OPE Analysis of Quasi-PDFs

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Outline

Talk is mostly based on work "Factorization Theorem Relating Euclidean and Light-Cone Parton Distributions" by **Taku Izubuchi**, **Xiangdong Ji**, **Luchang Jin**, **Iain Stewart**, and **Yong Zhao**. **arXiv:1801.03917**.

- **PDF**
- Quasi-PDF
- Matching between Quasi-PDF and PDF
- Matching in coordinate space
- Moments and Quasi-PDF
- Conclusion

Disclaimer: We limit the discussion to iso-vector (quasi-)PDF to avoid any issue related with gluon (quasi-)PDF.

Define PDF

$$O(\xi^-) \;=\; \bar{\psi}(\xi^-)\gamma^+ U(\xi^-,0)\psi(0) \tag{1}$$

where $U(\xi^-,0)$ is the gauge link along the light cone direction, $\xi^- = (t-z)/\sqrt{2}$ is the light cone coordinate and t is physical time, and $\gamma^+ = (\gamma^0 + \gamma^z)/\sqrt{2}$.

$$Q\left(\zeta = P^+\xi^-, \frac{\mu}{\Lambda_{\mathrm{QCD}}}\right) \;=\; \frac{1}{2\,P^+}\langle P|O(\xi^-)|P\rangle \tag{2}$$

$|P\rangle$ represent a single hadron state with its momentum along z direction. Renormalization is needed to evaluate the matrix elements. One can conveniently choose $\overline{\mathrm{MS}}$ renormalization scheme and denote the renormalization scale to be μ. Another scale Λ_{QCD} is introduced and abbreviate it as Λ.

$$q\left(x, \frac{\mu}{\Lambda_{\mathrm{QCD}}}\right) \;=\; \int \frac{d\,\zeta}{2\pi} e^{-ix\zeta} Q\left(\zeta, \frac{\mu}{\Lambda_{\mathrm{QCD}}}\right) \tag{3}$$

Finally, PDF of the hadron is defined above as $q(x,\, \mu/\Lambda)$. This definition has a special property:

$$\bar{q}(x, \mu/\Lambda_{\mathrm{QCD}}) \;=\; -q(-x, \mu/\Lambda_{\mathrm{QCD}}) \tag{4}$$

QCD factorization

$$F_1(x,Q^2) \;=\; \int \frac{d\xi}{|\xi|} C_1\left(\frac{x}{\xi},\frac{Q^2}{\mu^2}\right) q\left(\xi,\frac{\mu}{\Lambda_{\rm QCD}}\right) + \text{higher twist contributions} \tag{5}$$

where $F_1(x,Q^2)$ is commonly known as the deep-inelastic scattering (DIS) structure function.

Operator product expansion (OPE)

$$O(\xi^-) \;=\; \sum_n \frac{(i\,\xi^-)^n}{n!} O^{+\cdots+} \tag{6}$$

where $O^{+\cdots+}$ is a component of the following symmetric traceless operator:

$$O^{\mu_0\mu_1\cdots\mu_n} \;=\; \bar{\psi}\,\gamma^{(\mu_0}\,i\,D^{\mu_1}\ldots i\,D^{\mu_n)}\,\psi - \text{trace} \tag{7}$$

The moments of PDF $a_n(\mu/\Lambda)$ is related with OPE:

$$\frac{1}{2}\langle P|O^{\mu_0\mu_1\cdots\mu_n}|P\rangle \;=\; a_{n+1}\left(\frac{\mu}{\Lambda_{\rm QCD}}\right)(P^{\mu_0}P^{\mu_1}\ldots P^{\mu_n} - \text{trace}) \tag{8}$$

Here we demonstrate the reason we call $a_n(\mu/\Lambda_{\rm QCD})$ the moment of PDF.

