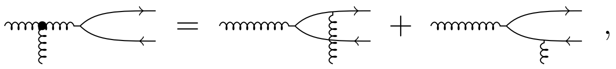
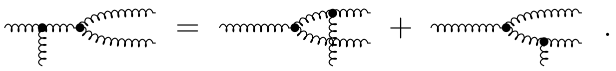
# 2 Color conservation, color rotations and SU($N$) irreps

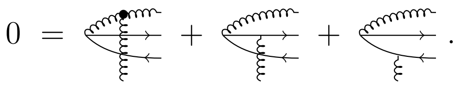
## 2.1 Color conservation pictorially

In lesson 1, we wrote the pictorial form of the Lie algebra, both in the fundamental and adjoint representation (see (25) and (28)),


(32)

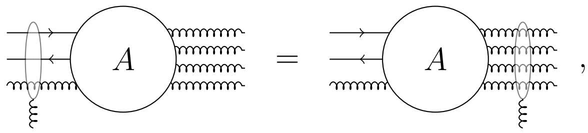

(33)

The latter equations can also be written by stretching the incoming gluon line to the final state (or the outgoing parton lines to the initial state). For instance, stretching the incoming gluon in (32) to the final state, one obtains (due to the antisymmetry of the lego bricks under the exchange of two lines)


(34)

Thus, for each lego the sum of gluon attachments to parton lines 'before' and 'after' the interaction vertex gives the same result, and this holds independently of the way the lego is represented in a time-ordered picture: either as a $1 \to 2$ splitting (as in (32) and (33)), or $2 \to 1$, or $3 \to 0$, or $0 \to 3$ (as in (34)).

This trivially generalizes to any operator constructed from the lego bricks, leading to the pictorial representation of *color conservation*,


(35)

where an ellipse crossed by a set of parton lines denotes the sum of all attachments of the "scattering gluon" to those lines.

**Exercise 13.** Although trivial, give a convincing proof of (35).

## 2.2 Color rotations

### 2.2.1 Finite SU($N$) transformations

The special unitary group SU($N$) is the Lie group of $N \times N$ unitary matrices with unit determinant,

$$U \in \mathrm{SU}(N) \;\Leftrightarrow\; UU^\dagger = \mathbb{1} \;\text{ and }\; \det U = 1\,. \qquad (36)$$

Any element of SU($N$) can be parametrized by

$$U(\alpha) = e^{i\alpha^a T^a}\,, \qquad (37)$$

where the matrices $T^a$ ($a = 1 \ldots N^2 - 1$) are the Hermitian matrices introduced in lesson 1 (hence the name of SU($N$) *generators* for those matrices). The real parameters $\alpha^a$ may be viewed as the "angles" of the "color rotation" $U(\alpha)$.

**Exercise 14.** Check that the matrix (37) indeed belongs to SU($N$). (In fact, the exponential parametrization (37) generates *all* elements of SU($N$), see mathematics textbooks for a proof.)

By construction, the QCD lagrangian is invariant under SU($N$) transformations or "color rotations". In order to address the color structure of QCD (in particular, to determine the invariant multiplets of a parton system), we first consider SU($N$) transformations of the quark, antiquark and gluon "color coordinates".

**Color rotations of quark and antiquark coordinates**

Let us start with quarks and antiquarks. Under a given color rotation $U(\alpha) \in$ SU($N$), the quark coordinates (denoted by an *upper* index according to our initial convention, see section 1.1) transform as

$$q' = U\, q \;\;\Leftrightarrow\;\; q'^{\,i} = U^i{}_j\, q^j\,. \tag{38}$$

When restricting to the color degree of freedom, antiquark coordinates are simply obtained from quark coordinates by complex conjugation. In the same color rotation of angles $\alpha^a$, antiquark coordinates thus transform as

$$q^{*\prime} = U^*\, q^* \;\;\Leftrightarrow\;\; (q^{*\prime})^i = (U^*)^i{}_j\, (q^*)^j\,. \tag{39}$$

