Lecture notes on birdtracks

Introduction to color in QCD: initiation to the birdtrack pictorial technique

Stéphane Peigné

Subatech, Nantes, France

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Preamble

These lectures are an initiation to the fun and quite intuitive "birdtrack technique", which aims to address the color structure of QCD in a pictorial way, drawing color graphs named birdtracks [1] rather than writing mathematical symbols carrying color indices. In four lessons we will learn the basic rules, discuss color conservation and infinitesimal color rotations, learn how to project on partonic color states, derive their Casimir charges... and at the same time learn a little bit of representation theory.

We will consider SU(N) (with $N \geq 3$) as the symmetry group of QCD, i.e., N quark colors. For general N, the birdtrack technique is not more complicated than for "real QCD" ($N = 3$). On the contrary, working with a general parameter N usually brings a better understanding than working with fixed $N = 3$. The birdtrack technique can be used as a handy tool in virtually any SU(N) color calculation, whether it is evaluating simple color factors or addressing complex color structures that may otherwise seem out of reach. I hope these lessons will give you a glimpse of the unspeakable and profound satisfaction that drawing birdtracks can bring.

Participation in these courses does not require any prior knowledge (not even in QCD), except perhaps some notions of linear algebra, which will be recalled anyway. The lessons include a list of trivial and very simple exercises to keep the focus. Doing these exercises is mandatory to avoid superficial reading, and to learn how to use birdtracks in practice.

To go beyond these introductory lessons, the birdtrack drawer is encouraged to consult the following helpful references. In Ref. [2], the color pictorial technique is illustrated by nice examples and exercises. Ref. [3] gives a pedagogical presentation of birdtracks and addresses a systematic method (different from that described in lessons 3 and 4 of the present notes) to find the projectors on the color multiplets of multiparton systems. Ref. [3] also contains a substantial birdtrack bibliography. The color pictorial technique is a useful tool for QCD, but it has much broader applications. The birdtrack lover should consult the valuable bible on the uses of birdtracks in group theory, Ref. [1].

References

[1] P. Cvitanović, *Group theory: Birdtracks, Lie's and Exceptional groups*, Princeton University Press, Princeton, NJ, 2008. URL: www.birdtracks.eu .

[2] Yu. L. Dokshitzer, *Perturbative QCD (and beyond)*, in *Strong Interactions Study Days Kloster Banz*, Germany, October 10-12, 1995, *Lect. Notes Phys.* **496** (1997) 87.

[3] S. Keppeler, *Birdtracks for SU(N)*, in *QCD Master Class 2017*, Saint-Jacut-de-la-Mer, France, June 18-24, 2017, *Sci. Post. Phys. Lect. Notes* **3** (2018) [1707.07280].

[4] H. Georgi, *Lie Algebras In Particle Physics : from Isospin To Unified Theories*, CRC Press, Taylor & Francis, 2000. URL: doi.org/10.1201/9780429499210 .

[5] M. Hamermesh, *Group Theory and its Application to Physical Problems*, Addison-Wesley, Reading, MA, 1962.

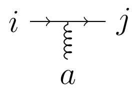
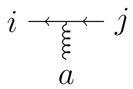
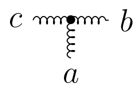
Contents

Color graphs are drawn with the TikZ LaTeX package (TikZ version 3.1.2).

1 Pedestrian introduction

1.1 The basic Lego bricks

The color structure of all interaction terms in the QCD Lagrangian can be expressed in terms of three elementary bricks, or "legos". Using these legos, the color structure of any QCD problem can in principle be worked out. The legos are defined and represented pictorially as

 $= (T^a)^j{}_i$;  $= -(T^a)^i{}_j$;  $= -i f_{abc}$. (1)

