Putative finite-volume effects for spatially-nonlocal operators

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To appear real soon…

Nonlocal operators and nucleon structure

Wilson-line operators

- Quasi PDFs
- Pseudo PDFs

Currents

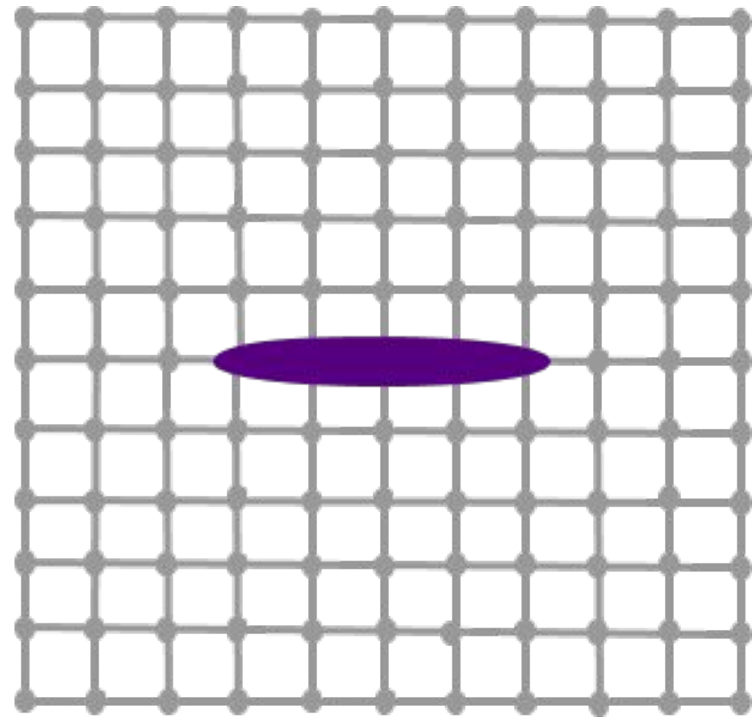
- Light quarks
- Auxiliary scalars
- Auxiliary heavy quarks