A standard convention is to denote complex conjugation by moving quark and antiquark indices up and down, namely,

$$(q^*)^i \equiv q_i\;;\quad (U^*)^i{}_j \equiv U_i{}^j\,, \tag{40}$$

a convention that we have implicitly used from the beginning by assigning *lower* color indices to antiquarks, see section 1.1. The transformation (39) of antiquark coordinates is then written as

$$q_i' = U_i{}^j\, q_j\,. \tag{41}$$

To complement the above convention, any quantity transforming as quark (antiquark) coordinates is assigned an upper (lower) index. We readily verify that a product of the form $A^iB_i$ (implicitly summed over $i$) is SU($N$) invariant. Indeed, $(A^iB_i)' = U^i{}_j\, U_i{}^k\, A^jB_k = A^iB_i$, since $U^i{}_j\, U_i{}^k = U^i{}_j(U^*)^i{}_k = U^i{}_j(U^\dagger)^k{}_i = (U^\dagger U)^k{}_j = \delta^k{}_j$.

Under two successive color rotations of angles $\alpha^a$ and $\beta^b$, quark coordinates transform as

$$q \;\xrightarrow{\alpha}\; U(\alpha)\, q \;\xrightarrow{\beta}\; U(\beta)U(\alpha)\, q = U(\gamma(\alpha,\beta))\, q\,. \tag{42}$$

Indeed, since SU($N$) is a group, the product $U(\beta)U(\alpha)$ must coincide with an element of SU($N$) of angles $\gamma^c$, the latter being fully determined by $\alpha^a$ and $\beta^b$.

**Exercise 15.** (To be done once in a lifetime.)

Let us recall the Baker-Campbell-Hausdorff formula for the product of two exponentials of matrices,

$$e^X\, e^Y = e^{X+Y+\frac{1}{2}[X,Y]+\frac{1}{12}([X,[X,Y]]-[Y,[X,Y]])+\ldots}\,, \tag{43}$$

where the dots stand for higher-order terms in $X$ and $Y$ (all being nested commutators of $X$ and $Y$). Using (43), show that the angles $\gamma^a$ defined by (42) are given by $\gamma^a(\alpha,\beta) = \alpha^a + \beta^a + \frac{1}{2} f_{abc}\alpha^b\beta^c + \ldots$, and find the next term in the series.

This exercise illustrates that the structure of SU($N$) (with respect to the multiplication law) is fully determined by the SU($N$) Lie algebra (4).

### Color rotations of gluon coordinates

How should the $N^2-1$ gluon coordinates $\Phi^a$ transform under a color rotation of angles $\alpha^a$, when the $N$ quark coordinates transform with the matrix $U(\alpha)$? For each $U(\alpha)$ acting in quark space, we must find a corresponding $(N^2-1)\times(N^2-1)$ matrix $\tilde{U}(\alpha)$ acting in gluon space, in such a way that the "representation" $U(\alpha)\to\tilde{U}(\alpha)$ preserves the group structure. Indeed, in the successive rotations of angles $\alpha$ and $\beta$, the gluon coordinates become

$$\Phi \xrightarrow{\alpha} \tilde{U}(\alpha)\,\Phi \xrightarrow{\beta} \tilde{U}(\beta)\tilde{U}(\alpha)\,\Phi\,, \tag{44}$$

but for consistency with (42), the same result should be obtained by a single rotation of angles $\gamma(\alpha,\beta)$, represented by $\tilde{U}(\gamma(\alpha,\beta))$ when acting in gluon space. We thus need

$$\tilde{U}(\beta)\tilde{U}(\alpha)=\tilde{U}(\gamma(\alpha,\beta))\,, \tag{45}$$

with the same function $\gamma(\alpha,\beta)$ as derived in Exercise 15.

