Particle number conservation in different matching procedures

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Outline

- Quark distributions and quark quasi-distributions
- Extracting quark distributions from the quasi-distributions
- Convoluting the matching kernels with the distributions
  - Ideal case
  - Real case
- Renormalizing the kernels
- Summary

Quark distributions

The most general form of the matrix element is:

$$\langle P|O^{\mu_1\mu_2\cdots\mu_n}|P\rangle = 2a_n^{(0)}\Pi^{\mu_1\mu_2\cdots\mu_n}$$

$$\Pi^{\mu_1\mu_2\cdots\mu_n} = \sum_{j=0}^{k}(-1)^j\frac{(2k-j)!}{2^j(2k)!}\{g\cdots gP\cdots P\}_{k,j}(P^2)^j$$

We use the following four-vectors

$$P = (P_0,0,0,P_3) \quad \lambda = (1,0,0,-1)/\sqrt{2} \longrightarrow \boxed{\lambda\cdot P = (P_0+P_3)/\sqrt{2} = P_+}$$

$$\lambda_{\mu_1}\lambda_{\mu_2}\left\langle P\middle|O^{\mu_1\,\mu_2}\middle|P\right\rangle = 2a_n^{(0)}\left(P^+P^+ - \lambda^2\frac{M^2}{4}\right) = 2a_n^{(0)}P^+P^+$$

In general, we have

$$\lambda_{\mu_1}\cdots\lambda_{\mu_n}\Pi^{\mu_1\cdots\mu_n} = (P^+)^n \longrightarrow \boxed{\langle P|O^{+\cdots+}|P\rangle = 2a_n^{(0)}(P^+)^n}$$

Taking the inverse Mellin transform

$$a_n^{(0)} = \int dx\, x^{n-1} q(x) \qquad\qquad q(x) = \frac{1}{2\pi i}\int_{-i\infty}^{+i\infty} dn\, x^{-n} a_n^{(0)}$$

Using $\quad a_n^{(0)} = \langle P|O^{+\cdots+}|P\rangle / 2(P^+)^n$

$$\boxed{q(x) = \int_{-\infty}^{+\infty} \frac{d\xi^-}{4\pi} e^{-ixP^+\xi^-} \langle P|\bar{\psi}(\xi^-)\gamma^+ W(\xi^-,0)\psi(0)|P\rangle}$$

$$W(\xi^-,0) = e^{-ig\int_0^{\xi^-} A^+(\eta^-)d\eta^-} \quad \text{(Wilson line)}$$

- Light cone correlations
- Equivalent to the distributions in the Infinite Momentum Frame
- Light cone dominated $\quad \xi^2 = t^2 - z^2 \sim 0$
- Not calculable on Euclidian lattice $\quad t^2 + z^2 \sim 0$

Quasi Distributions

X. Ji, "Parton Physics on a Euclidean Lattice," PRL 110 (2013) 262002.

Suppose we project outside of the light-cone:

$$\lambda = (0,0,0,-1) \qquad P = (P_0, 0, 0, P_3) \qquad \boxed{\lambda \cdot P = P_3}$$

We take n=2

$$\langle P|O^{33}|P\rangle = 2\tilde{a}_n^{(0)}(P^3P^3 - \cancel{\lambda^2}_{=-1} P^2/4) = 2\tilde{a}_n^{(0)}((P^3)^2 + P^2/4)$$

Mass terms contribute

In general,

$$\boxed{\langle P|O^{3\cdots3}|P\rangle = 2\tilde{a}_{2k}^{(0)}(P_3)^{2k}\sum_{j=0}^{k}\mu^j\frac{(2k-j)!}{j!\,(2k-2j)!} \equiv 2\tilde{a}_{2k}(P_3)^{2k}}$$

with $\boxed{\mu = M^2/4(P_3)^2}$

Defining $\tilde{a}_n^{(0)} = \int dx\, x^{n-1}\tilde{q}^{(0)}(x)$ $\quad$ $\tilde{a}_n = \int dx\, x^{n-1}\tilde{q}(x)$