On one hand:

$$\frac{1}{2}\langle P|O^{+\cdots+}|P\rangle = a_{n+1}\left(\frac{\mu}{\Lambda_{\rm QCD}}\right)(P^+)^{n+1} \tag{9}$$

$$Q\left(\zeta=P^+\xi^-,\frac{\mu}{\Lambda_{\rm QCD}}\right) = \sum_n \frac{(i\,\zeta)^n}{n!}a_{n+1}\left(\frac{\mu}{\Lambda_{\rm QCD}}\right) \tag{10}$$

$$\left(\frac{\partial}{\partial i\,\zeta}\right)^n Q\left(\zeta,\frac{\mu}{\Lambda_{\rm QCD}}\right)\bigg|_{\zeta=0} = a_{n+1}\left(\frac{\mu}{\Lambda_{\rm QCD}}\right) \tag{11}$$

On the other hand:

$$Q\left(\zeta,\frac{\mu}{\Lambda_{\rm QCD}}\right) = \int d\,x\,e^{ix\zeta}q\left(x,\frac{\mu}{\Lambda_{\rm QCD}}\right) \tag{12}$$

$$\left(\frac{\partial}{\partial i\,\zeta}\right)^n Q\left(\zeta,\frac{\mu}{\Lambda_{\rm QCD}}\right)\bigg|_{\zeta=0} = \int d\,x\,x^n\,q\left(x,\frac{\mu}{\Lambda_{\rm QCD}}\right) \tag{13}$$

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Calculate PDF on lattice

$$O^{\mu_0\mu_1\ldots\mu_n} = \bar{\psi}\gamma^{(\mu_0} iD^{\mu_1}\ldots iD^{\mu_n)}\psi - \mathrm{trace}$$

Moments of PDF

In principle, the moments of PDF can be calculated on lattice.

There are extensive lattice studies of moments of PDF.

However, because the lattice regulator breaks rotational symmetry, the operator $O^{\mu_0\mu_1\cdots\mu_n}$ ($n \geqslant 4$) mixes with lower dimensional operator. It is very hard to compute the 4th or higher order moments directly.

Calculate PDF on lattice

$$O^{\mu_0\mu_1\ldots\mu_n} = \bar{\psi}\gamma^{(\mu_0} i D^{\mu_1} \ldots i D^{\mu_n)} \psi - \text{trace}$$

Non-local operator

Using a non-local operator (or two local operators) does not have the mixing problem. The new problem is: in Euclidean space, non-local implies non-zero x^2, here x is the 4-vector characterizing the shape of the non-local operator, but the PDF is related to the light-cone region where $x^2 = 0$. Non-zero x^2 usually implies higher twist effects. The issue then become how to suppress these higher twist effects.

Quasi-PDF: large momentum hadron state

"Parton Physics on a Euclidean Lattice" by **Xiangdong Ji**, Phys.Rev.Lett. 110 (2013) 262002

Other approaches

- "Deep-inelastic scattering and the operator product expansion in lattice QCD" by **William Detmold** and **David Lin**, 2006.
- "Restoration of rotational symmetry in the continuum limit of lattice field theories" by **Zohreh Davoudi** and **Martin Savage**, Phys.Rev. D86 (2012) 054505.
- "Parton Distribution Function from the Hadronic Tensor on the Lattice" by **Keh-Fei Liu**, 2016.
- "Nucleon structure functions from lattice operator product expansion" by **QCDSF**, 2017.

Concept is introduced by **Xiangdong Ji** Phys.Rev.Lett. 110 (2013) 262002.

Introduce a non-local operator which is:

- Calculable on Euclidean spacetime lattice.
- Similar to the operator used to define PDF.

$$\tilde{O}(z) \;=\; \bar{\psi}(z)\gamma^z U(z,0)\psi(0) \tag{14}$$

Then we define its matrix elements and Fourier transformation similarly.

$$\tilde{Q}\left(\zeta = P^z z, \mu^2 z^2 = \frac{\mu^2\zeta^2}{(P^z)^2}, \frac{\mu}{\Lambda_{\mathrm{QCD}}}\right) \;=\; \frac{1}{2\,P^z}\langle P|\tilde{O}(z)|P\rangle \tag{15}$$

Note an important difference compare with PDF is the introduction of z^2.

$$\tilde{q}\left(\tilde{x}, \frac{\mu^2}{(P^z)^2}, \frac{\mu}{\Lambda_{\mathrm{QCD}}}\right) \;=\; \int \frac{d\,\zeta}{2\pi} e^{-i\tilde{x}\zeta}\,\tilde{Q}\left(\zeta, \frac{\mu^2\zeta^2}{(P^z)^2}, \frac{\mu}{\Lambda_{\mathrm{QCD}}}\right) \tag{16}$$

The hope is: the quasi-PDF $\tilde{q}(\tilde{x}, \mu^2/(P^z)^2, \mu/\Lambda_{\mathrm{QCD}})$ can be used to determine PDF if the momentum of the hadron P^z is large.

