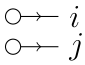
# 3 Diquark states, Schur's lemma and Casimir charges

In this lesson, we present a systematic method [1] to find the set of Hermitian projectors on the irreps of a parton system, which can in principle be applied to any parton system. Here it is explained in the very simple case of a $qq$ pair, and other examples will be addressed in lesson 4.

## 3.1 Irreps of diquark states

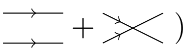
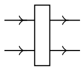
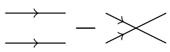
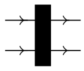
Pictorially, a diquark state is represented as



$$\equiv q^i q^j \, . \qquad (64)$$

As we saw in lesson 2, finding a basis of the vector space $\{q^i q^j\} \equiv V \otimes V$ (of dimension $N^2$) where all color rotations (represented by $N^2 \times N^2$ matrices) are block-diagonal (and cannot be further block-diagonalized) amounts to finding a maximal and complete set of Hermitian, mutually orthogonal projectors $\mathbb{P}_i$.

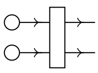
The case of diquarks being very simple and well known, let us immediately give the result for the relevant set of projectors. It is composed of two projectors corresponding to the symmetrizer and anti-symmetrizer (over the two quark indices), given respectively by:

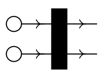
$$P_S = \frac{1}{2}\Big($$  $$\Big) \equiv$$  $$; \quad P_A = \frac{1}{2}\Big($$  $$\Big) \equiv$$  $$. \qquad (65)$$

For a system $\{q^i q^j \dots q^p\}$ made up only of quarks, representation theory [5] tells us that the bases of the SU($N$) invariant (and irreducible) subspaces are given by linear combinations of $q^i q^j \dots q^p$ having different symmetry properties in the permutation of indices. In the present case of two quarks, we can build either a totally symmetric or totally antisymmetric linear combination of $q^i q^j$, leading to the set (65) of projectors.

**Exercise 19.** Verify that $P_S$ and $P_A$ form a complete set of Hermitian projectors, which are mutually orthogonal. Calculate their ranks.

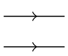
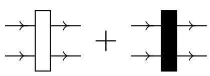
**Exercise 20.** img($P_S$) and img($P_A$) are the subspaces spanned by $V_S^{ij}$ and $V_A^{ij}$ defined by

$$V_S^{ij} = (P_S)^{ij}{}_{kl}\, q^k q^l \; = $$  $$= \frac{1}{2}(q^i q^j + q^j q^i) \, , \qquad (66)$$

$$V_A^{ij} = (P_A)^{ij}{}_{kl}\, q^k q^l \; = $$  $$= \frac{1}{2}(q^i q^j - q^j q^i) \, . \qquad (67)$$

Check that img($P_S$) and img($P_A$) are invariant under SU($N$) (which we know from lesson 2), by writing how $V_S^{ij}$ and $V_A^{ij}$ transform under finite color rotations (thus showing that they are SU($N$) tensors of rank 2).

In summary, we have the completeness relation

 $$= P_S + P_A =$$  $$, \qquad (68)$$

and the product of two fundamental representations decomposes into a sum of irreps as

$$N \otimes N \; = \; \frac{N(N+1)}{2} \oplus \frac{N(N-1)}{2} \, , \qquad (69)$$

where only the dimensions of the irreps are mentioned. Note that in general, knowing the dimension of an irrep is not sufficient to fully determine the irrep, as illustrated by the following exercise.

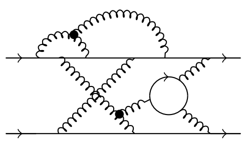
**Exercise 21.** For $N = 3$, the relation (69) reads $\mathbf{3}\otimes\mathbf{3} = \mathbf{6}\oplus\mathbf{3}$, but the generators of the irrep of dimension 3 acting on the subspace spanned by $V_A^{ij}$ are equivalent to $-(T^a)^* = -({}^tT^a)$, i.e., $V_A^{ij}$ does not transform under SU(3) as a quark, but as an antiquark. (Thus, (69) is commonly written as $\mathbf{3}\otimes\mathbf{3} = \mathbf{6}\oplus\bar{\mathbf{3}}$.) Prove this by trading the three independent components of $V_A^{ij}$ for the 3-vector $B_k \equiv \frac{1}{2}\epsilon_{ijk}V_A^{ij}$ (with $\epsilon_{ijk}$ the Levi-Civita tensor of rank 3) and by evaluating the shift $\delta B_k$ under an infinitesimal color rotation.