It is clear that (45) will be satisfied by the matrices

$$\tilde{U}(\alpha)=e^{i\alpha^a\tilde{T}^a}\,, \tag{46}$$

provided one can find $(N^2-1)\times(N^2-1)$ matrices $\tilde{T}^a$ $(a=1\dots N^2-1)$ having the same Lie algebra as the $T^a$'s, namely,

$$\left[\tilde{T}^a,\tilde{T}^b\right]=if_{abc}\tilde{T}^c\,. \tag{47}$$

We know from lesson 1 that such matrices exist: the matrices $t^a$ defined by (5) satisfy (29).

A few remarks:

- The set of matrices $U(\alpha)=e^{i\alpha^aT^a}$ (i.e., the SU($N$) group itself) acting on the quark and $\tilde{U}(\alpha)=e^{i\alpha^at^a}$ acting on the gluon are respectively called the *fundamental* and *adjoint* SU($N$) representations.
- The adjoint representation is real: $\tilde{U}(\alpha)^*=e^{-i\alpha^a(t^a)^*}=e^{i\alpha^at^a}=\tilde{U}(\alpha)$.
- If there are $d_R\times d_R$ matrices $T^a(R)$ $(a=1\dots N^2-1)$ satisfying the SU($N$) Lie algebra, $\left[T^a(R),T^b(R)\right]=if_{abc}T^c(R)$, the matrices $U_R(\alpha)=e^{i\alpha^aT^a(R)}$ define an SU($N$) representation of dimension $d_R$, acting on objects with $d_R$ components while preserving the group structure. The $T^a(R)$'s are the SU($N$) generators in the representation $R$.
- For $N>2$, SU($N$) representations do not exist for any dimension $d_R$. For $N=3$, the possible dimensions are $d_R=1,3,6,8,10,15\dots$
- When there is no risk of confusion, an SU($N$) (irreducible) representation is labelled by its dimension in the case $N=3$. For instance, the fundamental and adjoint SU($N$) representations are denoted by $R=\mathbf{3}$ and $R=\mathbf{8}$, with generators $T^a(\mathbf{3})=T^a$ and $T^a(\mathbf{8})=t^a$.
- The antiquark transforms under the complex conjugate of the fundamental representation, denoted by $R=\bar{\mathbf{3}}$ and given by the set of $N\times N$ matrices $U(\alpha)^*\equiv e^{i\alpha^aT^a(\bar{\mathbf{3}})}$, with generators $T^a(\bar{\mathbf{3}})=-(T^a)^*$. Although the representations $\mathbf{3}$ and $\bar{\mathbf{3}}$ have the same dimension $N$, they are not equivalent (for $N>2$), i.e., $U(\alpha)$ and $U(\alpha)^*$ are not related by a change of basis, and thus describe the transformations of different objects.

### 2.2.2 Infinitesimal color rotations

SU($N$) is a Lie group for which infinitesimal transformations capture most of the group structure [4]. In particular, it is sufficient to consider infinitesimal transformations to highlight SU($N$) representations (see section 2.3).

Let us consider a color rotation of infinitesimal angles $\delta\alpha^a$. According to (37) and (38), the quark transforms as

$$q'^{\,i} = q^i + i\delta\alpha^a\,(T^a)^i{}_j\,q^j\,, \tag{48}$$

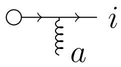
from which the transformation of the antiquark directly follows (take the complex conjugate, and recall that $(T^a)^* = {}^tT^a$):

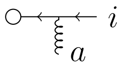
$$q'_i = q_i - i\delta\alpha^a\,q_j\,(T^a)^j{}_i\,. \tag{49}$$

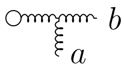
Using (46), the gluon transforms as:

$$\Phi'^{\,b} = \Phi^b + i\delta\alpha^a\,(t^a)_{bc}\,\Phi^c\,. \tag{50}$$

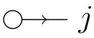
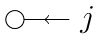
The *infinitesimal shifts* of the quark, antiquark and gluon coordinates thus read

$$\delta q^i \;\equiv\; q'^{\,i} - q^i \;=\; i\delta\alpha^a$$  , (51)

$$\delta q_i \;\equiv\; q'_i - q_i \;=\; i\delta\alpha^a$$  , (52)

