# Quasi transverse-momentum dependent PDFs

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# Outline

1. TMD factorization
2. Towards Quasi TMDPDFs
3. Obstructions in matching the soft function
4. (Naive) beam functions from lattice
5. Conclusion

# TMD factorization

# TMD factorization theorem

- Factorization theorem in Fourier space:

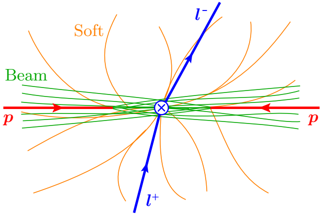
$$\sigma(\vec{q}_T) = H(Q) \int \mathrm{d}^2\vec{b}_T\, e^{\mathrm{i}\vec{b}_T\cdot\vec{q}_T} B_n(x_1, \vec{b}_T) B_{\bar{n}}(x_2, \vec{b}_T) S_{n\bar{n}}(\vec{b}_T)$$

- Can absorb soft function into TMDPDFs:

$$\sigma(\vec{q}_T) = H(Q) \int \mathrm{d}^2\vec{b}_T\, e^{\mathrm{i}\vec{q}_T\cdot\vec{b}_T}\, f_n^{\mathrm{TMD}}(x_1, \vec{b}_T)\, f_{\bar{n}}^{\mathrm{TMD}}(x_2, \vec{b}_T)$$

$$f_n^{\mathrm{TMD}}(x, \vec{b}_T) = B_n(x, \vec{b}_T)\sqrt{S_{n\bar{n}}(\vec{b}_T)}$$

- Beam functions $B_{n,\bar{n}}$: collinear radiation
  - Factorize into $n$ and $\bar{n}$ functions
- Soft function $S_{n\bar{n}}$: soft radiation
  - Depends on *both* $n$ and $\bar{n}$
  - Universal: same soft function for DIS



# Soft-subtracted TMDPDF

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$$f_n^{\mathrm{TMD}}(x, \vec{b}_T) = B_n(x, \vec{b}_T)\sqrt{S_{n\bar{n}}(\vec{b}_T)}$$

- $B_n$ are pure collinear matrix elements
  - In practice: Requires soft (zero-bin) subtraction $S^0_{n\bar{n}}$

$$f_n^{\mathrm{TMD}}(x, \vec{b}_T) = \frac{B_n^{(\mathrm{unsub})}(x, \vec{b}_T)}{S^0_{n\bar{n}}(\vec{b}_T)}\sqrt{S_{n\bar{n}}(\vec{b}_T)}$$

  - $f_n^{\mathrm{TMD}}$ also depends on $\bar{n}$ direction
- Multiple formulations in the literature [Collins, Soper, Sterman '85; Collins '11; Becher, Neubert '10; Echevarria, Idilbi, Scimemi '11; Chiu, Jain, Neill, Rothstein '12]
- Example 1: [Collins '11]

$$f_n^{\mathrm{TMD}}(x, \vec{b}_T, \zeta) = \lim_{\substack{y_A\to+\infty \\ y_B\to-\infty}} B_{y_A}^{(\mathrm{unsub})}(x, \vec{b}_T)\sqrt{\frac{S_{y_A,y_n}}{S_{y_A,y_B}S_{y_n,y_B}}}$$

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- Example 2: [Echevarria, Idilbi, Scimemi '11]

$$f_n^{\mathrm{TMD}}(x, \vec{b}_T, \zeta) = \frac{B_n^{(\mathrm{unsub})}(x, \vec{b}_T, \delta, \zeta)}{\sqrt{S_{n\bar{n}}(\vec{b}_T, \delta)}}$$

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- Example 3: [Chiu, Jain, Neill, Rothstein '12]

$$f_n^{\mathrm{TMD}}(x, \vec{b}_T, \zeta) = B_n^{(\mathrm{unsub})}(x, \vec{b}_T, \nu^2/\zeta)\sqrt{S_{n\bar{n}}(\vec{b}_T, \nu\, b_T)}$$

# TMD factorization theorem

- Beam function definition (unsubtracted):

$$B_n^{\mathrm{unsub}}(b^+, \vec{b}_T) = \langle P(P_n) \big| \bar{q}(b^+, \vec{b}_T) W^{(0,\vec{0}_T)}_{(b^+,\vec{b}_T)} \frac{\gamma^-}{2} q(0) \big| P(P_n) \rangle$$