As mentioned in the preamble, we consider the SU(N) symmetry group ($N \geq 3$). Quark and antiquark color indices denoted by $i, j \ldots$ thus vary from 1 to N, and gluon indices denoted by $a, b \ldots$ vary from 1 to N^2-1. The matrices T^a (with $a = 1 \ldots N^2-1$) are $N \times N$ Hermitian matrices ($(T^a)^\dagger \equiv {}^t(T^a)^* = T^a$), of zero trace,

$$\mathrm{Tr}\, T^a = 0\,, \tag{2}$$

and normalized so that

$$\mathrm{Tr}\,(T^a T^b) = \frac{1}{2}\delta_{ab}\,. \tag{3}$$

The f_{abc}'s are the SU(N) structure constants defining the Lie algebra of SU(N),

$$\left[T^a, T^b\right] = i f_{abc}\, T^c\,, \tag{4}$$

from which one can easily show that f_{abc} is totally antisymmetric in a, b, c.

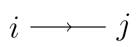
To memorize the three legos (1), one may view them as representing respectively quark, antiquark, and gluon scattering (the time arrow going from left to right) off an external gluon field (carrying the color index a) coupled *from below*, and remember that each lego corresponds to the SU(N) *generator* of index a in the representation of the scattered parton, namely, T^a for a quark, $-T^a$ for an antiquark, and t^a for a gluon, where the $(N^2-1)\times(N^2-1)$ matrices t^a are defined by

$$(t^a)_{bc} = -i f_{abc}\,. \tag{5}$$

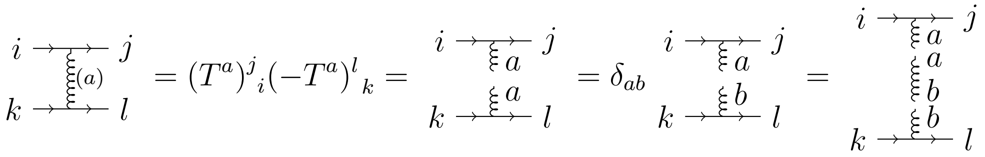
Note however that the meaning of a generator (which will be recalled in lesson 2) is not really useful at this stage.

An important feature of the rules (1) is that each lego is antisymmetric under the exchange of two lines. For instance, by exchanging any two lines of the first lego, one obtains a diagram looking like the second lego (antiquark scattering off a gluon coupled from below), which thus gets a minus sign. Let us also remark that in (1), the color index of an outgoing quark (or incoming antiquark) is written by convention as an upper index, and that of an outgoing antiquark (or incoming quark) as a lower index. The usefulness of this convention will become clear below.

In addition to (1), we introduce the pictorial notation for Kronecker's of color indices:

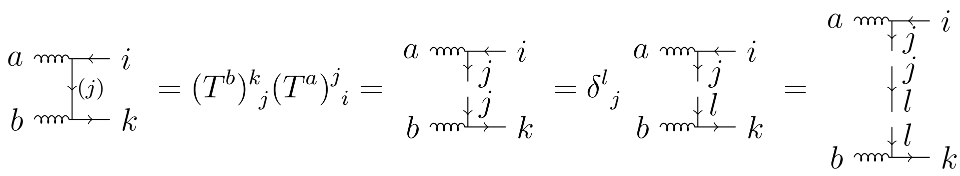
 $= \delta^j{}_i$;  $= \delta_{ab}$. (6)

For those who have not yet had the chance to get acquainted with QCD or even Feynman diagrams, let us mention that in a "lego construction" (called color graph or birdtrack in what follows) built from the basic legos, the color index of an internal line is summed over all its possible values. Together with Einstein summation convention, according to which an index appearing twice in an expression is implicitly summed, we thus have, for an internal gluon line:



$$= (T^a)^j{}_i(-T^a)^l{}_k = \quad = \delta_{ab} \quad = \quad , \qquad (7)$$

and similarly, for an internal quark line:



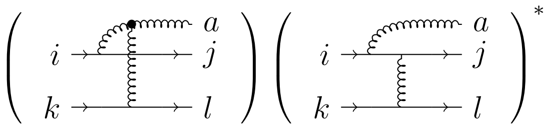
$$= (T^b)^k{}_j(T^a)^j{}_i = \quad = \delta^l{}_j \quad = \quad . \qquad (8)$$

Obviously, in pictorial notation summing over a repeated index amounts to connecting lines. The convention that a line with an arrow pointing out of (into) a vertex carries an upper (lower) index implies that for quarks and antiquarks, a repeated index always appears in both the upper and lower positions. In other words, the convention just serves to distinguish quarks and antiquarks and is consistent with the connection of quark lines respecting the direction of the arrow.