Smeared local operators

Nonlocal operators and nucleon structure

Wilson-line operators

- Quasi PDFs
- Pseudo PDFs

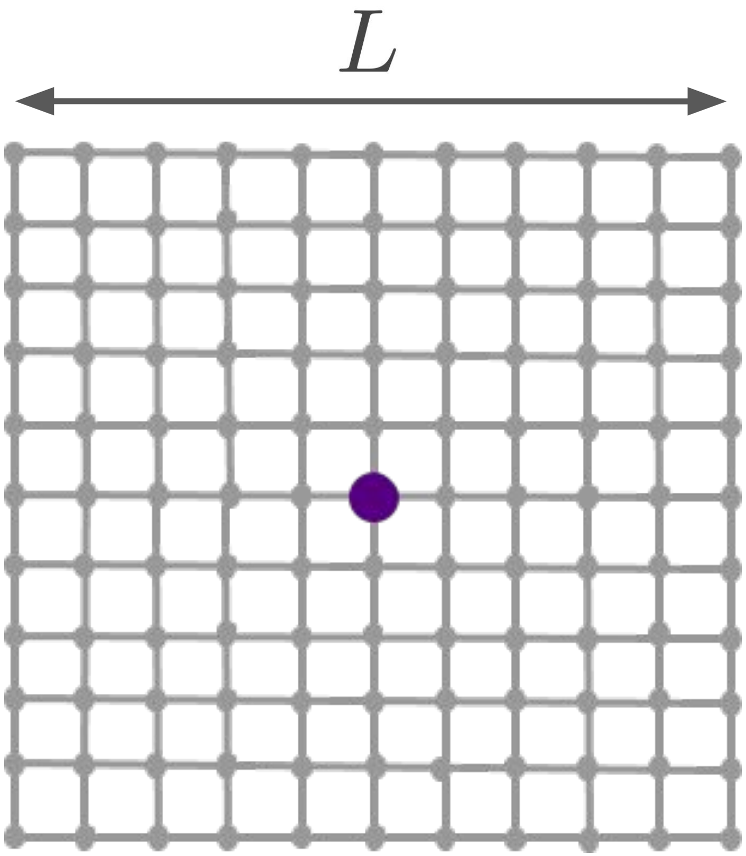


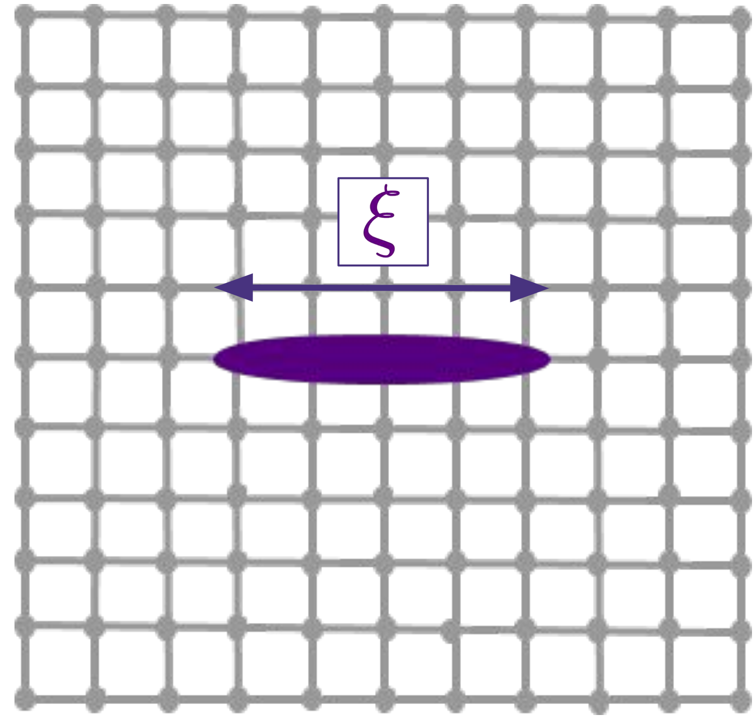
Currents

- Light quarks
- Auxiliary scalars
- Auxiliary heavy quarks

Smeared local operators

Finite-volume effects



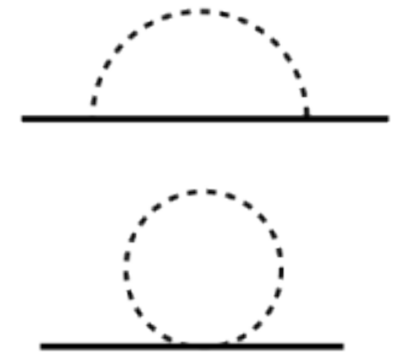


$$e^{-m_\pi L}$$

$$e^{-m_\pi |L-\xi|}$$

$$e^{-m_\pi \xi}$$

Local operators



Finite-volume effects - infrared degrees of freedom

- pions and nucleons
- chiral perturbation theory

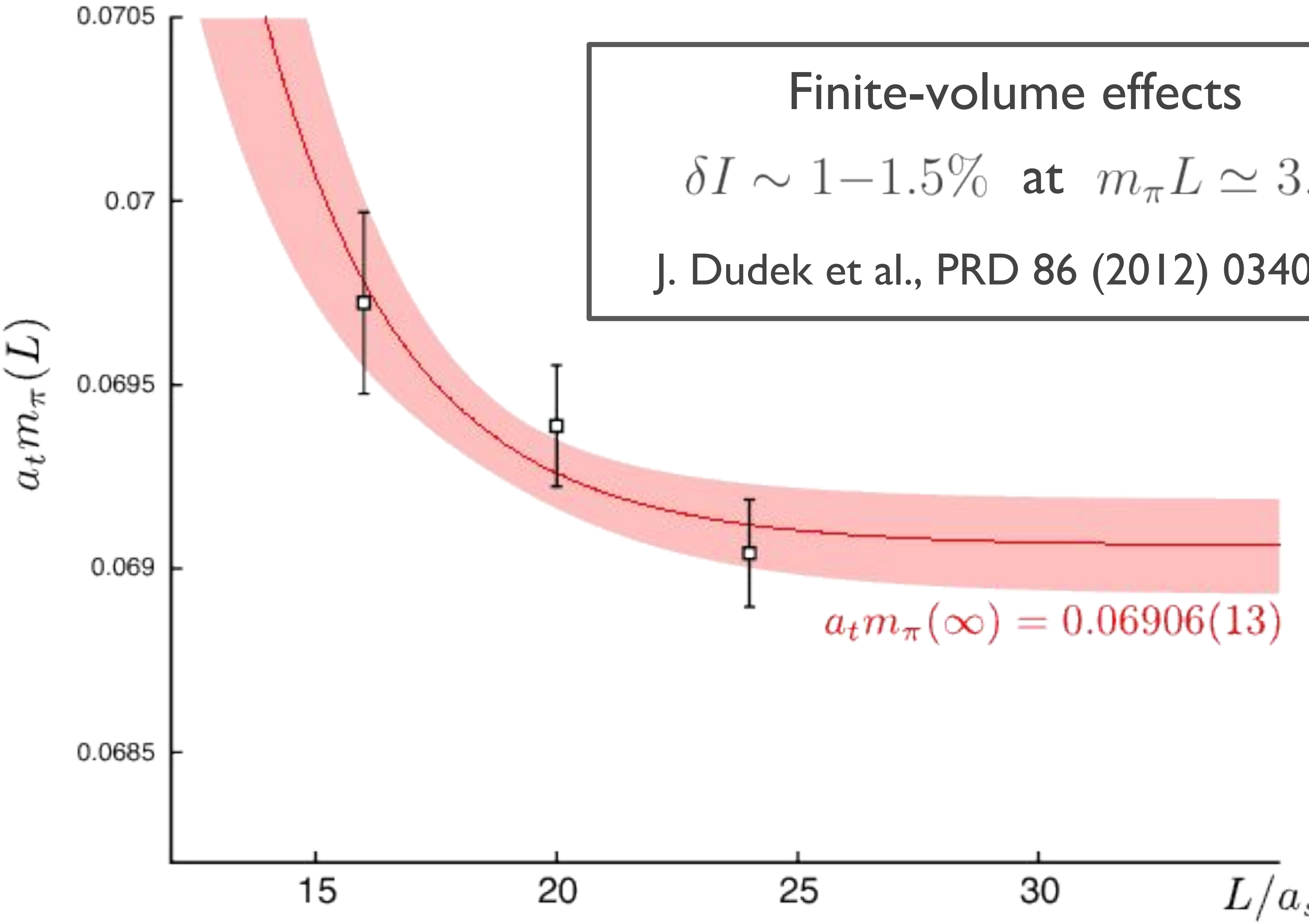
Consider pion tadpole correction to nucleon mass

- infinite volume $I_\infty = \int \frac{\mathrm{d}^4 k}{(2\pi)^4} \frac{1}{k^2 + m_\pi^2}$
- finite volume $I_{\mathrm{FV}} = \frac{1}{L^3} \sum_{\mathbf{k}} \int \frac{\mathrm{d}k_4}{2\pi} \frac{1}{k^2 + m_\pi^2}$
- finite-volume effects $\delta I = I_\infty - I_{\mathrm{FV}} = \sum_{\mathbf{n} \neq 0} \int \frac{\mathrm{d}^4 k}{(2\pi)^4} \frac{e^{i \mathbf{n} \cdot \mathbf{k} L}}{k^2 + m_\pi^2}$

$$\delta I = \frac{1}{(4\pi)^2} \sum_{\mathbf{n}} \left(\frac{4 m_\pi}{|\mathbf{n}| L} \right) K_1(|\mathbf{n}| L) \sim e^{-m_\pi L}$$

M. Lüscher, 1983

Local operators: pion mass



Finite-volume effects

$\delta I \sim 1\text{–}1.5\%$ at $m_\pi L \simeq 3.8$

J. Dudek et al., PRD 86 (2012) 034031

Also S. Beane et al., PRD 85 (2011) 034505

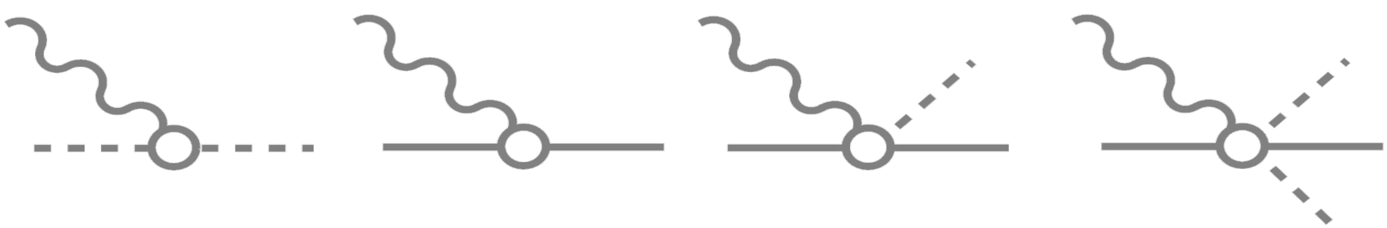
Our model

Couple two scalar particles $m_\varphi \ll m_\chi$:

- φ plays the role of the pion
- χ plays the role of the nucleon

Introduce external current

$$\mathcal{J}(x) = \frac{1}{2} Z_\varphi g_\varphi \varphi^2 + \frac{1}{2} Z_\chi g_\chi \chi^2 + \frac{1}{2} Z_{\chi\varphi} g_{\chi\varphi} \chi^2 \varphi + \frac{1}{4} Z_{\chi\varphi\varphi} g_{\chi\varphi\varphi} \chi^2 \varphi^2 + \cdots$$



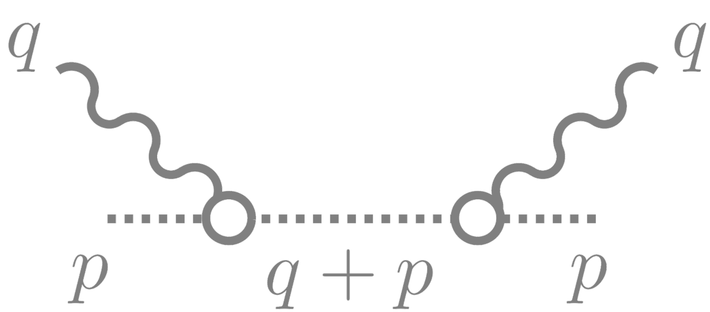
Consider

$$\mathcal{M}_\infty(\boldsymbol{\xi}, \mathbf{p}) \equiv \langle \mathbf{p} | \mathcal{J}(0, \boldsymbol{\xi}) \mathcal{J}(0) | \mathbf{p} \rangle$$

$$\delta \mathcal{M}_L(\boldsymbol{\xi}, \mathbf{p}) \equiv \mathcal{M}_L(\boldsymbol{\xi}, \mathbf{p}) - \mathcal{M}_\infty(\boldsymbol{\xi}, \mathbf{p})$$

Light particles

Leading order



$$\mathcal{M}_\infty(\boldsymbol{\xi}, \mathbf{p}) = g_\varphi^2 \int_{q_E} \frac{e^{i\mathbf{q}\cdot\boldsymbol{\xi}}}{(p_E + q_E)^2 + m_\varphi^2}$$

$$\delta\mathcal{M}_L(\boldsymbol{\xi}, \mathbf{p}) = g_\varphi^2 \sum_{\mathbf{n}\in\mathbb{Z}^3/\{\mathbf{0}\}} \int_q \frac{e^{i\mathbf{q}\cdot(\boldsymbol{\xi}+iL\mathbf{n})}}{(p_E + q_E)^2 + m_\varphi^2}$$