Taking the inverse Mellin transform $\tilde{q}^{(0)}(x) = \frac{1}{2\pi i}\int_{-i\infty}^{+i\infty} dn\, x^{-n}\tilde{a}_n^{(0)}$ $\quad$ $\tilde{q}(x) = \frac{1}{2\pi i}\int_{-i\infty}^{+i\infty} dn\, x^{-n}\tilde{a}_n$

$$\rightarrow \quad \tilde{q}(x) = \tilde{q}^{(0)}(\xi)/(1+\mu\xi^2) \; + antiquarks \qquad \xi = \frac{2x}{1+\sqrt{1+4\mu x^2}}$$

$$\rightarrow \quad \boxed{\tilde{q}(x, P_3) = \int_{-\infty}^{+\infty} \frac{dz}{4\pi} e^{izk_3}\langle P|\bar{\psi}(z)\gamma^3 W(z,0)\psi(0)|P\rangle}$$

$$W(z,0) = e^{-ig\int_0^z A^3(z')dz'}$$

$$k_3 = xP_3$$

- Nucleon moving with finite momentum in the z direction
- Pure spatial correlation
- Can be simulated on a lattice
- Can be related to the usual distributions via a matching procedure

The matrix elements of the qPDF´s, $\langle P|\bar{\psi}(z)\gamma^{3}W(z,0)\psi(0)|P\rangle$ contain standard log divergences with respect to the regulator a, and also power divergences related to the Wilson line, which resums into a multiplicative exponential form

These are renormalized the intermediate RI'-MOM scheme, with the computation of the renormalization functions $Z^{RI`}(z,\mu)$ (Martha´s talk)

From this point, we have two options to obtain the quark distributions in the $\overline{MS}$ scheme

Match the RI'-MOM quasi-distribution directly to the $\overline{MS}$ quark distribution;
(Zhao´s talk)

Or do a two step process:

1) Convert the ME from RI'-MOM to $\overline{MS}$ using perturbation theory (Martha´s talk)

2) Match the $\overline{MS}$ qPDF to the $\overline{MS}$ PDF

Extracting quark distributions from quark quasi-distributions

Infrared region untouched when going from
a finite to an infinite momentum

Infinite momentum: $p_3 \rightarrow \infty$ (before integrating over the quark transverse momentum k_T)

$$q(x,\mu) = q_{bare}(x)\left\{1 + \frac{\alpha_s}{2\pi} Z_F(\mu)\right\} + \frac{\alpha_s}{2\pi}\int_x^1 \Gamma\left(\frac{x}{y},\mu\right) q_{bare}(y)\frac{dy}{y} + \mathcal{O}(\alpha_s^2)$$

Finite momentum: $p_3 \;\; fixed$

$$\tilde{q}(x,P_3) = q_{bare}(x)\left\{1 + \frac{\alpha_s}{2\pi} \tilde{Z}_F\,(P_3)\right\} + \frac{\alpha_s}{2\pi}\int_{x/y_c}^1 \tilde{\Gamma}\left(\frac{x}{y},P_3\right) q_{bare}(y)\frac{dy}{y} + \mathcal{O}(\alpha_s^2)$$

$$\tilde{q}(\pm y_c) = 0$$

In principle, $y_c \rightarrow \infty$

Solving for the quark distributions

$$q(x,\mu) = \tilde{q}(x,p_3) - \frac{\alpha_s}{2\pi}\tilde{q}(x,p_3)\delta Z_F\left(\frac{\mu}{p_3},x_c\right) - \frac{\alpha_s}{2\pi}\int_{-x_c}^{-|x|/y_c}\delta\Gamma\left(y,\frac{\mu}{p_3}\right)\tilde{q}\left(\frac{x}{y},p_3\right)\frac{dy}{|y|}$$
$$- \frac{\alpha_s}{2\pi}\int_{+|x|/y_c}^{+x_c}\delta\Gamma\left(y,\frac{\mu}{p_3}\right)\tilde{q}\left(\frac{x}{y},p_3\right)\frac{dy}{|y|}$$

$$\delta\Gamma = \tilde{\Gamma} - \Gamma$$

$$\delta Z_F = \tilde{Z}_F - Z_F$$

The integral in x in the quasi-quark self-energy, $\tilde{Z}_F$, is left unintegrated, hence the dependence on the limits of integration, $\pm x_c$. At the end, $x_c \to \infty$ in δZ_F.

