}\epsilon^{ac}\left(\omega_a^{(m)}e_c^{(n)}\delta^k_{m+n+1}-\frac{1}{l}e_a^{(m)}e_c^{(n)}\delta_k^{m+n+1}\right)$$

$$R^a\left(e^{b(k)}\right)=de^{a(k)}+\sum_{m,n=1}^{N/2}\epsilon^{ac}\left(\omega^{(m)}e_c^{(n)}\delta^k_{m+n}+\tau^{(m)}\omega_c^{(n)}\delta^k_{m+n}-\frac{2}{l}\tau^{(m)}e_c^{(n)}\delta^k_{m+n}\right)$$

# Infinite-dimensional torsional Galilean gravity action

The NR gravity action considering the gauge connection one-form (5) for the $\mathfrak{tG}^{(N)}$ algebra and the non-vanishing components of the invariant tensor (4) is [P. Concha, E. Rodríguez, G.Rubio, P. Yañez (2022), arXiv:2204.11763 [hep-th]],

$$
\begin{aligned}
I_{\mathfrak{tG}^{(N)}} =& \frac{k}{4\pi}\sum_{i,m,n,k}^{N/2}\sigma_i\int\left(-\omega^{(m)}d\omega^{(n)}\delta^i_{m+n}+\omega^{(m)}_a d\omega^{a(m)}\delta^i_{m+n+1}-\epsilon^{ac}\omega^{(m)}\omega^{(n)}_a\omega^{(k)}_c\delta^i_{m+n+k+1}\right)\\
&+\frac{k}{2\pi}\sum_{i,m,n,k}^{N/2}\gamma_i\int\Big[-R\left(\omega^{(m)}\right)\tau^{(n)}\delta^i_{m+n}+R^a\left(\omega^{b(m)}\right)e^{(n)}_a\delta^i_{m+n+1}\\
&-\frac{1}{l}R\left(\tau^{(m)}\right)\tau^{(n)}\delta^i_{m+n}+\frac{1}{l}R^a\left(e^{b(m)}\right)e^{(n)}_a\delta^i_{m+n+1}+\frac{1}{l^2}\epsilon^{ac}\tau^{(m)}e^{(n)}_a e^{(k)}_c\delta^i_{m+n+k+1}\Big]
\end{aligned}
\tag{6}
$$

The CS action contains two sectors, one proportional to $\gamma_i$ and a second one proportional to $\sigma_i$, which corresponds to an exotic sector. Interestingly,considering only the non-degenerate cases given by even values of $N$, one can rewrite the CS action (6) as follows

$$
I_{\mathfrak{tG}^{(N)}} = I_{TNR} + I_{TEN} + \sum_{i=3}^{N/2} I_{\mathfrak{tG}^{(2i)}}
$$

# Infinite-dimensional torsional Galilean gravity equations

The field equations give the vanishing of the infinite-dimensional extension of the spin-connection curvatures $R\left(\omega^{(k)}\right)$ and $R^a\left(\omega^{b(k)}\right)$. In particular, the cosmological constant can be seen as a source for the spatial and temporal torsion $T^a\left(e^{b(k)}\right)$ and $T\left(\tau^{(k)}\right)$ respectively. The non-degeneracy of the invariant tensor for $\mathfrak{tG}^{(2i)}$ requires only $\gamma_i \neq 0$ and $\sigma_i \neq -\frac{l}{2}\gamma_i$ without imposing conditions on the other constants.

$$T^a\left(e^{b(k)}\right) = \frac{2}{l}\sum_{m,n=1}^{N/2} \epsilon^{ac}\tau^{(m)}e_c^{(n)}\delta^k_{m+n}$$

$$T\left(\tau^{(k)}\right) = \frac{1}{l}\sum_{m,n=1}^{N/2} \epsilon^{ac}e_a^{(m)}e_c^{(n)}\delta^{m+n+1}_k$$

where the spatial and temporal components of the infinite torsion are given by

$$T^a\left(e^{b(k)}\right) = de^{a(k)} + \sum_{m,n=1}^{N/2} \epsilon^{ac}\left(\omega^{(m)}e_c^{(n)}\delta^k_{m+n} + \tau^{(m)}\omega_c^{(n)}\delta^k_{m+n}\right)$$

$$T\left(\tau^{(k)}\right) = d\tau^{(k)} + \sum^{N/2} \epsilon^{ac}\left(\omega_a^{(m)}e_c^{(n)}\delta^k_{m+n+1}\right)$$

# Outline

# Comments and further developments

It is important to clarify that the torsional Newtonian gravity theory obtained here is different to the Newton-Cartan theory with torsion in which the time component of the torsion is non zero. [E. A. Bergshoeff, J. Hartong, and J. Rosseel (2015)], [E. Bergshoeff, J. Rosseel, and T. Zojer (2015)]

- Indeed, our procedure allows us to construct a gauge-invariant non-relativistic CS gravity action which can be seen as the teleparallel analogue of the Newtonian gravity.
- It would be interesting to study how our methodology could be extended in order to make contact with torsional Newton-Cartan formalism from a CS point of view.
- The Lie algebra expansion method could be useful to obtain Galiean or Carrollian (super)algebras along their extended versions in arbitrary spacetime dimensions.
- One way to go beyond Poincaré gravity without introducing a cosmological constant is possible through the Maxwell symmetry which has been introduced to describe a Minkowski space in the presence of a constant electromagnetic field background.[Robert Schrader (1972)]

Thank you!