R. Briceno, M Hansen & CJM, PRD 96 (2017) 014502

Finite-volume scaling

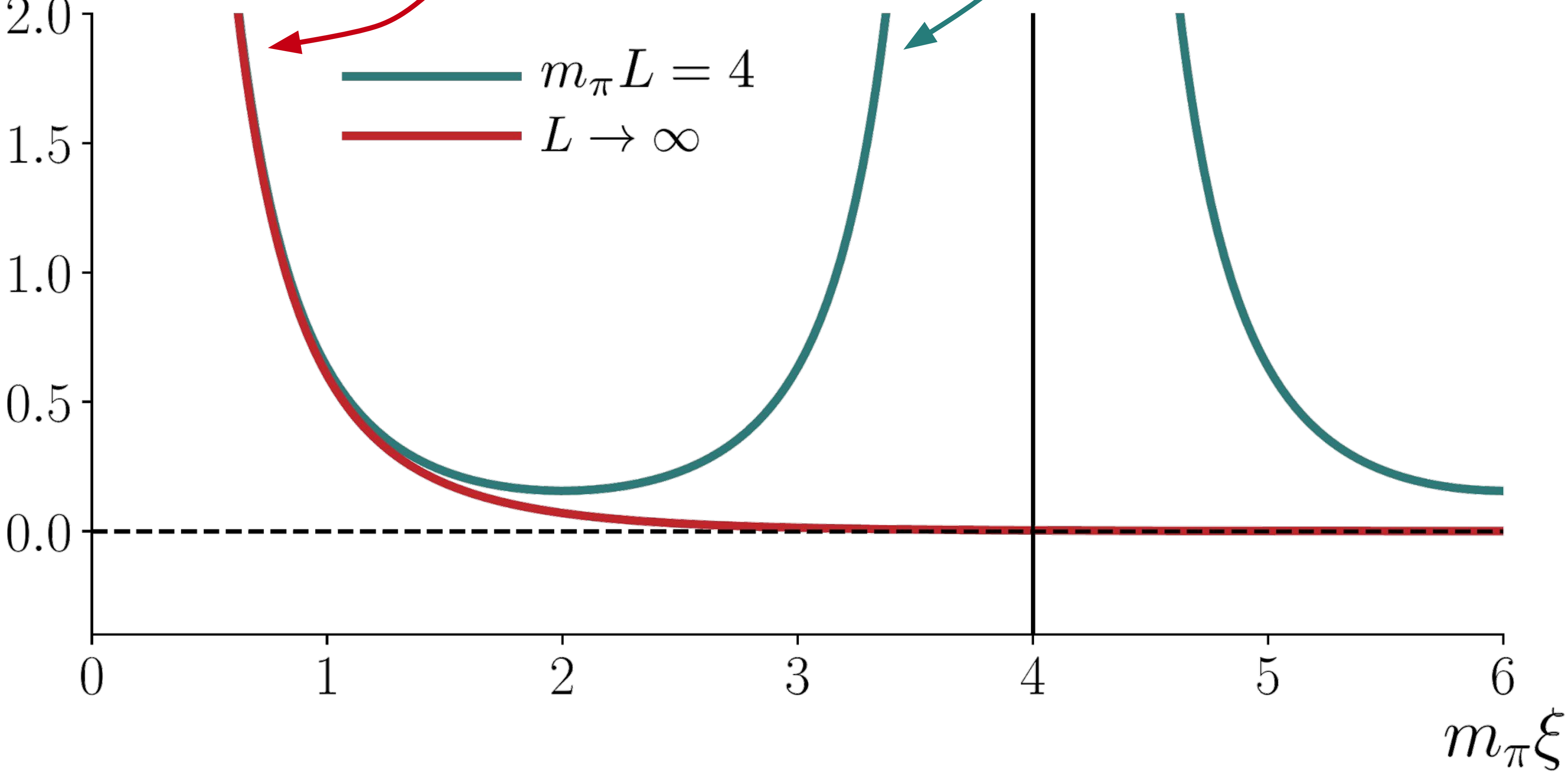
$$\delta\mathcal{M}_L(\boldsymbol{\xi}, \mathbf{0}) \propto \frac{e^{-m_\varphi(L-\xi)}}{(L-\xi)^{3/2}}$$

A. Cherman et al, PRD 95 (2017) 074512

Light particles

$$\delta\mathcal{M}_L(\xi,\mathbf{0}) = \frac{m_\pi}{4\pi^2}\sum_{\mathbf{n}\in\mathbb{Z}^3/\{\mathbf{0}\}}^{\mathbf{N}} \frac{K_1\left(m_\pi|\xi + L\mathbf{n}|\right)}{|\xi + L\mathbf{n}|}$$

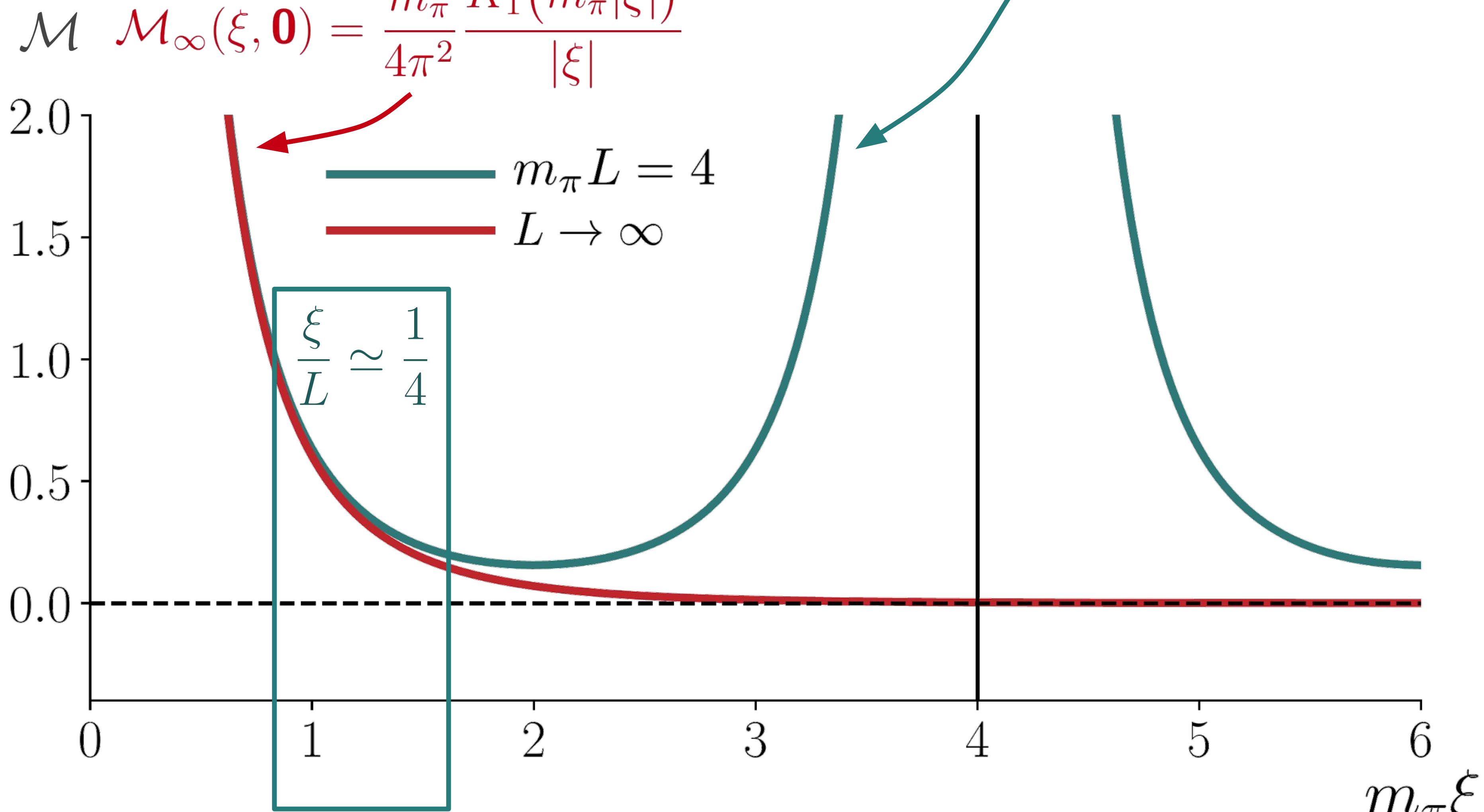
$\mathcal{M}$ $\quad \mathcal{M}_\infty(\xi,\mathbf{0}) = \frac{m_\pi}{4\pi^2}\frac{K_1\left(m_\pi|\xi|\right)}{|\xi|}$