Because quasi-quark vertex correction, $\tilde{\Gamma}$, only vanishes at the infinity, the range of integration in the vertex also extends to zero in the convolution as $y_c \to \infty$

In the $\overline{MS}$ scheme for the γ^3 case ($x_c \to \infty$)

$$\delta\Gamma\left(y, \frac{\mu}{p_3}\right) = \begin{cases} -\frac{1+y^2}{1-y} ln \frac{y-1}{y} + 1 & y > 1 \\ -\frac{1+y^2}{1-y} ln \frac{\mu^2}{4p_3^2 y(1-y)} + \frac{2-5y+y^2}{1-y} & 0 < y < 1 \\ -\frac{1+y^2}{1-y} ln \frac{y}{y-1} - 1 & y < 0 \end{cases}$$

$$\delta Z_F\left(\frac{\mu}{p_3}, x_c\right) = \begin{cases} \int_{-x_c}^{+x_c} d\eta \left(\frac{1+\eta^2}{1-\eta} ln \frac{\eta-1}{\eta} - 1\right) & \eta > 1 \\ \int_{-x_c}^{+x_c} d\eta \left(\frac{1+\eta^2}{1-\eta} ln \frac{\mu^2}{4p_3^2 \eta(1-\eta)} - \frac{2-5\eta+\eta^2}{1-\eta}\right) & 0 < \eta < 1 \\ \int_{-x_c}^{+x_c} d\eta \left(\frac{1+\eta^2}{1-\eta} ln \frac{\eta}{\eta-1} + 1\right) & \eta < 0 \end{cases}$$

W. Wang, S. Zhao and R. Zhu, ``Gluon quasidistribution function at one loop,'' Eur.\ Phys.\ J.\ C {78} (2018) no.2, 147, arXiv:1708.02458.

I. Stewart and Y. Zhao, ``Matching the Quasi Parton Distribution in a Momentum Subtraction Scheme,'' Phys. Rev. D 97 (2018) 054512 arXiv:1709.04933

F. Steffens, unpublished

Ideal Case: integrating from -∞ to +∞

Conservation of the quark number requires

$$\int_{-\infty}^{+\infty} q(x)dx = \int_{-\infty}^{+\infty} \tilde{q}(x)dx \qquad \tilde{q}(y_c \to \infty) = 0$$

And this implies in

$$\delta Z_F(x_c)\int_{-\infty}^{+\infty} dx\,\tilde{q}(x) = -\int_{-\infty}^{+\infty} dx \int_{-x_c}^{0} \delta\Gamma(y)\tilde{q}\left(\frac{x}{y}\right)\frac{dy}{|y|} - \int_{-\infty}^{+\infty} dx \int_{0}^{+x_c} \delta\Gamma(y)\tilde{q}\left(\frac{x}{y}\right)\frac{dy}{|y|}$$

For the positive y region:

$$\int_{-\infty}^{+\infty} dx \int_{0}^{+x_c} \delta\Gamma(y)\tilde{q}\left(\frac{x}{y}\right)\frac{dy}{|y|} = \left(\int_{0}^{+1}\delta\Gamma(y)dy + \int_{+1}^{+x_c}\delta\Gamma(y)dy\right)\int_{-\infty}^{+\infty}\tilde{q}(x)dx$$

Adding also the negative y region:

$$\delta Z_F(x_c)\int_{-\infty}^{+\infty} dx\,\tilde{q}(x) = -\int_{-1}^{+1}\delta\Gamma(y)\,dy\int_{-\infty}^{+\infty}\tilde{q}(t)\,dt$$

$$-\left(\int_{-x_c}^{-1}\delta\Gamma(y)\,dy+\int_{+1}^{+x_c}\delta\Gamma(y)\,dy\right)\int_{-\infty}^{+\infty}\tilde{q}(t)\,dt$$

We have just seen that:

$$\delta Z_F(x_c) = -\int_{-x_c}^{+x_c}\delta\Gamma(y)\,dy$$

Valid also in the limit $x_c \to +\infty$

Which implies in

$$\int_{-\infty}^{+\infty} q(x)\,dx = \int_{-\infty}^{+\infty}\tilde{q}(x)\,dx$$

As long as $\tilde{q}(x)$ extends to infinity

Integral in the self-energy can be extended to infinity;

Divergences are automatically cancelled in the convolution;

Particle number is conserved.

In practice: integration from -y_c to $+y_c$

$$\int_{-y_c}^{+y_c} q(x)dx = \int_{-y_c}^{+y_c} \tilde{q}(x)dx \qquad\qquad \tilde{q}(y_c\ finite) = 0$$

Then we must have

$$\delta Z_F(x_c)\int_{-y_c}^{+y_c} dx\, \tilde{q}(x,p_3) = -\int_{-y_c}^{+y_c} dx \int_{-x_c}^{-|x|/y_c} \delta\Gamma(y)\,\tilde{q}\left(\frac{x}{y}\right)\frac{dy}{|y|}$$
$$-\int_{-y_c}^{+y_c} dx \int_{+|x|/y_c}^{+x_c} \delta\Gamma(y)\,\tilde{q}\left(\frac{x}{y}\right)\frac{dy}{|y|}$$