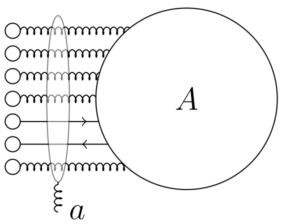
$$\delta \Phi^b \;\equiv\; \Phi'^{\,b} - \Phi^b \;=\; i\delta\alpha^a$$  , (53)

where we introduced the pictorial notation for coordinates:

 $\equiv q^j$ ;  $\equiv q_j$ ;  $\equiv \Phi^c$ . (54)

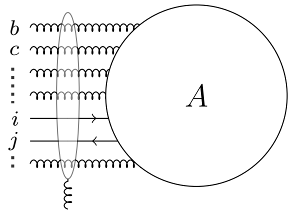
Our basic legos (1) are defined as the SU($N$) generators in the quark, antiquark and gluon representations. Up to the factor $i\delta\alpha^a$, the legos are thus nothing but the infinitesimal shift of the corresponding parton coordinates. In other words, in a color rotation of angles $\delta\alpha^a$ the infinitesimal shift of parton coordinates is obtained pictorially (up to the factor $i\delta\alpha^a$) by attaching a gluon of color $a$ from below to the corresponding line.

Let us now rewrite the "color conservation identity" (35) as

$i\delta\alpha^a$  $= 0$ . (55)

In (55), the sum of the infinitesimal shifts is obviously the infinitesimal shift of the incoming multi-parton state. Thus, a parton system which is fully contracted over parton color indices is SU($N$) invariant. Such a system is called a color singlet state. Color conservation (expressed pictorially as (35)) is equivalent to the SU($N$) invariance of color singlet systems.

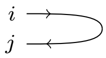
Note that if we do not contract with external parton coordinates, the identity (55) reads



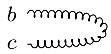

$$= 0\,, \tag{56}$$

with specified external indices $b, c, \ldots, i, j, \ldots$ Let us view the object carrying those indices, $A^{bc\ldots}{}_{i\ldots}{}^{j\ldots}$, as an SU($N$) tensor, thus transforming under SU($N$) as the product of parton coordinates $\Phi^b \Phi^c \ldots q_i \ldots q^j \ldots$ Eq. (56) gives an alternative formulation of color conservation, namely: all SU($N$) tensors (constructed from the basic legos) are in fact SU($N$) *invariant* tensors.

**Exercise 16.** Check explicitly that the tensors



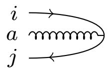

$$= \delta^j{}_i \quad ,$$




$$= \delta^{bc} \quad , \tag{57}$$

are invariant under finite color rotations.

**Exercise 17.** Express the SU($N$) invariance under finite color rotations of the tensor




$$= (T^a)^j{}_i \tag{58}$$

to obtain the relation

$$\tilde{U}_{bc} = 2\text{Tr}\,(T^b U T^c U^\dagger)\,, \tag{59}$$

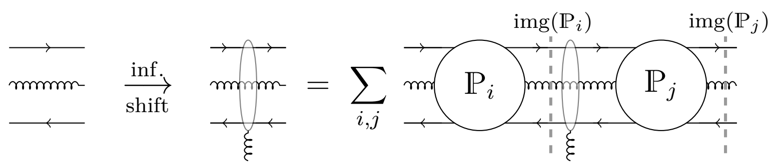
which determines the matrix elements $\tilde{U}_{bc}$ of a color rotation in the adjoint representation in terms of its fundamental representation $U$.

## 2.3 SU($N$) irreducible representations

Using the pictorial expression of color conservation and infinitesimal color rotations allows one to address SU($N$) irreducible representations in a rather intuitive way.

