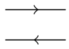
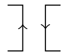
4 Color states of $q\bar{q}$, qg and qqq systems

Here we will apply the systematic method described in lesson 3 for diquark states to a few other systems, namely, $q\bar{q}$ pairs, qg pairs, and finally the qqq system. The latter is the simplest system for which the number of irreps is strictly smaller than the number of independent tensors, thus providing a simple illustration of color transitions between equivalent irreps.

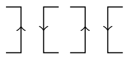
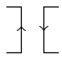
4.1 $q\bar{q}$

As a warming up, let us start with $q\bar{q}$ pairs. The systematic procedure for finding how $\mathbf{3} \otimes \bar{\mathbf{3}}$ decomposes into a sum of irreps is as follows.

(i) In the same way as was done for qq pairs in lesson 3, one can obtain a maximal set of independent tensors mapping $V \otimes \overline{V}$ to itself, for instance:

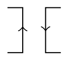
$\mathbb{1} \equiv$  ; $S \equiv$  . (84)

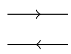
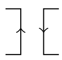
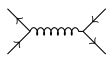
(ii) The multiplication table between these tensors has only one non-trivial entry:

$S^2 =$  $= N$  $= NS\,.$ (85)

Thus, S has for minimal polynomial $x^2 - Nx$, and for eigenvalues $\{\lambda_1, \lambda_2\} = \{N, 0\}$. In some basis of $\{q^i q_j\} \equiv V \otimes \overline{V}$, we have $S = \mathrm{diag}\,(\lambda_1 \ldots \lambda_1, \lambda_2 \ldots \lambda_2)$.

(iii) The projectors on the eigenspaces of S read

$P_{\lambda_1} = \dfrac{S - \lambda_2 \mathbb{1}}{\lambda_1 - \lambda_2} = \dfrac{S}{N} = \dfrac{1}{N}$  $\equiv \mathbb{P}_{\mathbf{1}}\,,$ (86)

$P_{\lambda_2} = \dfrac{S - \lambda_1 \mathbb{1}}{\lambda_2 - \lambda_1} = \mathbb{1} - \dfrac{S}{N} =$  $- \dfrac{1}{N}$  $= 2$  $\equiv \mathbb{P}_{\mathbf{8}}\,.$ (87)

We thus recover the projectors of ranks 1 and $N^2 - 1$ encountered when proving the Fierz identity (17) in lesson 1 (see Exercise 2). The small bonus of the systematic derivation is that the invariant subspace $\mathrm{img}(\mathbb{P}_{\mathbf{8}})$ is now shown to be irreducible (otherwise one would have found more than two independent tensors in step (i)).

The $q\bar{q}$ system thus decomposes into a sum of irreps as

$$\mathbf{3} \otimes \bar{\mathbf{3}} = \mathbf{1} \oplus \mathbf{8}\,, \tag{88}$$

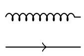
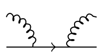
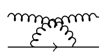
where $\mathbf{1}$ and $\mathbf{8}$ denote the trivial (or singlet) and adjoint representations of $\mathrm{SU}(N)$.

Exercise 25. Write the linear combinations of $q^i q_j$ spanning the invariant subspaces $\mathrm{img}(\mathbb{P}_{\mathbf{1}})$ and $\mathrm{img}(\mathbb{P}_{\mathbf{8}})$, and check explicitly that they transform as expected under $\mathrm{SU}(N)$.

4.2 qg

Finding the color states of a quark-gluon pair using the systematic procedure is fairly straightforward. Since you get used to it, several steps of the derivation are left as exercises.

(i) For a maximal set of independent tensors mapping $V \otimes A$ to itself, one can take:

$I \equiv$  ; $A \equiv$  ; $B \equiv$  . (89)

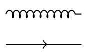
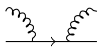
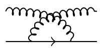
Exercise 26. Prove it by using the same method as for the qq case (see lesson 3), and paying attention to quark loops.