For the positive y region

$$\int_{-y_c}^{+y_c} dx \int_{+|x|/y_c}^{+x_c} \delta\Gamma(y)\,\tilde{q}\left(\frac{x}{y}\right)\frac{dy}{|y|} = \int_0^{+1} \delta\Gamma(y)dy \int_{-y_c}^{+y_c} \tilde{q}(t)dt + \int_1^{+x_c} \delta\Gamma(y)dy \int_{-y_c/|y|}^{+y_c/|y|} \tilde{q}(t)dt$$

Adding also the negative y region:

$$\delta Z_F(x_c)\int_{-y_c}^{+y_c}\tilde{q}(x)dx = -\int_{-1}^{+1}\delta\Gamma(y)dy\int_{-y_c}^{+y_c}\tilde{q}(t)dt$$

$$-\left(\int_{-x_c}^{-1}\delta\Gamma(y)dy+\int_{+1}^{+x_c}\delta\Gamma(y)dy\right)\int_{-y_c/|y|}^{+y_c/|y|}\tilde{q}(t)dt$$

Which is in general not satisfied, being dependent on the large y behaviour of $\delta\Gamma(y)$ and on the behaviour of $\tilde{q}(t\approx 0)$;

$\delta Z_F(x_c)$ diverges logarithmically with x_c; its cancellation with a similar log on the RHS of the above equation is not guaranteed anymore; Similar to what happens in the x by x computation.

To make this explicit, we notice that the vertex correction behaves as:

$$\delta\Gamma(y\to\infty)\to-\frac{3}{2y}$$

One possibility is to add a zero to the vertex, and redistribute the terms. For the positive y region, one has:

$$\int_{+1}^{+x_c}\left(\delta Z_F(y)-\frac{3}{2y}\right)dy\int_{-y_c}^{+y_c}\tilde{q}(x)dx=-\int_{+1}^{+x_c}\left(\delta\Gamma(y)+\frac{3}{2y}\right)dy\int_{-y_c/|y|}^{+y_c/|y|}\tilde{q}(t)dt$$

As long as

$$\int_{+1}^{+x_c}\left(\frac{1}{y}\right)dy\int_{-y_c/|y|}^{+y_c/|y|}\tilde{q}(t)dt=\int_{+1}^{+x_c}\left(\frac{1}{y}\right)dy\int_{-y_c}^{+y_c}\tilde{q}(t)dt$$

In this case, all integrals are separately finite as $x_c \to \infty$

For real data, the above condition seems to be satisfied

But we can, of course, do better than this….

Formally, the extra terms in the last slide can be seen as the result of renormalizing the UV divergences of the integrated vertex and self-energy corrections outside the physical region $0 < x < 1$.

The soft divergences (at $x = 1$) cancel between the real (vertex) and virtual (self-energy) corrections, as it is explicit through the plus prescription at $x = 1$.

The UV divergences, associated with the infinite momentum fraction in quasi-pdfs, however, have to be renormalized separately

$$Z_2^{-1} - 1 = -\frac{\partial \tilde{Z}_F\ (p_3)}{p_3}$$

$$\tilde{\Gamma}_\mu^R = (Z_2^{-1} - 1)\gamma_\mu + Z_2 \tilde{\Gamma}_\mu$$

The Ward identity requires that the renormalization of the vertex and self-energy be the same

Thus, the self-energy outside the physical region ($\eta > 1$) in DR:

$$\delta Z_F(\eta > 1) = \int_1^\infty d\eta\eta^{d-1}\left(\frac{1+\ \eta^2}{1-\eta} ln\frac{\eta-1}{\eta} - 1\right), \qquad d = 1-\epsilon$$

$$(Z_2^{-1} - 1)|_{div} = -\frac{3}{2\epsilon} = \int_1^\infty d\eta \frac{1}{\eta^{1+\epsilon}}$$

And the renormalized self-energy is

$$\delta Z_F^R(\eta > 1) = \int_1^\infty d\eta\eta\left(\frac{1+\ \eta^2}{1-\eta} ln\frac{\eta-1}{\eta} - 1 - \frac{3}{2\eta}\right)$$

Thus, the renormalized integrated vertex is

$$\delta\Gamma^R(\eta > 1) = \int_1^\infty d\eta\delta\Gamma(\eta) = \int_1^\infty d\eta\left(\frac{1+\eta^2}{1-\eta} ln\frac{\eta}{\eta-1} + 1 + \frac{3}{2\eta}\right)$$

How is the matching affected by this?