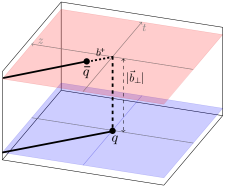
  - Proton matrix element, similar to collinear PDF

- Soft function definition:

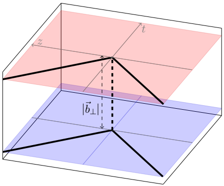
$$S_{n\bar{n}}(\vec{b}_T) = \langle 0 \big| [S^\dagger_{\bar{n}} S_n](\vec{b}_T) S^{(0,\vec{b}_T)}_{\perp,-\infty n} [S^\dagger_n S_{\bar{n}}](\vec{0}_T) S^{(0,\vec{b}_T)}_{\perp,-\infty\bar{n}} \big| 0 \rangle$$

  - Vacuum matrix element

- Wilson line paths:




Beam function

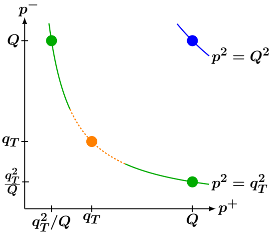



Soft function

# Rapidity (light-cone) divergences

$$\sigma(\vec{q}_T) = H(Q) \int \mathrm{d}^2\vec{b}_T\, e^{\mathrm{i}\vec{b}_T\cdot\vec{q}_T} B_n(x_1, \vec{b}_T) B_{\bar{n}}(x_2, \vec{b}_T) S_{n\bar{n}}(\vec{b}_T)$$

- Beam functions $B_{n,\bar{n}}$: collinear radiation
  - $p_n = (p_n^-, p_n^+, p_T) \sim (Q, q_T^2/Q, q_T)$
- Soft function $S_{n\bar{n}}$: soft radiation
  - $p_s = (p_s^-, p_s^+, p_T) \sim (q_T, q_T, q_T)$
- Beam and soft modes have virtuality $p^2 \sim q_T^2$
  - Induces *rapidity* (*light-cone*) singularities (not regulated by dimension regularization)




- Rapidity divergences arise from integrals of type

$$\int \mathrm{d}k^+ \mathrm{d}k^- \frac{f(k^+k^-)}{(k^+k^-)^{1+\epsilon}} = \int \frac{\mathrm{d}(k^+/k^-)}{2\,k^+/k^-} \int \mathrm{d}(k^+k^-) \frac{f(k^+k^-)}{(k^+k^-)^{1+\epsilon}}$$

  - Integrand depends only on product $k^+k^-$

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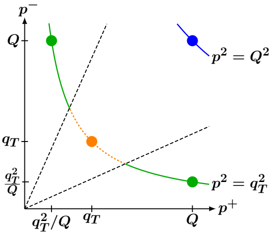


- Need additional rapidity regulator in both $B_n$ and $S_{n\bar{n}}$
  - Wilson lines off light cone [Collins '11]; analytic regulator [Becher, Bell '11]; $\delta$ regulator [Echevarria, Idilbi, Scimemi '11]; $\eta$ regulator [Chiu, Jain, Neill, Rothstein '12]; exponential regulator [Li, Neill, Zhu '16]
  - Final cross section $\sigma(\vec{q}_T)$ independent of regulator choice

# Rapidity (light-cone) divergences

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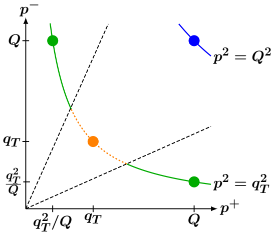



- Need additional rapidity regulator in both $B_n$ and $S_{n\bar{n}}$
  - Divergences cancel in TMDPDF $f_n^{\mathrm{TMD}} = \dfrac{B_n^{(\mathrm{unsub})}}{S_{n\bar{n}}^0}\sqrt{S_{n\bar{n}}}$
  - Regulator induces new scale $\zeta \sim Q^2$ from ratio of $B_n$ and $S_{n\bar{n}}$ rapidities

# Towards Quasi TMDPDFs

# Reminder: Collinear quasi PDF

- PDF:

$$f_q(x,\mu) = \int \frac{d\xi^+}{4\pi} e^{i\xi^+(xP_n^-)} \Big\langle P(P_n) \Big| \bar q(\xi^+) W_n(\xi^+,0)\,(\gamma_0+\gamma_3)\,q(0) \Big| P(P_n) \Big\rangle$$