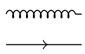
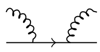
(ii) The non-trivial entries of the multiplication table are:

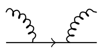
$$A^2 = C_F A \ \ ; \quad AB = BA = -\frac{1}{2N} A \ \ ; \quad B^2 = \frac{1}{4} I - \frac{1}{2N} A \,. \tag{90}$$

Exercise 27. Derive the relations (90) in a pictorial way. How many irreps can we expect from the set of tensors (89), and why? Infer the minimal polynomial of B, and explain why there is a basis of $\{q^i g^a\} \equiv V \otimes A$ where the operator B is represented by the diagonal matrix $B = \mathrm{diag}\,(\lambda_1 \ldots \lambda_1, \lambda_2 \ldots \lambda_2, \lambda_3 \ldots \lambda_3)$, with $\{\lambda_1, \lambda_2, \lambda_3\} = \left\{\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2N}\right\}$.

(iii) The projectors on the eigenspaces of B are given by

$P_{\lambda_1} = \frac{1}{2}$  $- \frac{1}{N+1}$  $+$  , (91)

$P_{\lambda_2} = \frac{1}{2}$  $- \frac{1}{N-1}$  $-$  , (92)

$P_{\lambda_3} = \frac{1}{C_F}$  . (93)

Exercise 28. Obtain the latter projectors using the formula

$$P_{\lambda_i} = \frac{(B-\lambda_j)(B-\lambda_k)}{(\lambda_i-\lambda_j)(\lambda_i-\lambda_k)}\,, \tag{94}$$

where P_{λ_i} is the projector on the eigenspace associated to the eigenvalue λ_i, and λ_j, λ_k are the two other eigenvalues. (Eq. (94) follows from a mere observation of B in its diagonal matrix form.)

By construction, the projectors (91)–(93) form a complete (and maximal) set of Hermitian, mutually orthogonal projectors, which thus project on the irreps of a qg pair.

Exercise 29. Show that the dimensions K_α and Casimir charges C_α of the irreps read:

$$K_\alpha = \left\{\tfrac{N(N+2)(N-1)}{2}, \tfrac{N(N-2)(N+1)}{2}, N\right\}, \tag{95}$$

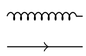
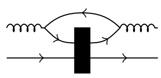
$$C_\alpha = \left\{\tfrac{(N+1)(3N-1)}{2N}, \tfrac{(N-1)(3N+1)}{2N}, C_F\right\}. \tag{96}$$

Naming the irreps by their dimensions in the case $N = 3$, we thus have

$$\mathbf{8} \otimes \mathbf{3} = \mathbf{15} \oplus \bar{\mathbf{6}} \oplus \mathbf{3}\,. \tag{97}$$

The second irrep is denoted as $\bar{\mathbf{6}}$ because for $N = 3$, it is equivalent to the *complex conjugate* of the irrep $\mathbf{6}$ appearing in the diquark case $\mathbf{3} \otimes \mathbf{3} = \mathbf{6} \oplus \bar{\mathbf{3}}$ (see lesson 3).

Exercise 30. Prove the latter statement, by first verifying that the projector P_{λ_2} can be rewritten as (for SU(N))

$P_{\lambda_2} = 2\,\Big[$  $- P_{\lambda_3}\Big]$  , (98)

then recalling the result of Exercise 21 for $N = 3$, and finally using Schur's lemma.

Let us emphasize that for $N > 3$, the irreps denoted by their "SU(3) names" $\mathbf{6}$ (appearing in $\mathbf{3} \otimes \mathbf{3} = \mathbf{6} \oplus \bar{\mathbf{3}}$) and $\bar{\mathbf{6}}$ (appearing in $\mathbf{8} \otimes \mathbf{3} = \mathbf{15} \oplus \bar{\mathbf{6}} \oplus \mathbf{3}$) have nothing in common and should be considered separately as defined by their associated projectors.