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$$\tilde{O}(z) \;=\; \bar{\psi}(z)\gamma^z U(z,0)\psi(0)$$

OPE can be performed for quasi-PDF operator ($z^2 \to 0$):

$$\tilde{O}(z) \;=\; \sum_n C_n(\mu^2 z^2)\,\frac{(i\,z)^n}{n!}\,O^{z\cdots z} + \text{higher twist contributions} \tag{17}$$

Similar to $O^{+\cdots+}$, $O^{z\cdots z}$ is also a component of the operator:

$$O^{\mu_0\mu_1\cdots\mu_n} \;=\; \bar{\psi}\,\gamma^{(\mu_0}\,i\,D^{\mu_1}\ldots i\,D^{\mu_n)}\,\psi - \text{trace}$$

$$\frac{1}{2}\langle P|O^{z\ldots z}|P\rangle \;=\; a_{n+1}\left(\frac{\mu}{\Lambda_{\text{QCD}}}\right)((P^z)^{n+1} - \text{trace}) \tag{18}$$

Use large momentum P^z to suppress the higher twist and trace contributions.

$$\tilde{Q}\left(\zeta, \frac{\mu^2\zeta^2}{(P^z)^2}, \frac{\mu}{\Lambda_{\text{QCD}}}\right) \;=\; \sum_n C_n\left(\frac{\mu^2\,\zeta^2}{(P^z)^2}\right)\frac{(i\,\zeta)^n}{n!}\,a_{n+1}\left(\frac{\mu}{\Lambda_{\text{QCD}}}\right) \tag{19}$$

$$\tilde{Q}\left(\zeta, \frac{\mu^2\zeta^2}{(P^z)^2}, \frac{\mu}{\Lambda_{\rm QCD}}\right) = \sum_n C_n\left(\frac{\mu^2\zeta^2}{(P^z)^2}\right)\frac{(i\,\zeta)^n}{n!}\,a_{n+1}\left(\frac{\mu}{\Lambda_{\rm QCD}}\right)$$
$$= \sum_n C_n\left(\frac{\mu^2\zeta^2}{(P^z)^2}\right)\frac{(i\,\zeta)^n}{n!}\int_{-1}^{1} dx\, x^n\, q\left(x, \frac{\mu}{\Lambda_{\rm QCD}}\right)$$

$$\tilde{q}\left(\tilde{x}, \frac{\mu^2}{(P^z)^2}, \frac{\mu}{\Lambda_{\rm QCD}}\right) = \int \frac{d\zeta}{2\pi} e^{-i\tilde{x}\zeta}\,\tilde{Q}\left(\zeta, \frac{\mu^2\zeta^2}{(P^z)^2}, \frac{\mu}{\Lambda_{\rm QCD}}\right)$$
$$= \int_{-1}^{1} dx\left[\int \frac{d\zeta}{2\pi} e^{-i\tilde{x}\zeta}\sum_n C_n\left(\frac{\mu^2\zeta^2}{(P^z)^2}\right)\frac{(i\,\zeta)^n}{n!}x^n\right] q\left(x, \frac{\mu}{\Lambda_{\rm QCD}}\right)$$

Define:

$$C\left(\frac{\tilde{x}}{x}, \frac{\mu^2}{(x\,P^z)^2}\right) = \int \frac{d\,(x\,\zeta)}{2\pi} e^{-i\frac{\tilde{x}}{x}(x\zeta)}\sum_n C_n\left(\frac{\mu^2\,(x\,\zeta)^2}{(x\,P^z)^2}\right)\frac{(i\,x\,\zeta)^n}{n!} \tag{20}$$

We have:

$$\tilde{q}\left(\tilde{x}, \frac{\mu^2}{(P^z)^2}, \frac{\mu}{\Lambda}\right) = \int_{-1}^{1}\frac{dx}{|x|}\,C\left(\frac{\tilde{x}}{x}, \frac{\mu^2}{(x\,P^z)^2}\right) q\left(x, \frac{\mu}{\Lambda}\right) \tag{21}$$

Determine the matching kernel

The same matching formula should also apply to quark state (quasi-)PDF. We denote the quark state (quasi-PDF) by the same function name but the argument $\mu/\Lambda_{\mathrm{QCD}}$ absent.

$$\tilde{q}\left(\tilde{x}, \frac{\mu^2}{(P^z)^2}\right) = \int_{-1}^{1} \frac{d\,x}{|x|} C\left(\frac{\tilde{x}}{x}, \frac{\mu^2}{(x\,P^z)^2}\right) q(x) \tag{22}$$

One can perturbatively compute $q(x)$ and $\tilde{q}(\tilde{x}, \mu^2/(P^z)^2)$, then solves the matching kernel C.