Let us now suppose that we do not know anything about representation theory, and that we therefore do not know from the start the set of Hermitian projectors. We describe below a systematic method to find them. In the $qq$ case, the method is obviously not the most economical, but an advantage of this method is that it can be applied to any parton system composed of quarks, antiquarks and gluons (as we will see in lesson 4).

The general procedure consists of three steps:

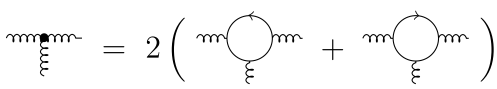
(i) Find the maximal number of linearly independent operators (built from the basic legos, thus being SU($N$) invariant tensors when specifying external indices) mapping the vector space to itself.

In the present case of the vector space $V \otimes V$, an operator (or tensor) of this type can be expressed in terms of graphs of the generic form

 . (70)

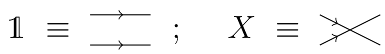
Such graphs can be replaced by (linear combinations of) simpler graphs using the following algorithm.

First, we can get rid of any three-gluon vertex appearing in the graph by using the identity (prove it!)

 . (71)

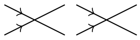
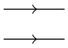
The graphs then reduce to (linear combinations of) graphs where any internal gluon connects at both ends to quark lines.

Second, every internal gluon can be removed with the help of the Fierz identity (17). So we end up with graphs with four external quark lines (together with airborne quark loops that simply contribute to an irrelevant global factor $N^\ell$) and without gluons. There are only two ways to connect the four external quark lines, and this proves that there are only two independent tensors mapping $V \otimes V$ to itself, namely,

 . (72)

(ii) Find the "multiplication table" between these operators, and infer the minimal polynomial and eigenvalues of the most interesting one(s).

The multiplication table of the set $\{\mathbb{1}, X\}$ has only one non-trivial entry,

$X^2 =$  $=$  $= \mathbb{1}\,.$ (73)

The characteristic equation of the operator $X$ is $X^2 - \mathbb{1} = 0$. The minimal polynomial of $X$ is thus $x^2 - 1 = (x-1)(x+1)$, which is split with simple roots. From basic linear algebra, it follows that $X$ can be diagonalized (which is not a surprise since $X$ is clearly Hermitian) and has eigenvalues $\{\lambda_1, \lambda_2\} = \{1, -1\}$.

In some basis of $\{q^i q^j\} \equiv V \otimes V$, the matrix representation of $X$ thus reads

$$X = \begin{pmatrix} \lambda_1 & & & & \\ & \ddots & & & \\ & & \lambda_1 & & \\ & & & \lambda_2 & & \\ & & & & \ddots & \\ & & & & & \lambda_2 \end{pmatrix} . \tag{74}$$

(iii) Express the projectors on the corresponding eigenspaces in terms of the SU$(N)$ invariant tensors.

In the above basis, the projectors $P_{\lambda_1}$ and $P_{\lambda_2}$ on the eigenspaces of $X$ are

$$P_{\lambda_1} = \begin{pmatrix} 1 & & & & \\ & \ddots & & & \\ & & 1 & & \\ & & & 0 & & \\ & & & & \ddots & \\ & & & & & 0 \end{pmatrix} \quad ; \quad P_{\lambda_2} = \begin{pmatrix} 0 & & & & \\ & \ddots & & & \\ & & 0 & & \\ & & & 1 & & \\ & & & & \ddots & \\ & & & & & 1 \end{pmatrix} . \tag{75}$$

Their explicitly SU$(N)$ invariant form follows from the identities $X = \lambda_1 P_{\lambda_1} + \lambda_2 P_{\lambda_2}$ and $\mathbb{1} = P_{\lambda_1} + P_{\lambda_2}$, or directly from a mere observation of the matrix $X$ given in (74),

$$P_{\lambda_1} = \frac{X - \lambda_2 \mathbb{1}}{\lambda_1 - \lambda_2} = \frac{1}{2}(\mathbb{1} + X) = P_S \quad ; \quad P_{\lambda_2} = \frac{X - \lambda_1 \mathbb{1}}{\lambda_2 - \lambda_1} = \frac{1}{2}(\mathbb{1} - X) = P_A\,. \tag{76}$$