4.3 qqq

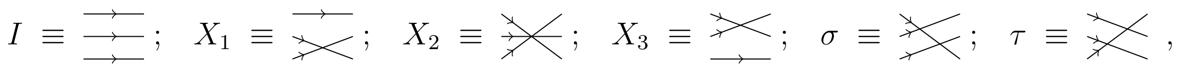
It is important to study the qqq system once in a lifetime. Indeed, for $N = 3$ the decomposition of qqq into multiplets contains the color singlet *baryons* of the real world. Moreover, it is a simple case allowing to discuss transition operators.

4.3.1 Decomposition into a sum of irreps

We follow the same systematic procedure.

(i) Complete set of independent tensors mapping $V \otimes V \otimes V \equiv V^{\otimes 3} \to V^{\otimes 3}$

Using the same algorithm as in previous cases, one finds a set of six tensors with only quark lines,



(99)

which correspond to the elements of the permutation group on three objects S_3.

This is the first example we encounter where some of the independent tensors (here σ and $\tau = \sigma^\dagger$) are not Hermitian. The six tensors cannot all be used in the construction of Hermitian projectors, and for the qqq system we must have $n_{\text{irreps}} < n_{\text{tensors}}$ (see the remarks in the end of section 3.1).

(ii) Multiplication table

The multiplication table is that of the group S_3.

$\cdot$	I	σ	τ	X_1	X_2	X_3
I	I	σ	τ	X_1	X_2	X_3
σ	σ	τ	I	X_2	X_3	X_1
τ	τ	I	σ	X_3	X_1	X_2
X_1	X_1	X_3	X_2	I	τ	σ
X_2	X_2	X_1	X_3	σ	I	τ
X_3	X_3	X_2	X_1	τ	σ	I

Table 1: Multiplication table of the set of tensors (99). To read this table correctly, note that $\sigma X_2 = X_3$, $X_2\,\sigma = X_1 \ldots$, each entry being easily checked pictorially.

In order to determine the qqq irreps and associated projectors, we have to find the maximal set of *commuting* Hermitian tensors which can be obtained from the set (99). From the above table we easily verify the following points:

❄ The tensors $\sigma+\tau$ and $\Sigma \equiv X_1+X_2+X_3$ are Hermitian and commute with every tensor of the set (99). We can thus take the tensors I, $\sigma+\tau$, Σ in the set we are looking for.

❄ The tensors X_1, X_2, X_3 are Hermitian, but do not commute with each other. Thus, only one of the X_i's can be added to the set. Let us choose X_3.

We therefore choose $\{I, \sigma+\tau, \Sigma, X_3\}$ as the maximal set from which we can find four irreps and the associated projectors. Note that the tensors $\{\sigma-\tau, X_2 - X_1\}$, which complete the former set to form a basis of six tensors mapping $V^{\otimes 3} \to V^{\otimes 3}$, are not necessary for the construction of irreps. Their interpretation will be given shortly.

In order to proceed to the explicit construction of projectors, we observe that the operator $\sigma+\tau$ has a simple characteristic equation:

$$(\sigma+\tau)^2 = \sigma+\tau+2I \,. \tag{100}$$

Its minimal polynomial is thus $x^2-x-2=(x+1)(x-2)$, and in some basis of $\left\{q^i q^j q^k\right\} \equiv V^{\otimes 3}$, it is represented by the matrix

$$\sigma+\tau = \begin{pmatrix} -1 & & & & \\ & \ddots & & & \\ & & -1 & & \\ & & & 2 & \\ & & & & \ddots \\ & & & & & 2 \end{pmatrix} \,. \tag{101}$$

(iii) Projectors on eigenspaces

The projectors on the eigenspaces of $\sigma+\tau$ associated to the eigenvalues -1 and 2 read

$$P_{(-1)} = \frac{1}{3}\left(2I-\sigma-\tau\right) \,, \tag{102}$$

$$P_{(2)} = \frac{1}{3}\left(I+\sigma+\tau\right) \,. \tag{103}$$

By construction, these projectors are Hermitian and mutually orthogonal invariant tensors, and satisfy the completeness relation $P_{(-1)}+P_{(2)} = I = \mathbb{1}_{V^{\otimes 3}}$. Thus, they split the vector space $V^{\otimes 3}$ into two SU(N) invariant subspaces. Since we know there are four *irreducible* representations, we continue the splitting of the space by considering the action of another tensor of our set of four, e.g. X_3, in each subspace already found.