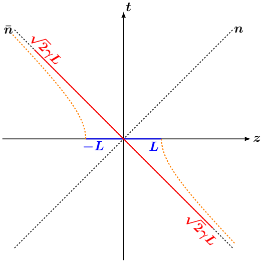
- Quasi PDF: Equal-time correlator [Ji '13, '14]

$$\tilde f_q(x,P_z,\mu) = \int \frac{dz}{4\pi} e^{iz(xP_z)} \big\langle P(P_z) \big| \bar q(z) W_z(z,0) \gamma_3 q(0) \big| P(P_z) \big\rangle$$

- Factorization theorem:

  [Xiong, Ji, Zhang, Zhao '13; Izubuchi, Ji, Jin, Stewart, Zhao '18]

$$\tilde f_i = C_{ij} \otimes f_j + \mathcal{O}\left(\frac{M^2}{P_z^2}, \frac{\Lambda^2_{\rm QCD}}{P_z^2}\right)$$




- Physical picture: equal-time correlation approaches lightlike correlation upon Lorentz boost

# Towards Quasi TMDPDFs

Known issues: [Ji, Sun, Xiong, Yuan 14; Ji, Jin, Yuan, Zhang, Zhao ’18]

1. Wilson line structure of quasi TMDPDF / quasi soft function on the lattice
   - Lattice size limits length $L$ of Wilson lines
2. Rapidity divergences
   - Affected by $L$

Our work:

3. Separately consider
   - unsubtracted quasi beam function ☺
   - (quasi) soft function ☹
4. Then consider quasi TMDPDF ☹

$$f_n^{\mathrm{TMD}} = \frac{B_n^{(\mathrm{unsub})}}{S_{n\bar{n}}^0} \sqrt{S_{n\bar{n}}}$$

# (Quasi) beam function at finite length

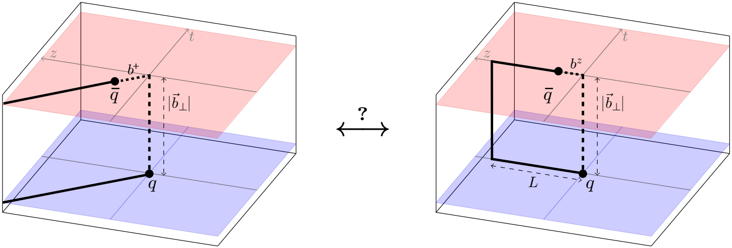
- Beam function definition:

$$B_n(b^+, \vec{b}_T) = \langle P(P_n) | \bar{q}(b^+, \vec{b}_T) W^{(0,\vec{0}_T)}_{(b^+,\vec{b}_T)} \frac{\gamma^-}{2} q(0) | P(P_n) \rangle$$

- Quasi beam function: (drop time dependence)

$$\tilde{B}_n(b^z, \vec{b}_T) = \langle P(P_z) | \bar{q}(b^z, \vec{b}_T) W^{(0,\vec{0}_T)}_{(b^z,\vec{b}_T)} \frac{\gamma^3}{2} q(0) | P(P_z) \rangle$$

- Path:




- Finite length $L$ requires to close Wilson line in transverse direction

# (Quasi) soft function at finite length

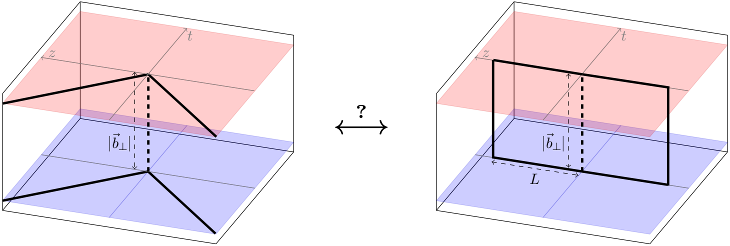
- Soft function definition:

$$S_{n\bar{n}}(\vec{b}_T) = \langle 0 \big| [S^\dagger_{\bar{n}} S_n](\vec{b}_T) S^{(0,\vec{b}_T)}_{\perp,-\infty n} [S^\dagger_n S_{\bar{n}}](\vec{0}_T) S^{(0,\vec{b}_T)}_{\perp,-\infty \bar{n}} \big| 0 \rangle$$