Sutracting the divergence at x_c

$$\delta\Gamma(y) = \frac{1+y^2}{1-y} ln\frac{y}{y-1} + 1 + \frac{3}{2}\delta(y - x_c)\ln(x_c)$$

But with the convolution:

$$\int_{+|x|/y_c}^{+x_c} \delta(y - x_c)\tilde{q}\left(\frac{x}{y}\right)\frac{dy}{|y|} = \left(-1 + 2\,\theta(x_x)\right)\frac{1}{x_c}\;\tilde{q}\left(\frac{x}{x_c}\right) \longrightarrow 0$$

In the limit that $x_c \to \infty$, as long as $\tilde{q}(x)$ is finite or diverges slower than $\frac{1}{x}$ as $x \to 0$

And

$$\int_1^{x_c} dy\left(\frac{1+y^2}{1-y} ln\frac{y}{y-1} + 1 + \frac{3}{2}\delta(y - x_c)\ln(x_c)\right)$$

Is finite as $x_c \to \infty$

In practice, however, when doing the convolution, the extra term in the delta function can be neglected

But the renormalized self-energy stands as it is:

$$\delta Z_F^R(\eta > 1) = \int_1^\infty d\eta\eta\left(\frac{1+\ \eta^2}{1-\eta} ln\frac{\eta-1}{\eta} - 1 - \frac{3}{2\eta}\right)$$

Because the self-energy part has been renormalized, the integrals can be extended to infinity.

For the physical region, for example, one then has

$$\delta Z_F^R(0 \le \eta \le 1) = \int_0^1 d\eta\left(\frac{1+\eta^2}{(1-\eta)_+}\left(ln\frac{\mu^2}{4p_3^2} - \ln\big(\eta(1-\eta)\big)\right) - \frac{2-5\eta+\eta^2}{(1-\eta)_+}\right)$$

$$= -\frac{3}{2} ln\frac{\mu^2}{4p_3^2} - 3 - \frac{7}{2}$$

While for the unphysical region, one has

$$\delta Z_F^R(\eta < 0) + \delta Z_F^R(\eta > 1) = +3$$

Putting everything together, one gets

$$q(x,\mu) = \tilde{q}(x,p_3) - \frac{\alpha_s}{2\pi}\tilde{q}(x,p_3)\delta Z_F^R\left(\frac{\mu}{p_3}\right) - \frac{\alpha_s}{2\pi}\int_{-\infty}^{-|x|/y_c}\delta\Gamma\left(y,\frac{\mu}{p_3}\right)\tilde{q}\left(\frac{x}{y},p_3\right)\frac{dy}{|y|}$$
$$- \frac{\alpha_s}{2\pi}\int_{+|x|/y_c}^{+\infty}\delta\Gamma\left(y,\frac{\mu}{p_3}\right)\tilde{q}\left(\frac{x}{y},p_3\right)\frac{dy}{|y|}$$

$$+\frac{\alpha_s}{2\pi}\left(\frac{3}{2}ln\frac{\mu^2}{4p_3^2}+\frac{7}{2}\right)\tilde{q}(x,p_3)$$

Same result if one uses Eq. (67) of Izubuchi et al. (1801.03971) for the convolution;

P_3 dependence of the integral? How strong for real data?

Subtracting in the integrand

We go back to the integrated vertex

$$\delta\Gamma^R(x>1) = \int_1^\infty dx \delta\Gamma(x) = \int_1^\infty dx \left(\frac{1+x^2}{1-x} ln \frac{x}{x-1} + 1 + \frac{3}{2x} \right)$$

We make the simplest choice:

$$\delta\Gamma^R(x) = \frac{1+x^2}{1-x} ln \frac{x}{x-1} + 1 + \frac{3}{2x}, \qquad x > 1$$

And similarly for the negative region

$$\delta\Gamma^R(x) = -\frac{1+x^2}{1-x} ln \frac{x}{x-1} - 1 + \frac{3}{2(1-x)}, \qquad x < 1$$

And the final matching can be written taking $x_c \to \infty$

$$\delta\Gamma^R\left(y, \frac{\mu}{p_3}\right) = \begin{cases} -\frac{1+y^2}{1-y} ln \frac{y-1}{y} + 1 + \frac{3}{2y} & y > 1 \\ -\frac{1+y^2}{1-y} ln \frac{\mu^2}{4p_3^2 y(1-y)} + \frac{2-5y+y^2}{1-y} & 0 < y < 1 \\ -\frac{1+y^2}{1-y} ln \frac{y}{y-1} - 1 + \frac{3}{2(1-y)} & y < 0 \end{cases}$$