Let us start with the action of X_3 in $\text{img}(P_{(-1)})$. Since $X_3^2 = I \Rightarrow X_3^2\, P_{(-1)} = P_{(-1)}$, quite trivially the minimal polynomial of X_3 restricted to $\text{img}(P_{(-1)})$ is x^2-1, and in some basis of $\text{img}(P_{(-1)})$ the operator X_3 is of the form

$$X_3|_{\text{img}(P_{(-1)})} = \begin{pmatrix} 1 & & & & \\ & \ddots & & & \\ & & 1 & & \\ & & & -1 & \\ & & & & \ddots \\ & & & & & -1 \end{pmatrix} \,. \tag{104}$$

The projectors on the respective eigenspaces are given by

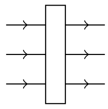
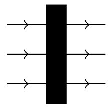
$$P_\pm = \frac{I \pm X_3}{2}\, P_{(-1)} = \frac{1}{6}\left[2I-\sigma-\tau \pm (2X_3 - X_1 - X_2)\right] \,. \tag{105}$$

Applying the same reasoning in the subspace $\text{img}(P_{(2)})$, we obtain two other projectors (on the two eigenspaces of X_3 restricted to $\text{img}(P_{(2)})$),

$$\tilde{P}_\pm = \frac{I \pm X_3}{2}\, P_{(2)} = \frac{1}{6}\left[I+\sigma+\tau \pm (X_1+X_2+X_3)\right] \,. \tag{106}$$

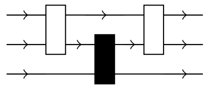
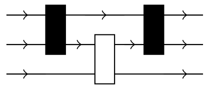
We have thus found four projectors satisfying all requirements, and thus the four irreps.

The projectors $\tilde{P}_\pm$ coincide respectively with the symmetrizer and antisymmetrizer over the three quark indices, defined and denoted pictorially as [3]:

$$\mathcal{S} \equiv \frac{1}{3!} \sum_{\pi \in S_3} \pi \equiv$$  ; $$\mathcal{A} = \frac{1}{3!} \sum_{\pi \in S_3} \text{sign}(\pi)\, \pi \equiv$$  . (107)

In (107) the sums are over all permutations π of S_3, and $\text{sign}(\pi)$ denotes the signature of the permutation (with $\text{sign}(\pi) = +1$ for $\pi = I, \sigma, \tau$ and $\text{sign}(\pi) = -1$ for $\pi = X_1, X_2, X_3$).

Exercise 31. Show that the projectors $P_\pm$ given by (105) can also be written as

$$P_+ = \frac{4}{3}$$  ; $$P_- = \frac{4}{3}$$  , (108)

corresponding to mixed symmetries in quark indices.

Exercise 32. Find the dimensions of the four irreps as a function of N.

Naming as usual the irreps by their dimensions for $N = 3$, the irreps associated with the projectors $\mathbb{P}_\alpha \equiv \{\tilde{P}_-, P_+, P_-, \tilde{P}_+\}$ are thus labelled by $\alpha = \{\mathbf{1}, \mathbf{8}^+, \mathbf{8}^-, \mathbf{10}\}$. With this notation the decomposition of SU(N) qqq states into a sum of irreps reads

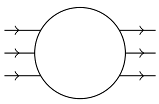
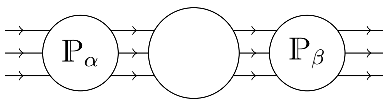
$$\mathbf{3} \otimes \mathbf{3} \otimes \mathbf{3} = \mathbf{1} \oplus \mathbf{8}^+ \oplus \mathbf{8}^- \oplus \mathbf{10}\ . \qquad (109)$$

Exercise 33. For $N = 3$, the irrep $R = \mathbf{1}$ is the representation of the color singlet baryons, and its Casimir must vanish. Calculate the Casimir of the SU(N) irrep $R = \mathbf{1}$ for general N.