In what follows, everything will be derived step by step, using only (1) and (6) for all knowledge.

1.2 First trivial rules

The simplest way color appears in QCD is in the form of numbers called *color factors*. For example, in the calculation of the $qq \to qqg$ partonic cross section, one of the contributions is proportional to



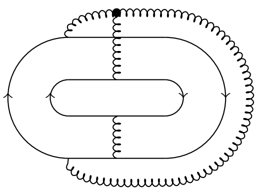
(9)

where each factor is a *matrix element* in color space (i.e., a color graph with specified external indices), and every initial and final color index is implicitly summed. To calculate such color factors in a pictorial way, we first need the following rule:

The complex conjugate of a color matrix element is obtained pictorially by taking the mirror image of the associated graph and reversing the arrows of quark and antiquark lines. (R1)

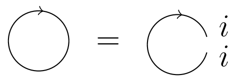
Exercise 1. Check that this rule indeed holds for each of the three legos, and then show that it must be true for any matrix element built from these legos.

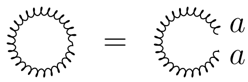
Using rule (R1) and then summing over repeated indices by connecting lines, the color factor (9) becomes

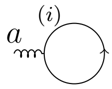


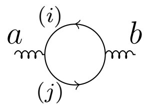
. (10)

Such graphs can be evaluated with the help of simple pictorial rules. The simplest ones read

 $= \delta^i{}_i = N$, (11)

 $= \delta_{aa} = N^2 - 1$, (12)

 $= (T^a)^i{}_i = 0$, (13)

 $= (T^a)^j{}_i (T^b)^i{}_j = \frac{1}{2}$  . (14)

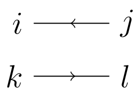
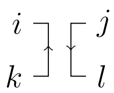
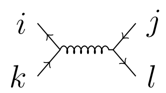
In the next sections, more interesting rules are obtained using the pictorial representations of the Fierz identity and the Lie algebra.

1.3 Fierz identity

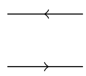
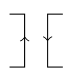
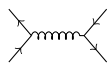
In index notation, the Fierz identity reads

$$\delta^i{}_j\, \delta^l{}_k = \frac{1}{N}\, \delta^i{}_k\, \delta^l{}_j + 2\, (T^a)^i{}_k\, (T^a)^l{}_j\,, \tag{15}$$

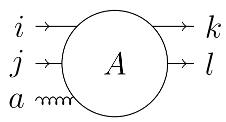
which corresponds pictorially to

 $= \frac{1}{N}$  $+ 2$  . (16)

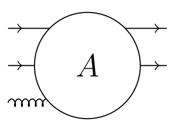
For more comfort, we may write the latter equation by removing the external indices:

 $= \frac{1}{N}$  $+ 2$  . (17)

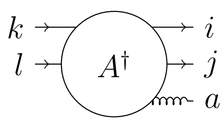
It is important to note that in doing so, the mathematical meaning of the pictorial equation is changed: a color graph represents a *color matrix element* when the external indices are specified, or the corresponding *linear map* (between the vector spaces spanned by the objects carrying the initial and final indices) when external indices are removed. For instance,

 $\equiv A^{kl}{}_{ija}$ (18)

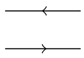
are the matrix elements of the operator

 $\equiv A\,.$ (19)

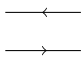
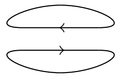
Note that the pictorial rule (R1) for complex conjugation of matrix elements corresponds to *Hermitian conjugation* at the operator level. Indeed, using (18) we have (using $A^* = {}^tA^\dagger$)

$(A^{kl}{}_{ija})^* = (A^\dagger)^{ija}{}_{kl} =$  , (20) but according to the rule (R1), this must coincide with the graph obtained by taking the mirror image of (18) and reversing arrows. Thus, that transformation applied to the operator A must give $A^\dagger$.