- Equal-time soft function definition: (drop time dependence)

$$\tilde{S}_{\hat{z},-\hat{z}}(\vec{b}_T) = \langle 0 \big| [S^\dagger_{-\hat{z}} S_{\hat{z}}](\vec{b}_T) S^{(0,\vec{b}_T)}_{\perp,-\infty \hat{z}} [S^\dagger_{\hat{z}} S_{-\hat{z}}](\vec{0}_T) S^{(0,\vec{b}_T)}_{\perp,\infty \hat{z}} \big| 0 \rangle$$

- Path:




- Finite length $L$ requires to close Wilson line in transverse direction

# Rapidity divergences at finite $L$

- Rapidity divergences arise from integrals of type

$$\int_0^\infty \mathrm{d}k^+ \mathrm{d}k^- \frac{f(k^+k^-)}{(k^+k^-)^{1+\epsilon}}$$

  - Integrand depends only on product $k^+k^-$

- Eikonal propagator for $L < \infty$:

$$\frac{1}{k^\pm + \mathrm{i}0} \to \frac{1 - e^{\mathrm{i}k^\pm L}}{k^\pm}$$

  - Finite $L$ fully regulates $k^\pm \to 0$
  - No rapidity divergences for finite $L$

- Explicitly checked at one loop

- ☺ No need to worry about rapidity divergences on lattice

- ☹ Hard to disentangle finite-$L$ effects from from rapidity divergences

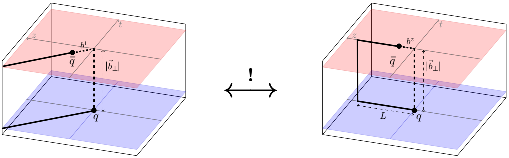
# Matching the unsub beam function

- (Quasi) beam function definition:

$$B_n(b^+, \vec{b}_T) = \langle P(P_n) \big| \bar{q}(b^+, \vec{b}_T) W^{(0,\vec{0}_T)}_{(b^+,\vec{b}_T)} \frac{\gamma^-}{2} q(0) \big| P(P_n) \rangle$$

$$\tilde{B}_n(b^z, \vec{b}_T) = \langle P(P_z) \big| \bar{q}(b^z, \vec{b}_T) W^{(0,\vec{0}_T)}_{(b^z,\vec{b}_T)} \frac{\gamma^3}{2} q(0) \big| P(P_z) \rangle$$

- Quasi beam function approaches beam function after Lorentz boost




  - Transverse separation not affected by boost
  - Boost argument should still apply for $\{LP_z\ ,\ L/b_T\ ,\ P_z b_T\} \gg 1$

$$\tilde{B}_i(x, \vec{b}_T; P^z, \tilde{\mu}, L) \sim \int_0^1 \frac{\mathrm{d}y}{y} C_{ij}\left(x, y; P^z, \tilde{\mu}, \mu, L, \nu\right) B_j(y, \vec{b}_T, \mu, \nu)$$

  - Spoiler: works only for some rapidity regulators $\to$ more details later

# Obstructions in matching the soft function

# Obstructions in matching the soft function

- Soft function and quasi soft function differ only by path:

$$S_{n\bar{n}}(\vec{b}_T) = \langle 0 \big| [S^\dagger_{\bar{n}} S_n](\vec{b}_T)\, S^{(0,\vec{b}_T)}_{\perp,-\infty n}\, [S^\dagger_n S_{\bar{n}}\, S^{(0,\vec{b}_T)}_{\perp,-\infty\bar{n}}](\vec{0}_T) \big| 0 \rangle$$