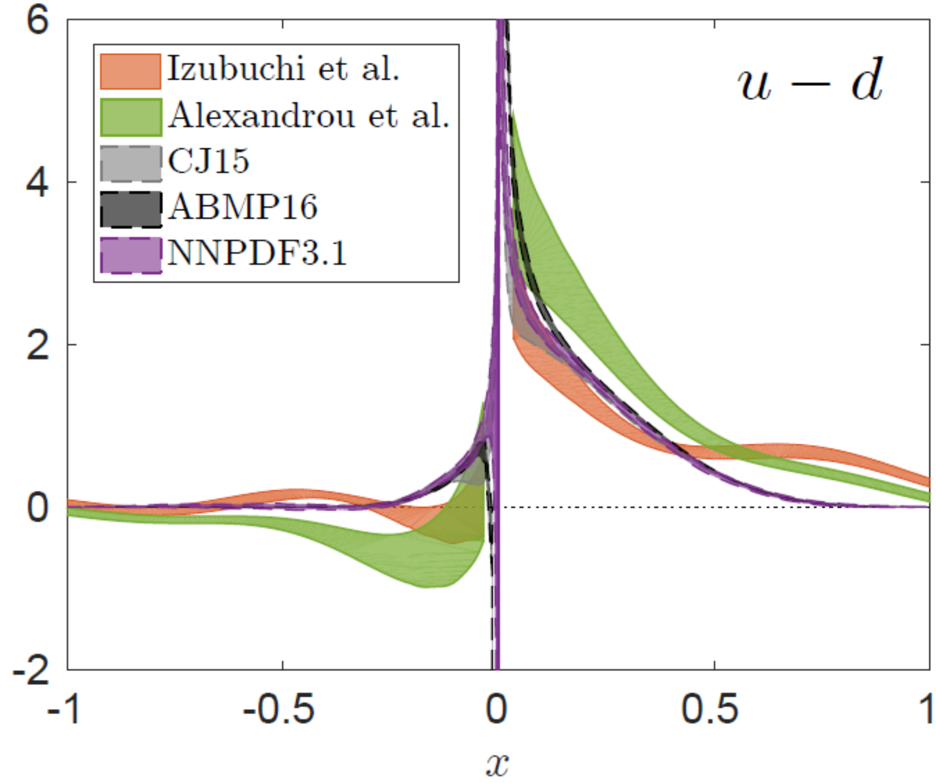
$$\delta Z_F^R\left(\frac{\mu}{p_3}\right) = \begin{cases} \int_{-\infty}^{+\infty} d\eta \left(\frac{1+\eta^2}{1-\eta} ln \frac{\eta-1}{\eta} - 1 - \frac{3}{2\eta}\right) & \eta > 1 \\ \int_{-\infty}^{+\infty} d\eta \left(\frac{1+\eta^2}{1-\eta} ln \frac{\mu^2}{4p_3^2 \eta(1-\eta)} - \frac{2-5\eta+\eta^2}{1-\eta}\right) & 0 < \eta < 1 \\ \int_{-\infty}^{+\infty} d\eta \left(\frac{1+\eta^2}{1-\eta} ln \frac{\eta}{\eta-1} + 1 - \frac{3}{2(1-\eta)}\right) & \eta < 0 \end{cases}$$

Renormalizes the whole momentum fraction in the unphysical region;

Automatically preserves quark number in all stages of the computation;

It is not scale dependent.

Unpolarized nonsinglet distribution at $10\pi/L$



Details on the computation of this plot will be shown in K. Cichy talk

When subtracting at x_c, integral of the distribution seems to be violated by $\sim 10\%$; violation increases with momentum

Summary

Matching between quasi PDF and light-cone PDF in the $\overline{MS}$ scheme

If $\tilde{q}(x)$ and the its integral extends to infinity, particle number is conserved

If $\tilde{q}(x)$ is set to vanish at some finite value, one has to worry about the infinities not only when doing the convolution, but also for the integrated distributions

Two ways to handle with the infinities:

1) Subtracting at the infinity (x_c)

   It seems, however, that the quark number increases with P_3; about 10% violation for ETMC data at $\frac{10\pi}{L}$, increasing with P_3

2) Subtracting in the integrand

   Always preserve quark number; modification in the unphysical region becomes more and more irrelevant as P_3 increases