The Fierz identity (17) is thus a relation between linear maps of the vector space $V \otimes \overline{V}$ to itself, where V and $\overline{V}$ are the quark and antiquark vector spaces, respectively. In particular, the l.h.s. of (17) is the identity operator:

 $= \mathbb{1}_{V\otimes\overline{V}}\,.$ (21)

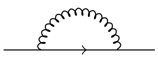
Exercise 2. Prove the Fierz identity (17) by using the following linear algebra result: *If E is a vector space of dimension n, p_i $(i = 1\dots m)$ are m projectors $(p_i^2 = p_i)$ such that $p_i\,p_j = 0$ for $i \neq j$ and $\sum_{i=1}^m \mathrm{rank}(p_i) = n$, then $\sum_{i=1}^m p_i = \mathbb{1}_E$.* (Recall that $\mathrm{rank}(f) \equiv \dim[\mathrm{img}(f)]$.) Also use the fact that the rank of a projector is equal to its trace, and that the trace is simply obtained pictorially by connecting the initial and final lines carrying the same type of indices. For instance, the rank of the identity projector (21) can be written as :

$\mathrm{rank}(\mathbb{1}_{V\otimes\overline{V}}) = \mathrm{Tr}\,(\mathbb{1}_{V\otimes\overline{V}}) = \mathrm{Tr}\,\Big($  $\Big) =$  $= N^2\,,$ (22)

which coincides as it should with $\dim(V \otimes \overline{V})$.

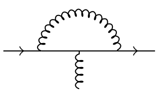
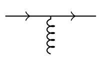
Let us now use the Fierz identity to derive simple pictorial rules.

Exercise 3. Multiply (16) by $i \rightarrow j\ = \delta^j{}_i$ and sum over repeated indices to show that

 $= C_F$ $\qquad (C_F = \dfrac{N^2-1}{2N})\,.$ (23)

Anticipating the next lessons, let us note that Eq. (23) can be written as $T^aT^a = C_F\mathbb{1}_V$, where T^aT^a is called the Casimir operator in the fundamental (quark) representation, and C_F the (squared) quark color charge or simply "quark Casimir". (In any representation R, the Casimir operator $T^a(R)T^a(R)$ is proportional to the identity in that representation, as a consequence of Schur's lemma, see lesson 3.)

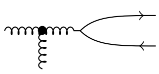
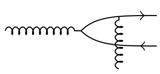
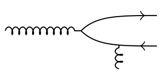
Exercise 4. Use the Fierz identity to obtain:

 $= -\dfrac{1}{2N}$  $.$ (24)

1.4 Lie algebra

1.4.1 Lie algebra in the fundamental representation

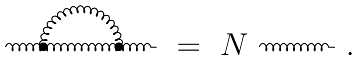
The matrices T^a (called the *generators* of SU(N) in the fundamental representation, see lesson 2) satisfy the Lie algebra (4), which can be expressed pictorially as

 $=$  $+$  $.$ (25)

Exercise 5. Check it!

The identity (25) is the first example of "color conservation" that we encounter (see lesson 2). We will now use this identity to find new rules.

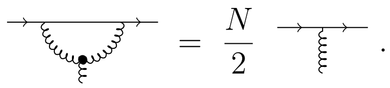
Exercise 6. Using *only* the pictorial rules derived so far, show that



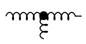
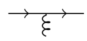
(26)

(Hint: to start, trade a gluon for a quark-antiquark pair by using (14).)

Exercise 7. Starting from (25), show in two different ways the following rule:

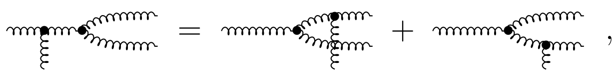


(27)

(Hint: multiply (25) either to the left by  , or to the right by  , and sum over appropriate indices.)

1.4.2 Lie algebra in the adjoint representation

A nice identity is the so-called Jacobi identity,



(28)

which is another manifestation of color conservation.