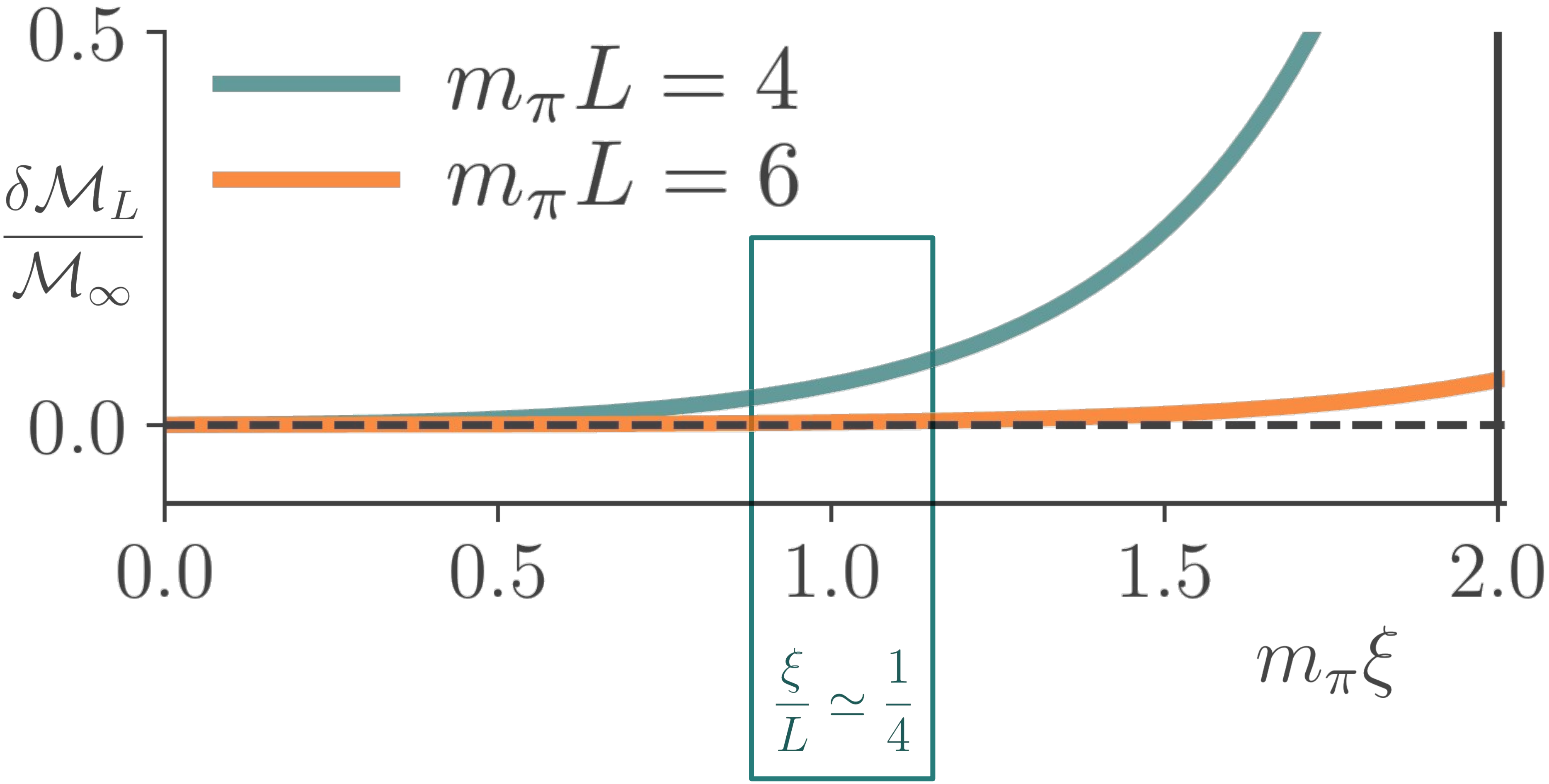
Light particles

$$\delta\mathcal{M}_L(\xi,\mathbf{0}) = \frac{m_\pi}{4\pi^2}\sum_{\mathbf{n}\in\mathbb{Z}^3/\{\mathbf{0}\}}^{\mathbf{N}} \frac{K_1\left(m_\pi|\xi+L\mathbf{n}|\right)}{|\xi+L\mathbf{n}|}$$

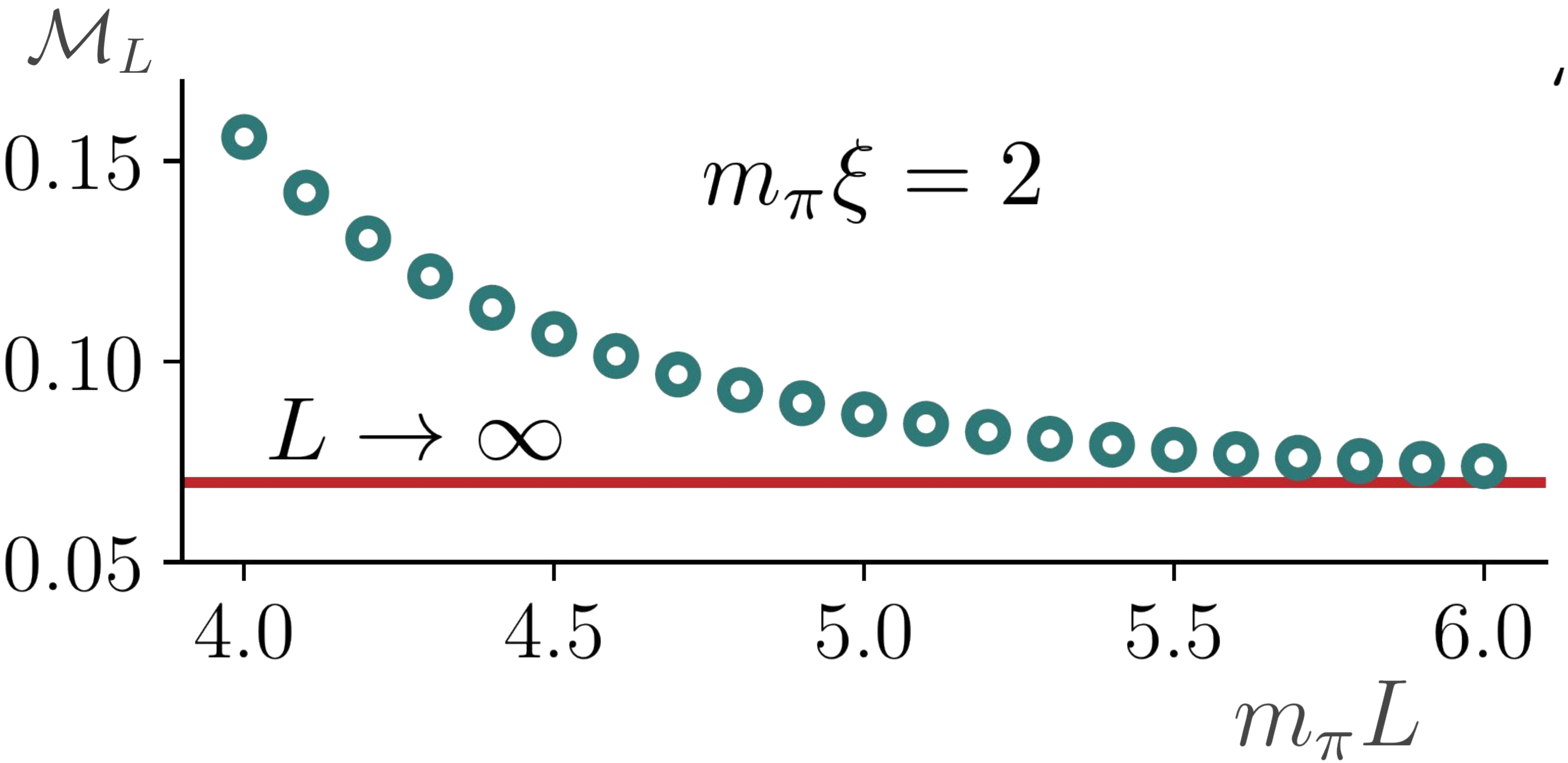
$$\mathcal{M}_\infty(\xi,\mathbf{0}) = \frac{m_\pi}{4\pi^2}\frac{K_1\left(m_\pi|\xi|\right)}{|\xi|}$$