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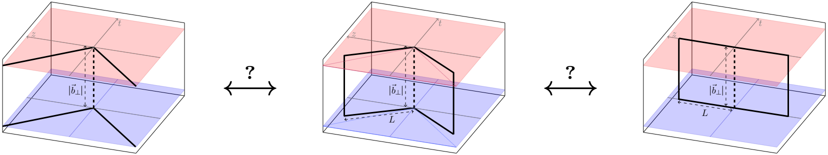
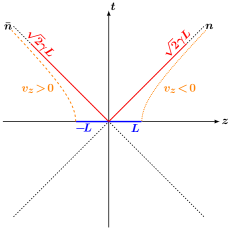
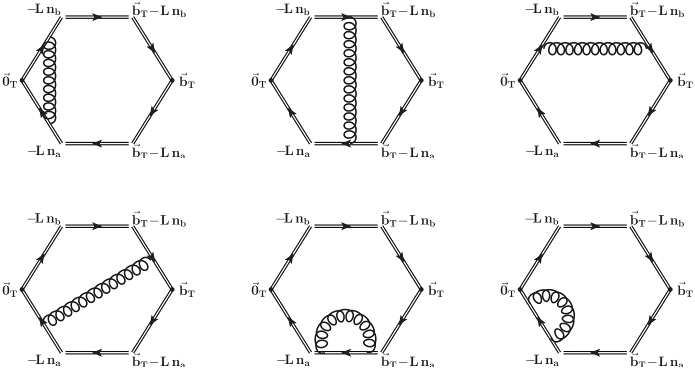



- Soft function depends on both $n$ and $\bar{n}$
- Simple boost argument fails:
  Can not simultaneously boost $\hat{z} \to n^\mu$ , $-\hat{z} \to \bar{n}^\mu$
  - Can we still derive a matching relation?



# Perturbative comparison of (quasi?) soft function

- $S_{n\bar{n}}$ and $\tilde{S}_{\hat{z},-\hat{z}}$ must describe the same IR physics
  - $\vec{b}_T$ dependence must be identical (since $b_T \sim \Lambda_{\rm QCD}^{-1}$)
- Compare at one loop for finite $L$, but take $L \gg b_T \sim \Lambda_{\rm QCD}^{-1}$
- Diagrams:



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- Compare at one loop for finite $L$, but take $L \gg b_T \sim \Lambda_{\mathrm{QCD}}^{-1}$
- Result:

$$S_{n\bar{n}} = -\frac{\alpha_s C_F}{\pi}\left[\frac{1}{\epsilon^2} + \frac{1}{\epsilon}(L_L - 1) - \frac{1}{2}{L_b}^2 + L_b(L_L - 1) + \cdots\right]$$

$$\tilde{S}_{\hat{z},-\hat{z}} = \frac{\alpha_s C_F}{\pi}\left[ + \frac{2}{\epsilon} + L_b + 2\pi\frac{L}{b_T}\right]$$

- Different UV physics
  $\rightarrow$ taken care of by matching
- Different dependence on rapidity renormalization parameter
  $\rightarrow$ taken care of by matching
- Different IR physics ($\vec{b}_T$)
  $\rightarrow$ No nonperturbative matching!

$$L_L = \ln\frac{L^2\mu^2}{e^{-2\gamma_E}}$$

$$L_b = \ln\frac{b_T^2\mu^2}{4e^{-2\gamma_E}}$$

# Comparison to previous work

- Quasi TMDPDF was previously studied in [Ji, Jin, Yuan, Zhang, Zhao '18]
  - do not separately consider $\tilde{B}_{\hat{z}}$ and $\tilde{S}_{\hat{z},-\hat{z}}$
  - but directly absorb $\tilde{f}_{\hat{z}}^{\mathrm{TMD}} = \dfrac{B_{\hat{z}}^{\mathrm{unsub}}}{\sqrt{S_{\hat{z},-\hat{z}}}}$

- Matching relation at NLO:

$$\tilde{f}_{\hat{z}}^{\mathrm{TMD}}(x, \vec{b}_T; \zeta) = e^{-S_w^q(\zeta, \vec{b}_T)} \left[1 - \frac{\alpha_s C_F}{\pi}\right] \tilde{f}_n^{\mathrm{TMD}}(x, \vec{b}_T; \zeta)$$

- Matching kernel involves nonperturbative component for $b_T \sim \Lambda_{\mathrm{QCD}}^{-1}$

$$e^{-S_w^q(\zeta, \vec{b}_T)} = \exp\left[\int_{c_0/b_T}^{\zeta} \frac{\mathrm{d}\mu}{\mu} \frac{\alpha_s(\mu') C_F}{\pi}\right]$$

- Interpretation: matching requires nonperturbative knowledge of $S_w^q$!