Using pure dimensional regularization (canceling UV and IR poles with $\epsilon_{\mathrm{UV}} = \epsilon_{\mathrm{IR}}$), massless quark, the quark state PDF does not depend on any quantity that has a scale, therefore all the loop contribution will be zero and it is equal to the tree level result:

$$q(x) = \delta(x-1) \tag{23}$$

Immediately we have (in $\overline{\mathrm{MS}}$ scheme):

$$C\left(\tilde{x}, \frac{\mu^2}{(P^z)^2}\right) = \tilde{q}\left(\tilde{x}, \frac{\mu^2}{(P^z)^2}\right) \tag{24}$$

One loop calculation for $C(\tilde{x}, \mu^2/(P^z)^2)$ is already been done.

Fourier transform differently

Quasi-PDF is defined to be a Fourier transform of the coordinate space matrix elements:

$$\tilde{q}\left(\tilde{x}, \frac{\mu^2}{(P^z)^2}, \frac{\mu}{\Lambda_{\mathrm{QCD}}}\right) = \int \frac{d\,\zeta}{2\pi}\, e^{-i\tilde{x}\zeta}\, \tilde{Q}\left(\zeta, \frac{\mu^2\,\zeta^2}{(P^z)^2}, \frac{\mu}{\Lambda_{\mathrm{QCD}}}\right) \tag{25}$$

A.V. Radyushkin discovered that one can perform a slightly different Fourier transformation, he call the resulting quantity **Pseudo-PDF** (which is only non-zero when $|x| \leqslant 1$):

$$\mathcal{P}\left(x, \mu^2 z^2, \frac{\mu}{\Lambda_{\mathrm{QCD}}}\right) = \int \frac{d\,\zeta}{2\pi}\, e^{-ix\zeta}\, \tilde{Q}\left(\zeta, \mu^2 z^2, \frac{\mu}{\Lambda_{\mathrm{QCD}}}\right) \tag{26}$$

Similarly, we can prove the matching formula using OPE:

$$\mathcal{P}\left(x, \mu^2 z^2, \frac{\mu}{\Lambda_{\mathrm{QCD}}}\right) = \int_0^1 \frac{d\,x}{|x|}\, \mathcal{C}\left(\frac{\tilde{x}}{x}, \mu^2 z^2\right) q\left(x, \frac{\mu}{\Lambda_{\mathrm{QCD}}}\right) \tag{27}$$

Again, with dimensional regularization and massless quark, the matching kernel is the same as the quark state pseudo-PDF:

$$\mathcal{C}(x, \mu^2 z^2) = \mathcal{P}(x, \mu^2 z^2) = \int_0^1 \frac{d\,\zeta}{2\pi}\, e^{-ix\zeta}\, \tilde{Q}(\zeta, \mu^2 z^2) \tag{28}$$

This can be determined perturbatively and one-loop results is obtained.

Comparing the matching formulae

DIS structure function:

$$F_1(x, Q^2) \;=\; \int_{-1}^{1} \frac{d\xi}{|\xi|}\, C_1\!\left(\frac{x}{\xi}, \frac{Q^2}{\mu^2}\right) q\!\left(\xi, \frac{\mu}{\Lambda_{\text{QCD}}}\right) + \text{higher twist contributions}$$

Quasi-PDF:

$$\tilde{q}\!\left(\tilde{x}, \frac{\mu^2}{P_z^2}, \frac{\mu}{\Lambda_{\text{QCD}}}\right) \;=\; \int_{-1}^{1} \frac{d\,x}{|x|}\, C\!\left(\frac{\tilde{x}}{x}, \frac{\mu^2}{(x\,P^z)^2}\right) q\!\left(x, \frac{\mu}{\Lambda_{\text{QCD}}}\right) + \text{higher twist contributions}$$