Light particles



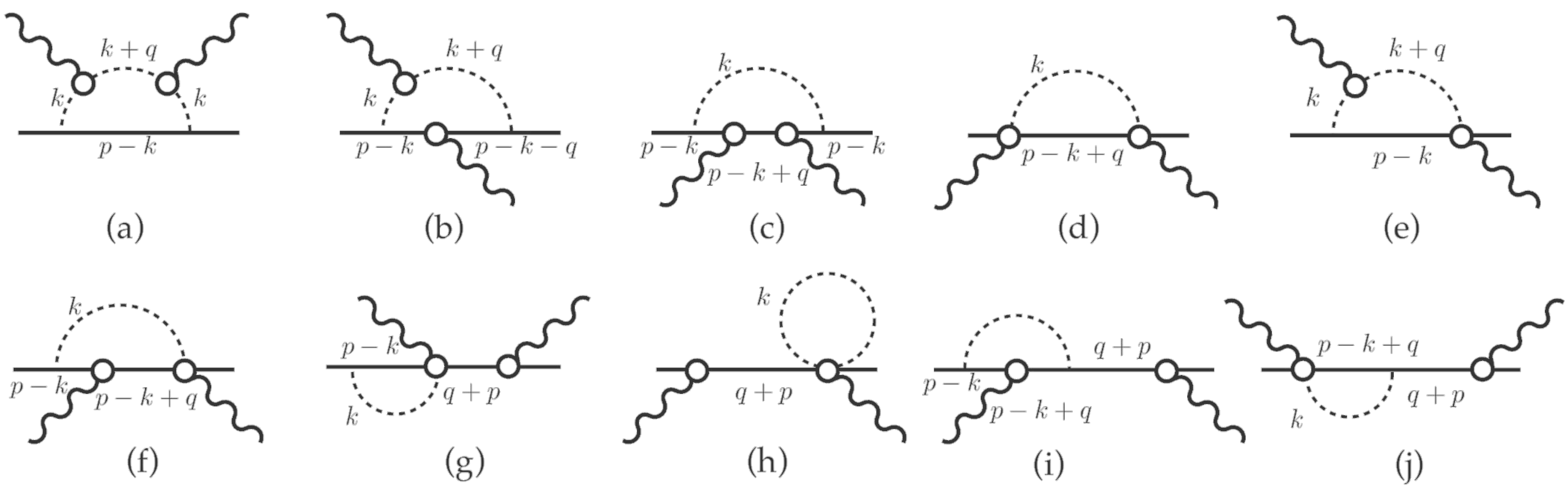
Light particles



Heavy particles

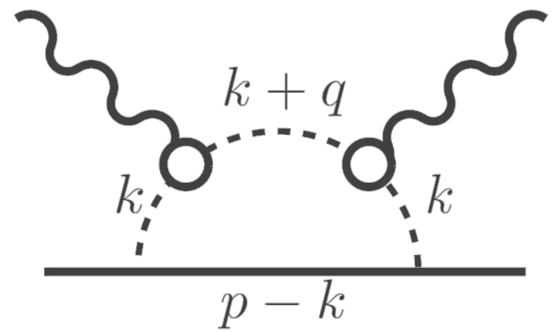
Leading order $\quad \delta\mathcal{M}_L(\boldsymbol{\xi},\mathbf{0}) \propto \dfrac{e^{-m_\chi(L-\xi)}}{(L-\xi)^{3/2}} \ll e^{-m_\varphi L}$

One loop



Heavy particles

Dominant diagram



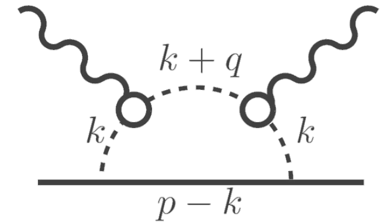
$$\delta\mathcal{M}_L^{(a)}(\boldsymbol{\xi},\mathbf{0}) \sim \frac{g^2 g_\varphi^2}{128\pi^3 m_\varphi}\left[\frac{\xi^{1/2}}{(L-\xi)^{3/2}}H_{x,3/2}(\xi) + \frac{(L-\xi)^{1/2}}{\xi^{3/2}}H_{x,3/2}(L-\xi)\right]e^{-m_\varphi L}$$

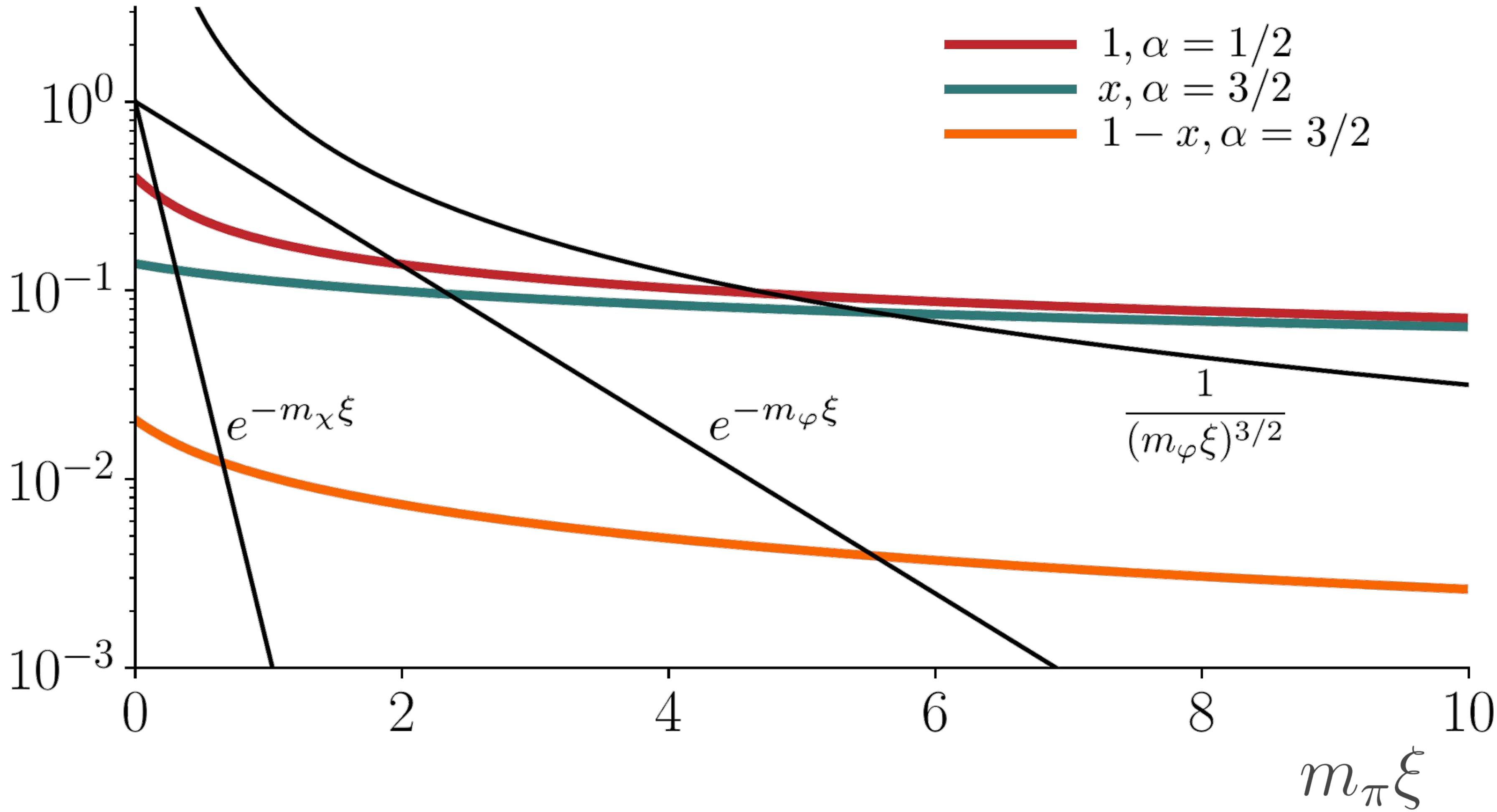
$$H_{f(x),\alpha}(\xi) = \int_0^1 \mathrm{d}x f(x)\,\frac{m_\varphi^\alpha}{M(x)^\alpha}e^{-\xi(M(x)-m_\varphi)}$$