Consider a multi-parton system spanning a color vector space $E$ of dimension $n$, and suppose we have at disposal $m$ projectors $\mathbb{P}_i$ constructed from the basic legos and satisfying the conditions $\mathbb{P}_i \cdot \mathbb{P}_j = 0$ for $i \neq j$ and $\sum_{i=1}^m \text{rank}(\mathbb{P}_i) = n$, implying the completeness relation $\sum_{i=1}^m \mathbb{P}_i = \mathbb{1}_E$. (An explicit case was given in lesson 1 when proving the Fierz identity, see Exercise 2.) We also suppose the projectors to be Hermitian, $\mathbb{P}_i^\dagger = \mathbb{P}_i$.

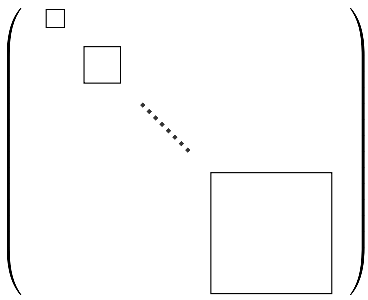
Let us apply an infinitesimal color rotation to the parton state (for the argument it is sufficient to keep only the infinitesimal shift and drop the factor $i\delta\alpha^a$), and then insert on the left and right the completeness relation :




$$, \tag{60}$$

where a dashed vertical line indicates to which subspace the corresponding intermediate multi-parton state belongs (here img($\mathbb{P}_i$) or img($\mathbb{P}_j$)).

Using color conservation and $\mathbb{P}_i \cdot \mathbb{P}_j = 0$ for $i \neq j$, only the terms with $i = j$ remain in the double sum. As a consequence, the image space $\mathrm{img}(\mathbb{P}_i)$ of the projector $\mathbb{P}_i$ is invariant under any infinitesimal color rotation, and thus under SU($N$). In a basis of $E$ obtained by joining bases of the invariant subspaces $\mathrm{img}(\mathbb{P}_i)$ (which due to the hermiticity of $\mathbb{P}_i$ are orthogonal to each other), any SU($N$) color rotation will be block-diagonal,

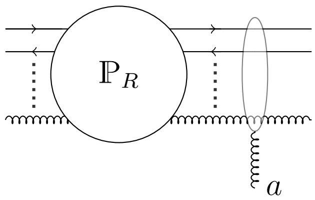
$U_{(E)} =$  . (61)

If each block cannot be further block-diagonalized, i.e., if the chosen set of projectors is of maximal cardinality, each invariant subspace $\mathrm{img}(\mathbb{P}_i)$ is said to transform under an irreducible representation (irrep) $R_i$ of SU($N$). The tensor product describing the parton system $\{q\bar{q}g\ldots\}$ is decomposed into a sum of irreps:

$$\mathbf{3} \otimes \bar{\mathbf{3}} \otimes \mathbf{8} \otimes \ldots \;=\; \bigoplus_{i=1}^{m} R_i \,. \tag{62}$$

In order to determine all irreps (also called multiplets) of a parton system, we need to find a maximal, complete set of Hermitian and mutually orthogonal projectors (constructed from the basic legos).

To conclude this lesson, let us give the pictorial representation of the SU($N$) generators $T^a(R)$ ($a = 1 \ldots N^2 - 1$) in the representation $R$ associated to the projector $\mathbb{P}_R$ (i.e., acting in the invariant subspace $\mathrm{img}(\mathbb{P}_R)$),

$T^a(R) \;=$  . (63)



Indeed, $T^a(R)$ defined in this way is a map of $\mathrm{img}(\mathbb{P}_R) \to \mathrm{img}(\mathbb{P}_R)$, and $i\delta\alpha^a\, T^a(R)$ acting on a parton state in $\mathrm{img}(\mathbb{P}_R)$ is the infinitesimal shift of this state under the infinitesimal color rotation of angles $\delta\alpha^a$.

**Exercise 18.** Check pictorially that the $T^a(R)$'s satisfy the SU($N$) Lie algebra.