# Almost-lightlike soft function

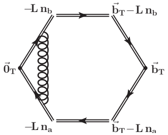
- Study smooth transition from quasi soft function to soft function

$$n^\mu = (1,0,0,+1) \quad \longrightarrow \quad n_v = (v,0,0,+1)$$
$$\bar{n}^\mu = (1,0,0,-1) \quad \longrightarrow \quad \bar{n}_v = (v,0,0,-1)$$

  - $v = 1$: soft function $S_{n\bar{n}}$
  - $v = 0$: spacelike soft function $S_{\hat{z},-\hat{z}}$

- Study limit $v \to 1$
  - $v < 1$ spacelike $\to$ accessible on lattice ?

- No smooth behavior of UV divergences:
  - $v = 1$: $\quad S \supset -\frac{\alpha_s C_F}{\pi}\left[\frac{1}{\epsilon^2} + \cdots\right]$
  - $v \to 1$: $\quad S \supset -\frac{\alpha_s C_F}{\pi}\left[\frac{1}{2\epsilon v}\ln\frac{1+v}{1-v} + \cdots\right]$
  - Different UV physics is absorbed in matching coefficient



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- Study limit $v \to 1$
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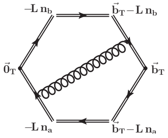
- Real corrections smoothly approach lightlike limit:

$$S \supset \frac{\alpha_s C_F}{2\pi}(n_v \cdot \bar{n}_v)\left[-\mathrm{Li}_2\left(\frac{-4L^2}{b_T^2}\right) + \mathcal{O}(v-1)\right]$$



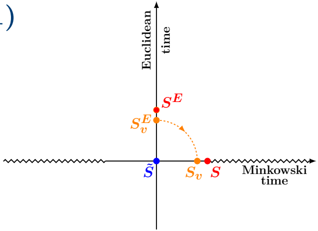

- $v \to 1$ can be matched onto lightlike result

# Almost-lightlike soft function from lattice

- Study smooth transition from quasi soft function to soft function

$$n^\mu = (1,0,0,+1) \quad \longrightarrow \quad n_v = (v,0,0,+1)$$
$$\bar{n}^\mu = (1,0,0,-1) \quad \longrightarrow \quad \bar{n}_v = (v,0,0,-1)$$

  - $v = 1$: soft function $S_{n\bar{n}}$
  - $v = 0$: spacelike soft function $S_{\hat{z},-\hat{z}}$




- Lattice can only calculate in Euclidean time
  - Analytical continuation required
- Illustration:

$$S_E \supset \frac{2\alpha_s C_F}{\pi}\left[ \qquad + \frac{1}{\epsilon} + 2\frac{\sqrt{2}L}{b_T} \arctan\frac{\sqrt{2}L}{b_T} + \cdots \right]$$
$$S_{n\bar{n}} \supset \frac{\alpha_s C_F}{\pi}\left[ -\frac{1}{\epsilon^2} - \frac{1}{\epsilon}\ln(L^2\mu^2) + \frac{1}{2}\ln^2(b_T^2\mu^2) + \cdots \right]$$

- No simple analytical continuation from lattice to Minkowski soft function

# (Naive) beam functions from lattice

# Naive beam functions from lattice

- Beam function:

$$B_n(b^+, \vec{b}_T) = \langle P(P_n) \big| \bar{q}(b^+, \vec{b}_T) W^{(0,\vec{0}_T)}_{(b^+,\vec{b}_T)} \frac{\gamma^-}{2} q(0) \big| P(P_n) \rangle$$

  - Can (in principle) be calculated from quasi beam function

- Soft function:

$$S_{n\bar{n}}(\vec{b}_T) = \langle 0 \big| [S^\dagger_{\bar{n}} S_n](\vec{b}_T) S^{(0,\vec{b}_T)}_{\perp,-\infty n} [S^\dagger_n S_{\bar{n}}](\vec{0}_T) S^{(0,\vec{b}_T)}_{\perp,-\infty \bar{n}} \big| 0 \rangle$$

  - Can not be calculated from quasi soft function

- TMDPDF:

$$f_n^{\mathrm{TMD}} = \frac{B_n^{(\mathrm{unsub})}}{S^0_{n\bar{n}}} \sqrt{S_{n\bar{n}}}$$

  - Can not be calculated from quasi TMDPDF due to soft subtraction

- Can still consider **ratios** where soft subtraction cancels!