Exercise 8. Prove the Jacobi identity in two ways:

(i) Verify that $\left[T^a,\left[T^b,T^c\right]\right]+\left[T^b,[T^c,T^a]\right]+\left[T^c,\left[T^a,T^b\right]\right]=0$ to infer the relation $f_{abe}f_{cde}+f_{bce}f_{ade}+f_{cae}f_{bde}=0$, and check that the latter is equivalent to (28);

(ii) The previous proof indicates that (28) is a direct consequence of the Lie algebra in the fundamental representation (4), i.e. of (25). Thus, it should be possible to prove the Jacobi identity (28) using only the pictorial rules derived so far. Try it! (Hint: use a similar start as in Exercise 6.)

Exercise 9. Check that the Jacobi identity is nothing but the expression of the SU(N) Lie algebra in the adjoint representation, namely,

$$\left[t^a,t^b\right]=if_{abc}\,t^c\,, \tag{29}$$

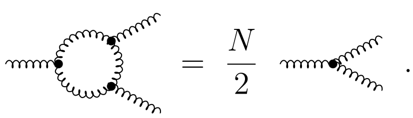
where the matrices t^a are defined by (5).

Anticipating the next lessons, we note that since the t^a matrices obey the Lie algebra and moreover satisfy $(t^a)^\dagger=t^a$ and $\mathrm{Tr}\,t^a=0$, they are SU(N) generators (lesson 2) in a representation of dimension N^2-1 called the gluon or *adjoint* representation. The operator t^at^a is thus the *Casimir operator* (lesson 3) in the adjoint representation, and from *Schur's lemma* (lesson 3) we must have $t^at^a=C_A\mathbb{1}_A$ (with A the gluon vector space of dimension N^2-1, and C_A the "gluon Casimir"). In fact, we already found t^at^a, since Eq. (26) reads (in index notation):

$$f_{acd}f_{bcd}=(t^c)_{ad}(t^c)_{db}=(t^ct^c)_{ab}=N\delta_{ab}\quad\Rightarrow\quad t^ct^c=N\,\mathbb{1}_A\,. \tag{30}$$

Therefore, the gluon Casimir is $C_A=N$.

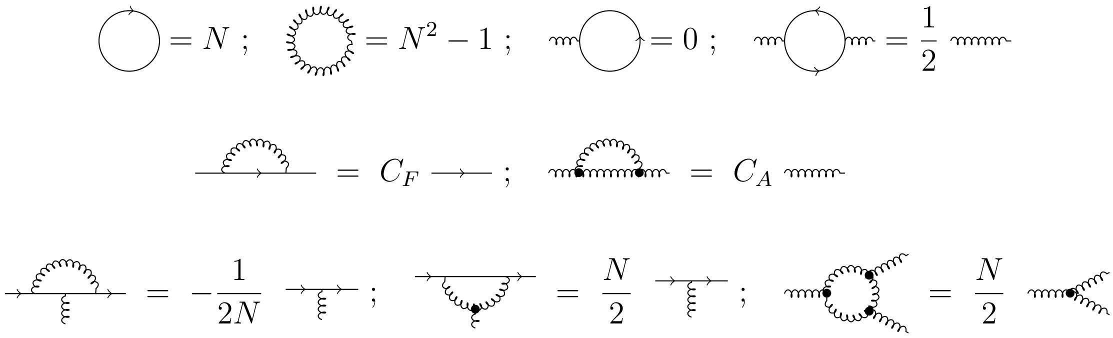
Exercise 10. Use the Jacobi identity to obtain the rule:



(31)

1.5 Sum up

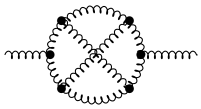
In this lesson we have obtained the following set of simple pictorial rules :



(R2)

These rules will prove very useful in the next lessons. When one of the above loops appears as a sub-diagram in a color graph, the latter can be simplified by contracting the loop to its simpler r.h.s. expression given in (R2).

Exercise 11. Using the rules (R2), calculate the color factor (10).

Exercise 12. Evaluate the color graph  . (Borrowed from Ref. [2].)