Pseudo-PDF:

$$\mathcal{P}\!\left(x, \mu^2 z^2, \frac{\mu}{\Lambda_{\text{QCD}}}\right) \;=\; \int_{0}^{1} \frac{d\,x}{|x|}\mathcal{P}\left(\frac{\tilde{x}}{x}, \mu^2 z^2\right) q\!\left(x, \frac{\mu}{\Lambda_{\text{QCD}}}\right) + \text{higher twist contributions}$$

- The three scales are Q^2, $(x\,P^z)^2$, $1\,/\,z^2$. Note that they are all related with **single parton**.
- One may view quasi-PDF and pseudo-PDF as lattice observables which can be "factorized" into PDF. One may find other relevant and perhaps better lattice observables. **Yan-Qing Ma** and **Jian-Wei Qiu**, arXiv:1404.6860.
- In particular, $F_1(x,\ Q^2)$ can also be computed on lattice. Many different approaches: **QCDSF** Phys.Rev.Lett. 118 (2017) no.24, 242001; **Keh-Fei Liu** arXiv:1603.07352; **William Detmold** and **David Lin** Phys.Rev. D73 (2006) 014501.

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It would be interesting to see what the matching formula looks like in coordinate space.

$$Q\left(\zeta = P^+\xi^-, \frac{\mu}{\Lambda_{\rm QCD}}\right) = \sum_n \frac{(i\,\zeta)^n}{n!} a_{n+1}\left(\frac{\mu}{\Lambda_{\rm QCD}}\right) \tag{29}$$

$$\begin{aligned}
\tilde{Q}\left(\zeta = P^z\, z, \mu^2 z^2, \frac{\mu}{\Lambda_{\rm QCD}}\right) &= \sum_n C_n(\mu^2 z^2)\, \frac{(i\,\zeta)^n}{n!}\, a_{n+1}\left(\frac{\mu}{\Lambda_{\rm QCD}}\right) \\
&= \sum_n \left[\int_0^1 d\,\alpha\, \mathcal{C}(\alpha, \mu^2 z^2)\, \alpha^n\right] \frac{(i\,\zeta)^n}{n!}\, a_{n+1}\left(\frac{\mu}{\Lambda_{\rm QCD}}\right) \\
&= \int_0^1 d\,\alpha\, \mathcal{C}(\alpha, \mu^2\, z^2) \sum_n \alpha^n \frac{(i\,\zeta)^n}{n!}\, a_{n+1}\left(\frac{\mu}{\Lambda_{\rm QCD}}\right) \\
&= \int_0^1 d\,\alpha\, \mathcal{C}(\alpha, \mu^2\, z^2)\, Q\left(\alpha\,\zeta, \frac{\mu}{\Lambda_{\rm QCD}}\right)
\end{aligned}$$

where

$$C_n(\mu^2 z^2) = \int_0^1 d\,\alpha\, \mathcal{C}(\alpha, \mu^2 z^2)\, \alpha^n \tag{30}$$

Matching in coordinate space

$$\tilde{Q}\left(\zeta = P^z z, \mu^2 z^2, \frac{\mu}{\Lambda_{\mathrm{QCD}}}\right) = \int_0^1 d\alpha\, \mathcal{C}(\alpha, \mu^2 z^2)\, Q\left(\alpha\zeta, \frac{\mu}{\Lambda_{\mathrm{QCD}}}\right) \tag{31}$$

The form of the matching formula does not imply z is fixed in the matching. In fact, we can easily change the parameter in the above formula:

$$\tilde{Q}\left(\zeta = P^z z, \frac{\mu^2\zeta^2}{(P^z)^2}, \frac{\mu}{\Lambda_{\mathrm{QCD}}}\right) = \int_0^1 d\alpha\, \mathcal{C}\left(\alpha, \frac{\mu^2\zeta^2}{(P^z)^2}\right) Q\left(\alpha\zeta, \frac{\mu}{\Lambda_{\mathrm{QCD}}}\right) \tag{32}$$

- One can use these coordinate space matching formula to derive the matching formula for quasi-PDF.
- Or, one can apply the coordinate space matching formula directly to the lattice data, which are also measured in the coordinate space. One can then Fourier transform the one-loop matched, light-cone, Ioffe-time distribution Q to obtain PDF.