# Ratios of TMDPDFs

- TMDPDF:

$$f_n^{\mathrm{TMD}} = \frac{B_n^{(\mathrm{unsub})}}{S_{n\bar{n}}^0} \sqrt{S_{n\bar{n}}}$$

- Can still consider **ratios** where soft subtraction cancels

$$\frac{\tilde{f}_{i/N}^{\mathrm{TMD}}}{\tilde{f}_{i/N'}^{\mathrm{TMD}}} = \frac{\tilde{B}_{i/N}^{\mathrm{unsub}}}{\tilde{B}_{i/N'}^{\mathrm{unsub}}} = \frac{\sum_j C_{ij} \otimes B_{j/N}^{\mathrm{unsub}}}{\sum_j C_{ij'} \otimes B_{j'/N'}^{\mathrm{unsub}}} \stackrel{?}{=} \frac{\sum_{j'} C_{ij} \otimes f_{j/N}^{\mathrm{TMD}}}{\sum_{j'} C_{ij'} \otimes f_{j'/N'}^{\mathrm{TMD}}}$$

- Caveat 1: Soft function differs for quarks and gluons
  - must not have mixing of quarks and gluons $\rightarrow$ isovector $i = u - d$
  - Example: Spin dependence ☺
  - Example: Nucleon dependence, e.g. $N = p, N' = n$ ☺
- Caveat 2: Both $\tilde{B}^{\mathrm{unsub}}$ and $B^{\mathrm{unsub}}$ depend on rapidity regulator
  - Lightcone rapidity regulator must be "boost-friendly":
    Regulator should only affect same direction as the Wilson line

# Boost-friendly rapidity regulators

- Rapidity regulator must not spoil the boost argument
- Illustration: Dependence on $\mathcal{L} = \ln(b_T \mu)$ in soft and beam functions

| Scheme | Regulator | $S_{n\bar{n}}$ | $B_n^{\text{unsub}}$ |
|---|---|---|---|
| Wilson lines off lightcone | $n_1 = (1, e^{-2y_1}, \vec{0}_T)$ <br> $n_2 = (e^{+2y_2}, 1, \vec{0}_T)$ | $\mathcal{L}$ | $\mathcal{L}^2$ |
| $\delta$ regulator | $k^- \to k^- + \mathrm{i}\delta^-$ | $\mathcal{L}^2$ | $\mathcal{L}$ |
| $\eta$ regulator | $k^- \to k^- \lvert k^-/\nu \rvert^\eta$ | $\mathcal{L}^2$ | $\mathcal{L}$ |
| Exp. regulator | $\mathrm{d}^d k \to \mathrm{d}^d k \, e^{-\tau k^0}$ | $\mathcal{L}^2$ | $\mathcal{L}^2$ |
| Analytic regulator | $k^+ \to (k^+)^{1+\alpha}$ | $1$ | $\mathcal{L}^2$ |
| Quasi TMDPDF | $L < \infty$ | $\mathcal{L}$ | $\mathcal{L}$ |

# Conclusion

# Conclusion

## TMD Factorization

- Involves both collinear and soft functions: $\sigma \sim B_n B_{\bar{n}} S_{n\bar{n}}$
- Wilson lines extend to infinity
  - Must close Wilson lines at length $L$ on lattice
- Must handle rapidity divergences
  - Regulated by Wilson line length $L$ on lattice

## Quasi TMDPDFs

- Can match unsubtracted beam function
- Can not match quasi soft function onto TMD soft function
  - Depends on both $n$ and $\bar{n}$ $\to$ boost argument fails
  - Hence can not simply match quasi TMDPDF onto TMDPDF!
- Ratios of unsubtracted beam functions work ...
  - because the soft function is universal and cancels
  - as long as quarks and gluon do not mix
  - matching simplest with a boost-friendly rapidity regulator

# Backup slides

# More details on rapidity regulators

## Collinear/soft Wilson lines

$$W_n = P\exp\left[ig\int_{-\infty}^{0} \mathrm{d}s\,\bar{n}\cdot A(\bar{n}s)\right] = \sum_{\text{perms}} \exp\left[-g\frac{\bar{n}\cdot A(k)}{\bar{n}\cdot k}\right]$$

$$S_n = P\exp\left[ig\int_{-\infty}^{0} \mathrm{d}s\,n\cdot A(ns)\right] = \sum_{\text{perms}} \exp\left[-g\frac{n\cdot A(k)}{n\cdot k}\right]$$