Finite-volume scaling

$$\delta\mathcal{M}_L(\boldsymbol{\xi},\mathbf{0}) \propto \frac{\xi^{1/2}}{(L-\xi)^{3/2}}e^{-m_\varphi L}$$

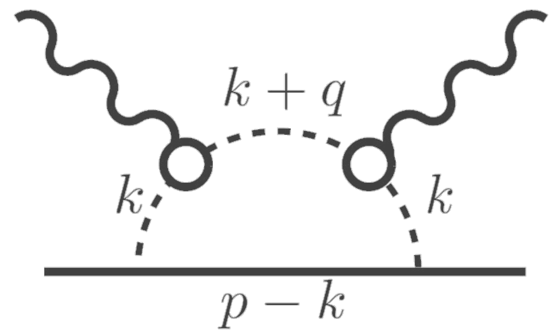
Heavy particles





Heavy particles

Dominant diagram



$$\delta\mathcal{M}_L^{(a)}(\boldsymbol{\xi},\mathbf{0}) \sim \frac{g^2 g_\varphi^2}{128\pi^3 m_\varphi}\left[\frac{\xi^{1/2}}{(L-\xi)^{3/2}} H_{x,3/2}(\xi) + \frac{(L-\xi)^{1/2}}{\xi^{3/2}} H_{x,3/2}(L-\xi)\right] e^{-m_\varphi L}$$

$$H_{f(x),\alpha}(\xi) = \int_0^1 \mathrm{d}x\, f(x)\, \frac{m_\varphi^\alpha}{M(x)^\alpha} e^{-\xi(M(x)-m_\varphi)}$$

Finite-volume scaling

$$\delta\mathcal{M}_L(\boldsymbol{\xi},\mathbf{0}) \propto \frac{\xi^{1/2}}{(L-\xi)^{3/2}} e^{-m_\varphi L}$$

Summary

Spatially extended operators

- introduce new IR scales
- potentially modify finite volume effects

Toy model matrix elements

- light particle external states

$$\delta\mathcal{M}_L(\boldsymbol{\xi},\mathbf{0}) \propto \frac{e^{-m_\varphi(L-\xi)}}{(L-\xi)^{3/2}}$$

- heavy particle external states

$$\delta\mathcal{M}_L(\boldsymbol{\xi},\mathbf{0}) \propto \frac{\xi^{1/2}}{(L-\xi)^{3/2}}e^{-m_\varphi L}$$

Looking forward

- chiral perturbation theory

Thank you

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Light particles

More generally

$$\mathcal{M}_{\infty}^{(d)}(\boldsymbol{\xi}, \mathbf{p}) = \int_{q_E} e^{i\mathbf{q}\cdot\boldsymbol{\xi}} \int_{k_{1,E}} \cdots \int_{k_{n-1,E}} D_E^{(d)}(p_E, q_E, k_{1,E}, \cdots, k_{n,E})$$

$$\delta\mathcal{M}_{L}^{(d)}(\boldsymbol{\xi}, \mathbf{p}) \equiv \mathcal{M}_{L}^{(d)}(\boldsymbol{\xi}, \mathbf{p}) - \mathcal{M}_{\infty}^{(d)}(\boldsymbol{\xi}, \mathbf{p})$$

$$= \sum_{\mathbf{M}\in\mathbb{Z}^{3n}/\{\mathbf{0}\}} \int_{K_E} e^{i\mathbf{q}\cdot\boldsymbol{\xi}+i\mathbf{K}\cdot L\mathbf{M}} D_E^{(d)}(p_E, K_E)$$

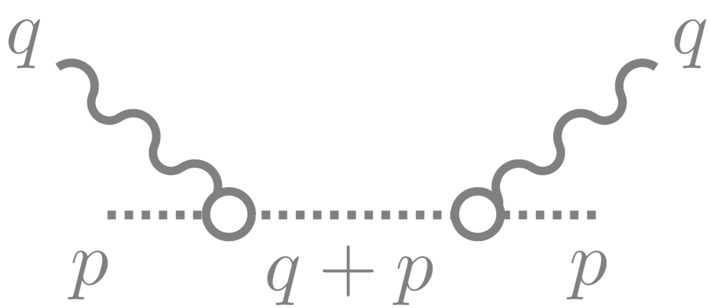
$$K_E \equiv \{q_E, k_{1,E}, \cdots, k_{n-1,E}\}$$

$$\mathbf{M} = \{\mathbf{n}, \mathbf{m}_1, \cdots, \mathbf{m}_{n-1}\}$$

R. Briceno, M Hansen & CJM, PRD 96 (2017) 014502

Light particles

Leading order



$$\mathcal{M}_\infty(\boldsymbol{\xi}, \mathbf{p}) = g_\varphi^2 \int_{q_E} \frac{e^{i\mathbf{q}\cdot\boldsymbol{\xi}}}{(p_E + q_E)^2 + m_\varphi^2}$$

$$\delta\mathcal{M}_L(\boldsymbol{\xi}, \mathbf{p}) = g_\varphi^2 \sum_{\mathbf{n}\in\mathbb{Z}^3/\{\mathbf{0}\}} \int_q \frac{e^{i\mathbf{q}\cdot(\boldsymbol{\xi}+iL\mathbf{n})}}{(p_E + q_E)^2 + m_\varphi^2}$$

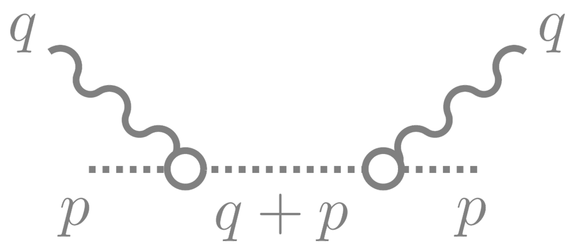
Finite volume scaling

$$\delta\mathcal{M}_L(\boldsymbol{\xi}, \mathbf{p}) = \frac{m_\varphi g_\varphi^2}{4\pi^2} \sum_{\mathbf{n}\in\mathbb{Z}^3/\{\mathbf{0}\}} e^{-i\mathbf{p}\cdot(\boldsymbol{\xi}+L\mathbf{n})} \frac{K_1\big(m_\varphi|\boldsymbol{\xi} + L\mathbf{n}|\big)}{|\boldsymbol{\xi} + L\mathbf{n}|} \sim \frac{m_\varphi g_\varphi^2}{4\pi^2} \frac{K_1\big(m_\varphi|L - \xi|\big)}{|L - \xi|}$$