We thus recover the projectors (65) without any prior knowledge of representation theory. Note that in the above derivation, the resulting projectors satisfy all requirements *by construction*: they are Hermitian and mutually orthogonal, they form a complete set ($P_S + P_A = \mathbb{1}$), and they cannot be reduced into a sum of more SU$(N)$ invariant projectors (since this would imply that there are more than two independent tensors mapping $V \otimes V$ to itself). Therefore, $P_S$ and $P_A$ must project onto the irreducible representations of diquark states.

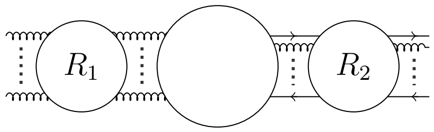
Let us end this section by two important remarks:

❖ Obviously, the number of projectors (i.e., the number of irreps) cannot exceed the number of independent tensors determined in step (i), $n_{\text{irreps}} \leq n_{\text{tensors}}$. For diquarks, we have $n_{\text{irreps}} = n_{\text{tensors}}$, but for more complicated systems we may have $n_{\text{irreps}} < n_{\text{tensors}}$. This happens when some of the independent tensors are not Hermitian and therefore cannot contribute to the construction of Hermitian projectors. This will be the case for the $qqq$ system considered in lesson 4.

❖ When $n_{\text{irreps}} < n_{\text{tensors}}$, one might naively think that $n_{\text{irreps}}$ coincides with the number of independent tensors that are Hermitian, but this is not the case. Indeed, the projectors are linear combinations of some subset of the independent tensors, and the tensors of this subset are thus linear combinations of the projectors. Since the projectors are not only Hermitian but also commute between them, the same must be true for the independent tensors of the subset. We infer that in general, it is the largest subset of *commuting* Hermitian operators found among the independent tensors that is used to construct the projectors, and thus determines the number of irreps.

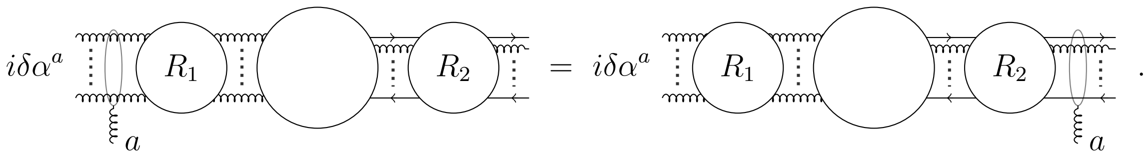
## 3.2 Schur's lemma

Consider an invariant tensor $A$ mapping the irreps $R_1$ and $R_2$ of respective vector spaces $W_1$ and $W_2$,

$A =$ 

. (77)

Since $A$ is an invariant tensor we can use color conservation:



. (78)

In the l.h.s. of (78), the infinitesimal color rotation acts in the irrep $R_1$ (as is pictorially obvious, it maps $\text{img}(\mathbb{P}_{R_1})$ to itself, see lesson 2). The same color rotation acts in the irrep $R_2$ in the r.h.s. of (78). For finite color rotations, (78) thus reads

$$\forall\, U(\alpha) \in \text{SU}(N)\,, \quad A\, U_{R_1}(\alpha) \;=\; U_{R_2}(\alpha)\, A\,. \tag{79}$$

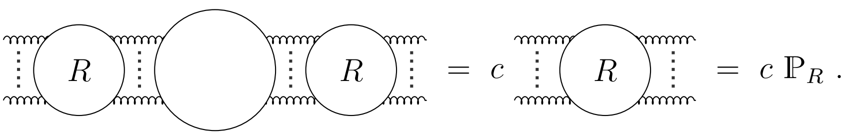
The condition (79) is the starting assumption for stating Schur's lemma, which consists of two parts (see e.g. Ref. [4] for a proof):

❖ if $R_1$ and $R_2$ are inequivalent irreducible representations, then $A = 0$.