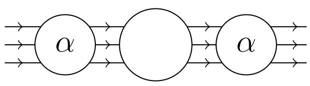
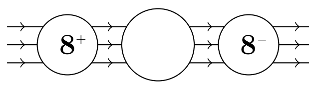
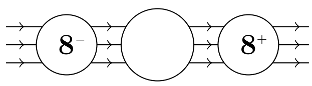
4.3.2 Transition operators

Among the six independent tensors contributing to the construction of all possible maps of $V^{\otimes 3} \to V^{\otimes 3}$, the set $\{I, \sigma + \tau, \Sigma, X_3\}$ (which can be traded for the $\mathbb{P}_\alpha$'s defined just above) allows to build the four irreps. What is the interpretation of the remaining tensors $\sigma - \tau$ and $X_2 - X_1$?

Let us repeat the counting of independent tensors as follows. Using the completeness relation $\mathbb{1} = \sum_\alpha \mathbb{P}_\alpha$, any $V^{\otimes 3} \to V^{\otimes 3}$ map can be put in the form

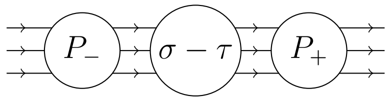
 $$= \sum_{\alpha,\beta}$$  . (110)

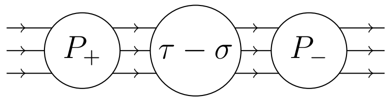
From Schur's lemma, in (110) only the terms for which α and β are equivalent irreps can contribute. Since equivalent irreps must at least have the same dimension, all possible maps are encompassed by the structures

 $\propto \mathbb{P}_\alpha$,  ,  . (111)

We infer that the two latter structures span the same subspace as $\{\sigma - \tau, X_2 - X_1\}$. In particular, they must be non-zero for some choice of the middle blob, thus defining *transition operators* (and proving in passing that $\mathbf{8}^+$ and $\mathbf{8}^-$ are equivalent). The operators $\sigma - \tau$ and $X_2 - X_1$ are responsible for the transitions $\mathbf{8}^+ \leftrightarrow \mathbf{8}^-$.

We have seen in lesson 3 that a transition operator is uniquely defined, see Exercise 22. In order to find the $\mathbf{8}^+ \leftrightarrow \mathbf{8}^-$ transition operators, we just need to insert in the middle blob some tensor giving a non-zero result. We can choose $\sigma - \tau$, and thus take

$$Q_a \equiv P_+(\sigma - \tau)P_- =$$  , (112)

$$Q_b \equiv Q_a^\dagger = P_-(\tau - \sigma)P_+ =$$  , (113)

for the $\mathbf{8}^- \to \mathbf{8}^+$ and $\mathbf{8}^+ \to \mathbf{8}^-$ transition operators, respectively. Note that Q_a and Q_b are nilpotent operators ($Q_a^2 = P_+(\sigma - \tau)P_-P_+(\sigma - \tau)P_- = 0$ since $P_-P_+ = 0$), as is the case for any transition operator mapping two irreps of the same vector space.

Exercise 34. Express Q_a and Q_b as linear combinations of $\sigma - \tau$ and $X_2 - X_1$.

Exercise 35. Check that $Q_aQ_b = 3P_+$ and $Q_bQ_a = 3P_-$, i.e., Q_aQ_b and Q_bQ_a are proportional to the identity operators in the subspaces $\mathrm{img}(P_+)$ and $\mathrm{img}(P_-)$, respectively.

Exercise 36. Find all *unitary* similarity transformations between the irreps $\mathbf{8}^-$ and $\mathbf{8}^+$.

Acknowledgments

Many thanks to the organizers of the school for their kind invitation and to all the participants for the friendly and relaxed atmosphere!