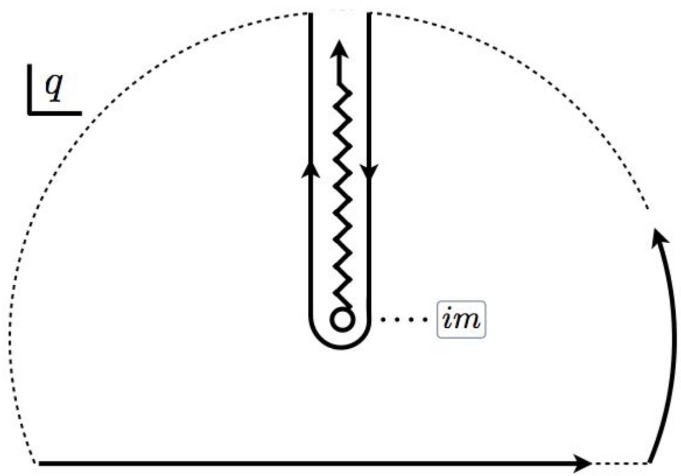
$$\delta\mathcal{M}_L(\boldsymbol{\xi}, \mathbf{0}) \propto \frac{e^{-m_\varphi(L-\xi)}}{(L - \xi)^{3/2}}$$

Bessel functions

Recall

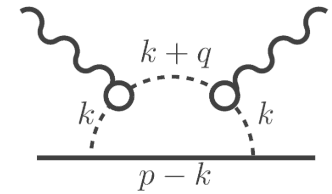


$$\mathcal{M}_\infty(\boldsymbol{\xi}, \mathbf{p}) = g_\varphi^2 \int_{q_E} \frac{e^{i\mathbf{q}\cdot\boldsymbol{\xi}}}{(p_E + q_E)^2 + m_\varphi^2}$$



$$I_{\mathrm{FV}} = \sum_{\mathbf{n}} \int \frac{\mathrm{d}^4 k}{(2\pi)^4} \frac{e^{i\mathbf{n}\cdot\mathbf{k}L}}{k^2 + m_\pi^2}$$

Heavy particles



Dominant diagram

$$\delta\mathcal{M}_L^{(a)}(\boldsymbol{\xi},\mathbf{p}) = g^2 g_\varphi^2 \sum_{\{\mathbf{n},\mathbf{m}\}\neq\mathbf{0}} \int_{q_E,k_E} e^{i\mathbf{q}\cdot(\boldsymbol{\xi}+L\mathbf{n})} e^{iL\mathbf{k}\cdot\mathbf{m}} \frac{1}{[k_E^2+m_\varphi^2]^2} \frac{1}{(k_E+q_E)^2+m_\varphi^2} \frac{1}{(p_E-k_E)^2+m_\chi^2}$$

Shift momentum variables, introduce Feynman parameters

$$\delta\mathcal{M}_L^{(a)}(\boldsymbol{\xi},\mathbf{p}) = 2g^2 g_\varphi^2 \int_0^1 \mathrm{d}x\, x \sum_{\mathbf{n},\mathbf{m}} e^{i(1-x)\mathbf{p}\cdot[L(\mathbf{m}-\mathbf{n})-\boldsymbol{\xi}]} \int_{q_E} \frac{e^{i\mathbf{q}\cdot(\boldsymbol{\xi}+L\mathbf{n})}}{q_E^2+m_\varphi^2} \int_{k_E} \frac{e^{i\mathbf{k}\cdot[L(\mathbf{m}-\mathbf{n})-\boldsymbol{\xi}]}}{[k_E^2+M(x)^2]^3}$$

$$M(x)^2 \equiv x m_\varphi^2 + (1-x)m_\chi^2 + x(1-x)p_E^2 = x m_\varphi^2 + (1-x)^2 m_\chi^2$$

Carry out momentum integrals

$$\delta\mathcal{M}_L^{(a)}(\boldsymbol{\xi},\mathbf{0}) = 2g^2 g_\varphi^2 \sum_{\mathbf{n},\mathbf{m}} \mathcal{I}_1\left[|L\mathbf{n}-\boldsymbol{\xi}|; m_\varphi\right] \left[\int_0^1 \mathrm{d}x\, x\, \mathcal{I}_3\left[|L\mathbf{m}-\boldsymbol{\xi}|; M(x)\right]\right]$$

Asymptotically

$$\delta\mathcal{M}_L^{(a)}(\boldsymbol{\xi},\mathbf{0}) \sim \frac{g^2 g_\varphi^2}{128\pi^3 m_\varphi}\left[\frac{\xi^{1/2}}{(L-\xi)^{3/2}} H_{x,3/2}(\xi) + \frac{(L-\xi)^{1/2}}{\xi^{3/2}} H_{x,3/2}(L-\xi)\right] e^{-m_\varphi L}$$

$$H_{f(x),\alpha}(\xi) = \int_0^1 \mathrm{d}x\, f(x)\, \frac{m_\varphi^\alpha}{M(x)^\alpha} e^{-\xi(M(x)-m_\varphi)}$$

PDFs

J.-W. Chen et al, 1804.01483

- pion: HISQ 2+1+1, a=0.12fm, L=2.9fm, m=310MeV, P = {0.86,1.32,1.74} GeV

J.W. Chen et al, 1803.04393

- HISQ 2+1+1, a=0.09fm, L=5.8fm, m=135MeV, P={2.2,2.6,3.0} GeV

C. Alexandrou et al, 1803.02685

- Tw 2, a=0.09fm, L=4.5fm, m=130MeV, P={0.83,1.11,1.38} GeV

H.-W. Lin et al, 1708.05301

- HISQ 2+1+1, a=0.09fm, L=5.8fm, m=135MeV, P={2.2,2.6,3.0} GeV

C. Alexandrou et al, PRD 96 (2017) 014513

- Tw 2+1+1, a=0.082fm, L=2.6fm, m=370MeV, P={0.49,0.98,1.47,1.96,2.45} GeV

J.-W. Chen et al, NPB 911 (2016) 246

- HISQ 2+1+1, a=0.12fm, L=2.9fm, m=310MeV, P={0.43,0.86,1.29} GeV

C. Alexandrou et al, PRD 92 (2015) 014502

- Tw 2+1+1, a=0.082fm, L=2.6fm, m=370MeV, P={0.49,0.98,1.47} GeV

H.-W. Lin et al, PRD 91 (2015) 054510

- HISQ 2+1+1, a=0.12fm, L=2.9fm, m=310MeV, P={0.43,0.86,1.29} GeV

PDAs

G.S. Bali et al, EPJC 78 (2018) 217

- pion: Wilson 2, a=0.071fm, L=2.27fm, m=295MeV, P={1.08,1.53,1.88} GeV
- L/a=32, mL=3.4, zmax/a=12 (?)

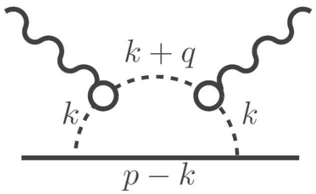
J.-W. Chen et al, 1712.10025

- kaon: HISQ 2+1+1, a=0.12fm, L=2.9fm, m=310MeV, P={0.86,1.29,1.72} GeV
- L/a=24, mL=4.5, zmax/a=14 (?!)