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The coefficients of the Taylor expansion of Q gives the moments of PDF.

$$Q\left(\zeta = P^+\xi^-, \frac{\mu}{\Lambda_{\rm QCD}}\right) = \sum_n \frac{(i\,\zeta)^n}{n!} a_{n+1}\left(\frac{\mu}{\Lambda_{\rm QCD}}\right) \tag{33}$$

However, the moments of quasi-PDF, in general, do not exist. (Note that the moments of pseudo-PDF do exist, but require non-trivial matching coefficients, given by $C_n(\mu^2 z^2)$, which we have computed explicitly at one-loop.) Based on OPE:

$$\tilde{Q}\left(\zeta = P^z\,z, \mu^2 z^2, \frac{\mu}{\Lambda_{\rm QCD}}\right) = \sum_n C_n(\mu^2 z^2)\,\frac{(i\,\zeta)^n}{n!} a_{n+1}\left(\frac{\mu}{\Lambda_{\rm QCD}}\right) \tag{34}$$

The reason is that C_n is singular in the $z \to 0$ limit. Especially, for the first moment:

$$\tilde{Q}\left(\zeta = P^z\,z, \mu^2\,z^2, \frac{\mu}{\Lambda_{\rm QCD}}\right)\Bigg|_{P^z z \to 0} = C_0(\mu^2 z^2) a_1\left(\frac{\mu}{\Lambda_{\rm QCD}}\right) = C_0(\mu^2 z^2) \tag{35}$$

In $\overline{\rm MS}$ scheme at one-loop order:

$$C_0(\mu^2 z^2) = 1 + \frac{\alpha_s C_F}{2\pi}\left[\frac{3}{2}\log\left(\frac{\mu^2 z^2\, e^{2\gamma_E}}{4}\right) + \frac{7}{2}\right]. \tag{36}$$

In the $z \to 0$ limit (P^z fixed):

$$\tilde{Q}\left(\zeta = P^z z, \mu^2 z^2, \frac{\mu}{\Lambda_{\text{QCD}}} \right)\Bigg|_{z\to 0} = C_0(\mu^2 z^2) = 1 + \frac{\alpha_s C_F}{2\pi}\left[\frac{3}{2}\log\left(\frac{\mu^2 z^2 e^{2\gamma_E}}{4} \right) + \frac{7}{2} \right] \quad (37)$$

However, one might have expected a different result:

$$\tilde{Q}\left(\zeta = P^z z, \mu^2 z^2, \frac{\mu}{\Lambda_{\text{QCD}}} \right)\Bigg|_{z\to 0} = \frac{1}{2 P^z}\langle P|\bar{\psi}(0)\gamma^z \psi(0)|P\rangle = 1\,? \quad (38)$$

The singularity is a result of the $\overline{\text{MS}}$ scheme subtraction. Before renormalization:

$$\tilde{Q}\left(\zeta = P^z z, \mu^2 z^2, \frac{\mu}{\Lambda_{\text{QCD}}} \right)\Bigg|_{z\to 0} - 1 \sim \frac{(\mu^2 z^2)^\epsilon}{\epsilon} \to 0 \quad (39)$$

However, if we expand on ϵ first

$$\frac{(\mu^2 z^2)^\epsilon}{\epsilon} = \frac{1}{\epsilon} + \log(\mu^2 z^2) \quad (40)$$

The result is logarithmic divergent after $\overline{\text{MS}}$ subtraction. (Not true for some other scheme.)

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Using the operator product expansion, we proved the large-momentum factorization of the quasi-Parton Distribution Function. The matching formula between quasi-PDF and PDF is:

$$\tilde{q}\left(\tilde{x}, \frac{\mu^2}{P_z^2}, \frac{\mu}{\Lambda_{\rm QCD}}\right) = \int_{-1}^{1} \frac{d\,x}{|x|}\, C\left(\frac{\tilde{x}}{x}, \frac{\mu^2}{(x\,P^z)^2}\right) q\left(x, \frac{\mu}{\Lambda_{\rm QCD}}\right) + \mathcal{O}\left(\frac{\Lambda_{\rm QCD}^2}{(P^z)^2}\right)$$

or the coordinate space version:

$$\tilde{Q}\left(\zeta = P^z\, z, \frac{\mu^2\zeta^2}{(P^z)^2}, \frac{\mu}{\Lambda_{\rm QCD}}\right) = \int_0^1 d\,\alpha\, \mathcal{C}\left(\alpha, \frac{\mu^2\zeta^2}{(P^z)^2}\right) Q\left(\alpha\,\zeta, \frac{\mu}{\Lambda_{\rm QCD}}\right) + \mathcal{O}\left(\frac{\Lambda_{\rm QCD}^2}{(P^z)^2}\right)$$

Thank You!