## $\eta$ regulator [Chiu, Jain, Neill, Rothstein '12]

$$W_n \to \sum_{\text{perms}} \exp\left[-g w^2 \frac{|\bar{n}\cdot\mathcal{P}|^{-\eta}}{\nu^{-\eta}}\frac{\bar{n}\cdot A(k)}{\bar{n}\cdot k}\right]$$

$$S_n \to \sum_{\text{perms}} \exp\left[-g w \frac{|2\mathcal{P}_z|^{-\eta/2}}{\nu^{-\eta/2}}\frac{n\cdot A(k)}{n\cdot k}\right]$$

# More details on rapidity regulators

## Collinear/soft Wilson lines

$$W_n = P\exp\left[ig\int_{-\infty}^{0} \mathrm{d}s\, \bar{n}\cdot A(\bar{n}s)\right] = \sum_{\text{perms}} \exp\left[-g\frac{\bar{n}\cdot A(k)}{\bar{n}\cdot k}\right]$$

$$S_n = P\exp\left[ig\int_{-\infty}^{0} \mathrm{d}s\, n\cdot A(ns)\right] = \sum_{\text{perms}} \exp\left[-g\frac{n\cdot A(k)}{n\cdot k}\right]$$

## $\delta$ regulator [Echevarria, Idilbi, Scimemi '11; Echevarria, Scimemi, Vladimirov '16]

$$W_n \to P\exp\left[ig\int_{-\infty}^{0} \mathrm{d}s\, \bar{n}\cdot A(\bar{n}s)e^{-\delta^+ s}\right]$$

$$S_n \to P\exp\left[ig\int_{-\infty}^{0} \mathrm{d}s\, n\cdot A(ns)e^{+\delta^- s}\right]$$

# More details on rapidity regulators

## Collinear/soft Wilson lines

$$W_n = P \exp\left[ \mathrm{i}g \int_{-\infty}^{0} \mathrm{d}s\, \bar{n} \cdot A(\bar{n}s) \right] = \sum_{\text{perms}} \exp\left[ -g \frac{\bar{n} \cdot A(k)}{\bar{n} \cdot k} \right]$$

$$S_n = P \exp\left[ \mathrm{i}g \int_{-\infty}^{0} \mathrm{d}s\, n \cdot A(ns) \right] = \sum_{\text{perms}} \exp\left[ -g \frac{n \cdot A(k)}{n \cdot k} \right]$$

## Exponential regulator [Li, Neill, Zhu '16]

- Does not modify Wilson lines, but phase space integrals ($b_0 = 2e^{-\gamma_E}$):

$$\int \mathrm{d}^d k \to \lim_{\tau \to 0} \int \mathrm{d}^d k \, e^{-k^0 \tau\, b_0}$$

# More details on rapidity regulators

## Collinear/soft Wilson lines

$$W_n = P \exp\left[ig \int_{-\infty}^{0} \mathrm{d}s\, \bar{n}\cdot A(\bar{n}s)\right] = \sum_{\text{perms}} \exp\left[-g \frac{\bar{n}\cdot A(k)}{\bar{n}\cdot k}\right]$$

$$S_n = P \exp\left[ig \int_{-\infty}^{0} \mathrm{d}s\, n\cdot A(ns)\right] = \sum_{\text{perms}} \exp\left[-g \frac{n\cdot A(k)}{n\cdot k}\right]$$

## Analytic regulator [Becher, Neubert '10]

- Replace eikonal propagator ($p$ = momentum along Wilson line)

$$\frac{1}{n\cdot k - \mathrm{i}0} \to \frac{\nu_1^{2\alpha}\, \bar{n}\cdot p}{[(n\cdot k)(\bar{n}\cdot p) - \mathrm{i}0]^{1+\alpha}}$$

$$\frac{1}{\bar{n}\cdot k - \mathrm{i}0} \to \frac{\nu_2^{2\beta}\, n\cdot p}{[(\bar{n}\cdot k)(n\cdot p) - \mathrm{i}0]^{1+\beta}}$$

- $S_{n\bar{n}} = 1$ in this regulator
- Asymmetric: take first $\beta \to 0$, then $\alpha \to 0$ (or vice versa)
  $\to S_{n\bar{n}} = 1$ absorbed into *one* TMDPDF