J.H. Zhang et al, PRD 95 (2017) 094514

- pion: HISQ 2+1+1, a=0.12fm, L=2.9fm, m=310MeV
- L/a=24, mL=4.5, zmax/a=

Finite-volume references



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S. Beane & M. Savage, PRD 70 (2004) 074029

$$\delta\mathcal{M}_L^{(a)}(\boldsymbol{\xi},\mathbf{0}) = 2g^2 g_\varphi^2 \sum_{\mathbf{n},\mathbf{m}} \mathcal{I}_1\left[|L\mathbf{n}-\boldsymbol{\xi}|;m_\varphi\right] \left[\int_0^1 \mathrm{d}x\, x\, \mathcal{I}_3\left[|L\mathbf{m}-\boldsymbol{\xi}|;M(x)\right]\right]$$

$$\delta\mathcal{M}_L^{(b)}(\boldsymbol{\xi},\mathbf{0}) = g^2 g_\varphi g_\chi \sum_{\{\mathbf{n},\mathbf{m}\}\neq\mathbf{0}} \left[\int_0^1 \mathrm{d}x\, \mathcal{I}_2\left[|L\mathbf{n}-\boldsymbol{\xi}|;M(x)\right]\right] \left[\int_0^1 \mathrm{d}y\, \mathcal{I}_2\left[|L\mathbf{m}-\boldsymbol{\xi}|;M(y)\right]\right] ,$$

$$\delta\mathcal{M}_L^{(c)}(\boldsymbol{\xi},\mathbf{0}) = 2g^2 g_\chi^2 \sum_{\{\mathbf{n},\mathbf{m}\}\neq\mathbf{0}} \mathcal{I}_1\left[|L\mathbf{n}-\boldsymbol{\xi}|;m_\chi\right] \left[\int_0^1 \mathrm{d}x\,(1-x)\, \mathcal{I}_3\left[|L\mathbf{m}-\boldsymbol{\xi}|;M(x)\right]\right] ,$$

$$\delta\mathcal{M}_L^{(d)}(\boldsymbol{\xi},\mathbf{0}) = g_{\chi\varphi}^2 \sum_{\{\mathbf{n},\mathbf{m}\}\neq\mathbf{0}} \mathcal{I}_1\left[|L\mathbf{n}-\boldsymbol{\xi}|;m_\chi\right] \mathcal{I}_1\left[|L\mathbf{m}-\boldsymbol{\xi}|;m_\varphi\right] ,$$

$$\delta\mathcal{M}_L^{(e)}(\boldsymbol{\xi},\mathbf{0}) = g g_\varphi g_{\chi\varphi} \sum_{\{\mathbf{n},\mathbf{m}\}\neq\mathbf{0}} \mathcal{I}_1\left[|L\mathbf{n}-\boldsymbol{\xi}|;m_\varphi\right] \left[\int_0^1 \mathrm{d}x\, \mathcal{I}_2\left[|L\mathbf{m}-\boldsymbol{\xi}|;M(x)\right]\right] ,$$

$$\delta\mathcal{M}_L^{(f)}(\boldsymbol{\xi},\mathbf{0}) = g g_\chi g_{\chi\varphi} \sum_{\{\mathbf{n},\mathbf{m}\}\neq\mathbf{0}} \mathcal{I}_1\left[|L\mathbf{n}-\boldsymbol{\xi}|;m_\chi\right] \left[\int_0^1 \mathrm{d}x\, \mathcal{I}_2\left[|L\mathbf{m}-\boldsymbol{\xi}|;M(x)\right]\right] ,$$

$$\delta\mathcal{M}_L^{(g)}(\boldsymbol{\xi},\mathbf{0}) = g g_{\chi\varphi} g_\chi \sum_{\{\mathbf{n},\mathbf{m}\}\neq\mathbf{0}} \mathcal{I}_1\left[|L\mathbf{n}-\boldsymbol{\xi}|;m_\chi\right] \left[\int_0^1 \mathrm{d}x\, \mathcal{I}_2\left[|L\mathbf{m}|;M(x)\right]\right] ,$$

$$\delta\mathcal{M}_L^{(h)}(\boldsymbol{\xi},\mathbf{0}) = \frac{1}{2} g_\chi g_{\chi\varphi\varphi} \sum_{\{\mathbf{n},\mathbf{m}\}\neq\mathbf{0}} \mathcal{I}_1\left[|L\mathbf{n}-\boldsymbol{\xi}|;m_\chi\right] \mathcal{I}_1\left[|L\mathbf{m}|;m_\varphi\right] .$$

$$\mathcal{I}_\gamma\left[|\boldsymbol{\xi}|;m\right] \equiv \int_{k_E} \frac{e^{i\mathbf{k}\cdot\boldsymbol{\xi}}}{[k_E^2+m^2]^\gamma} = \frac{1}{8\pi^2\Gamma(\gamma)} \left(\frac{|\boldsymbol{\xi}|}{2m}\right)^{\gamma-2} K_{\gamma-2}\left(|\boldsymbol{\xi}|m\right)$$

$$M(x)^2 \equiv x m_\varphi^2 + (1-x)m_\chi^2 + x(1-x)p_E^2 = x m_\varphi^2 + (1-x)^2 m_\chi^2$$

$$\delta\mathcal{M}_L^{(a)}(\boldsymbol{\xi},\mathbf{0}) \sim \frac{g^2 g_\varphi^2}{128\pi^3 m_\varphi}\left[\frac{\xi^{1/2}}{(L-\xi)^{3/2}}H_{x,3/2}(\xi) + \frac{(L-\xi)^{1/2}}{\xi^{3/2}}H_{x,3/2}(L-\xi)\right]e^{-m_\varphi L}\,,$$

$$\delta\mathcal{M}_L^{(b)}(\boldsymbol{\xi},\mathbf{0}) \sim \frac{g^2 g_\varphi g_\chi}{64\pi^3 m_\varphi}\left[\frac{1}{\xi^{1/2}(L-\xi)^{1/2}}H_{1,1/2}(\xi)H_{1,1/2}(L-\xi)\right]e^{-m_\varphi L}\,,$$

$$\delta\mathcal{M}_L^{(c)}(\boldsymbol{\xi},\mathbf{0}) = \frac{g^2 g_\chi^2}{128\pi^3}\frac{m_\chi^{1/2}}{m_\varphi^{3/2}}\left[\frac{(L-\xi)^{1/2}}{\xi^{3/2}}H_{1-x,3/2}(L-\xi)\right]e^{-\xi(m_\chi-m_\varphi)}e^{-m_\varphi L}\,,$$

$$\delta\mathcal{M}_L^{(d)}(\boldsymbol{\xi},\mathbf{0}) = \frac{g_{\chi\varphi}^2 m_\chi^{1/2} m_\varphi^{1/2}}{32\pi^3}\left[\frac{1}{\xi^{3/2}(L-\xi)^{3/2}}\right]e^{-\xi(m_\chi-m_\varphi)}e^{-m_\varphi L}\,,$$

$$\delta\mathcal{M}_L^{(e)}(\boldsymbol{\xi},\mathbf{0}) = \frac{g g_\varphi g_{\chi\varphi}}{64\pi^3}\left[\frac{1}{\xi^{1/2}(L-\xi)^{3/2}}H_{1,1/2}(\xi) + \frac{1}{\xi^{3/2}(L-\xi)^{1/2}}H_{1,1/2}(L-\xi)\right]e^{-m_\varphi L}\,,$$

$$\delta\mathcal{M}_L^{(f)}(\boldsymbol{\xi},\mathbf{0}) = \frac{g g_\chi g_{\chi\varphi} m_\chi^{1/2}}{64\pi^3 m_\varphi^{1/2}}\left[\frac{1}{\xi^{3/2}(L-\xi)^{1/2}}H_{1,1/2}(L-\xi)\right]e^{-\xi(m_\chi-m_\varphi)}e^{-m_\varphi L}\,,$$

$$\delta\mathcal{M}_L^{(g)}(\boldsymbol{\xi},\mathbf{0}) = \frac{g g_{\chi\varphi} g_\chi m_\chi^{1/2}}{64\pi^3 m_\varphi^{1/2}}\left[\frac{1}{\xi^{3/2}L^{1/2}}H_{1,1/2}(L)\right]e^{-\xi m_\chi}e^{-m_\varphi L}\,,$$

$$\delta\mathcal{M}_L^{(h)}(\boldsymbol{\xi},\mathbf{0}) = \frac{g_\chi g_{\chi\varphi\varphi} m_\varphi^{1/2} m_\chi^{1/2}}{64\pi^3}\left[\frac{1}{\xi^{3/2}L^{3/2}}\right]e^{-m_\chi\xi}e^{-m_\varphi L}\,,$$

$$H_{f(x),\alpha}(\xi) = \int_0^1 \mathrm{d}x\, f(x)\,\frac{m_\varphi^\alpha}{M(x)^\alpha}e^{-\xi(M(x)-m_\varphi)}$$

$$M(x)^2 \equiv x m_\varphi^2 + (1-x)m_\chi^2 + x(1-x)p_E^2 = x m_\varphi^2 + (1-x)^2 m_\chi^2$$