This can be proven by showing that if $A \neq 0$, $A$ must be an invertible square matrix, hence $\exists\, A : \forall\, \alpha, U_{R_1}(\alpha) = A^{-1} U_{R_2}(\alpha) A$, i.e., $R_1$ and $R_2$ are simply related by a change of basis, which is the definition of *equivalent* representations.

Viewing the tensor $A$ represented pictorially in (77) as the "evolution" of a parton system, we see that in absence of interaction with external color fields, a parton system may change its composition, but always remains in equivalent irreps.

❖ if $W_1 = W_2$ (i.e., the initial and final parton content is the same) and $R_1 = R_2 \equiv R$, then $A$ is proportional to the identity operator in the irrep $R$ (given by $\mathbb{1}_R = \mathbb{P}_R$),



(80)

The latter equation can be interpreted as follows. Suppose we prepare an incoming multiplet $R$ and try to mix the basis states of this multiplet with the help of an invertible matrix (represented by the middle blob in the l.h.s. of (80)), which is thus a map of img($\mathbb{P}_R$) $\to$ img($\mathbb{P}_R$). Due to Schur's lemma (80), up to an overall factor we get exactly the same states. The basis states of a multiplet are uniquely defined.

Both parts of Schur's lemma are very important results, useful for simplifying calculations and also for intuition.

To end this section, let us mention that when $R_1 \neq R_2$, a non-zero tensor $A$ of the form (77) is called a *transition operator* (for the transition $R_1 \to R_2$). In this case, Schur's lemma can be reformulated as:

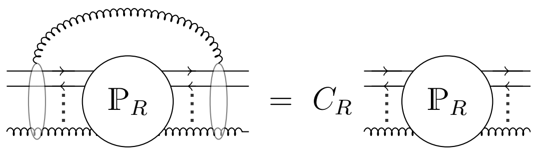
❖ There is a transition operator $A$ between $R_1$ and $R_2 \neq R_1$ *if and only if* $R_1$ and $R_2$ are equivalent irreps, and $A$ is then a similarity transformation between the two irreps, $U_{R_1}(\alpha) = A^{-1} U_{R_2}(\alpha) A$. (The reciprocal is trivial: if $R_1$ and $R_2$ are equivalent, there exists a non-zero operator $A$ such that $U_{R_1} = A^{-1} U_{R_2} A$, thus mapping img($\mathbb{P}_{R_1}$) $\to$ img($\mathbb{P}_{R_2}$), i.e., there is a transition operator $A$ between $R_1$ and $R_2$.)

**Exercise 22.** When $R_1$ and $R_2$ are different but equivalent irreps, show that the transition operator between $R_1$ and $R_2$ is uniquely defined (up to an overall factor).

We will see examples of transition operators in lesson 4, when discussing the $qqq$ system.

## 3.3 Casimir charges

In lesson 2 we gave the pictorial expression (63) of SU($N$) generators $T^a(R)$ in the irrep $R$. The Casimir operator in this representation is defined by $T^a(R)T^a(R)$, which from Schur's lemma (80) must be proportional to $\mathbb{1}_R = \mathbb{P}_R$, with a proportionality coefficient named the Casimir charge $C_R$,

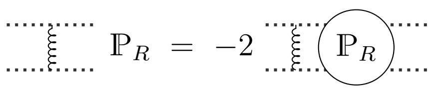
$T^a(R)T^a(R) =$  . (81)

Unlike the generators $T^a(R)$, the Casimir operator commutes with all SU($N$) transformations. In lesson 1, we met the Casimir operators $T^aT^a = C_F \mathbb{1}_V$ and $t^at^a = C_A \mathbb{1}_A$ in the fundamental (quark) and adjoint (gluon) representations.

**Exercise 23.** Show that the global Casimir charge $C_R$ of a color state $R$ of two partons (of individual Casimir charges $C_1$ and $C_2$) is given by

$$C_R = C_1 + C_2 + v_{12}(R)\,, \tag{82}$$

where the color "interaction potential" $v_{12}(R)$ of the parton pair in irrep $R$ is defined by

$v_{12}(R)\,\mathbb{P}_R \equiv -2$  . (83)

What is the generalization of (82) to a system of $n > 2$ partons?

**Exercise 24.** Calculate the Casimir charges of the two diquark irreps (associated with the projectors $P_S$ and $P_A$) as a function of $N$. In which color state is the color interaction potential attractive?