Renormalization of quasi-PDF

$$\tilde{O}(z) \;=\; \bar{\psi}(z)\gamma^z U(z,0)\psi(0)$$

This is not the first time we deal with this kind of operator. In heavy quark effective theory (HQET), we also have this kind of operator.

According to **X. Ji, J.H. Zhang, Y. Zhao** arXiv:1706.08962 and **T. Ishikawa, Y.Q. Ma, J.W. Qiu** arXiv:1707.03107, the non-local operator renormalize multiplicatively:

$$\big[\tilde{O}(z)\big]^{\overline{\rm MS}} \;=\; Z(\mu^2 z^2, \mu\, a)\big[\tilde{O}(z)\big]^{\rm Lat} \tag{41}$$

However, there is a subtle caveat discovered by **M. Constantinou** and **H. Panagopoulos** arXiv:1705.11193. The above renormalization formula is only correct for Chiral lattice fermion actions, e.g. domain wall fermion (DWF). For some other popular fermion actions like Wilson fermion, there is a mixing:

$$\big[\tilde{O}(z)\big]^{\overline{\rm MS}} \;=\; Z(\mu^2 z^2, \mu\, a)\big[\tilde{O}(z)\big]^{\rm Lat} + Z'(\mu^2 z^2, \mu\, a)\big[\tilde{O}'(z)\big]^{\rm Lat} \tag{42}$$

$$\tilde{O}'(z) \;=\; \bar{\psi}(z)U(z,0)\psi(0) \tag{43}$$

In practice, it is convenient to renormalize the lattice operator first and measure the renormalized matrix elements on the lattice. However, it should be noted that the form of the matching formula is unchanged when applied to the bare lattice results, since the initial OPE formula has the same shape:

$$\left[\tilde{Q}\left(\zeta = P^z z, \left(\frac{z}{a}\right)^2, \frac{1}{a\,\Lambda_{\rm QCD}}\right)\right]^{\rm Lat} = \sum_n \frac{C_n(\mu^2 z^2)}{Z(\mu^2 z^2, \mu\, a)} \frac{(i\,\zeta)^n}{n!} a_{n+1}\left(\frac{\mu}{\Lambda_{\rm QCD}}\right) \qquad (44)$$

To avoid computing the renormalization function Z, which depends on the lattice action, one can compute the ratio of two matrix elements using the same lattice operator. The numerator is simply $\left[\tilde{Q}\left(\zeta = P^z z, \left(\frac{z}{a}\right)^2, \frac{1}{a\,\Lambda_{\rm QCD}}\right)\right]^{\rm Lat}$, for the denominator, there are two common choices

- RI/MOM: use the matrix elements of a large Euclidean momentum (off-shell) quark state.

  I. Stewart and **Y. Zhao**.

- Reduced Ioffe-time distribution: $\left[\tilde{Q}\left(0, \left(\frac{z}{a}\right)^2, \frac{1}{a\,\Lambda_{\rm QCD}}\right)\right]^{\rm Lat} = \frac{C_0(\mu^2 z^2)}{Z(\mu^2 z^2, \mu\, a)}$.

  A. V. Radyushkin.

The requirement for the denominator is that the corresponding matrix elements in the continuum renormalization scheme can be calculated perturbatively.

Other operators

Light cone:

$$O(\xi^-) \;=\; \bar{\psi}(\xi^-)\gamma^+ U(\xi^-, 0)\psi(0) \tag{45}$$

Initial version:

$$\tilde{O}(z) \;=\; \bar{\psi}(z)\gamma^z U(z, 0)\psi(0) \tag{46}$$

- Use t direction gamma matrix:

$$\bar{\psi}(z)\gamma^t U(z, 0)\psi(0) \tag{47}$$

- Two vector current operators **Y.Q. Ma**, **J.W. Qiu** (possibly flavor changing **W. Detmold**, **D. Lin**):

$$J_\mu(z) J_\nu(0) \tag{48}$$